15-451 — Algorithms — Spring 2006

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Assignment 1

Due: Tuesday, January 31, 2006.

Some Reminders:

- You may discuss these problems with others, in small groups. However we strongly recommend that you think for a while about them yourself before starting such discussions.
- The work that you turn in must be your own, written by you in your own words. We are allowing handwritten solutions, although typeset ones are preferred. If you handwrite, WRITE CLEARLY, or we will revert to the old system of requiring you to typeset solutions.
- The cover page of your submission must clearly display the assignment number, your name, your recitation section and your Andrew ID.

1 Counting Passwords Revisited

Charlie (from Mini 1) still manages the security of a network, despite his incompetence. He sets up a new system (new means better, right?) in which a valid password consists of a string of length n from the alphabet $\{0...9\}$ that does not contain two adjacent 0s. Give a recurrence in one variable that describes the number of valid passwords of length n.

2 The NEW Price is Right!

You are a contestant on The **NEW** Price is Right. Come on down! There is a prize hidden behind door number 1. The value of the prize is a positive integer N, which you don't know. To win the prize, you have to guess N. Your goal is to do it in as few guesses as possible. You start with a number of chips (specified below). Each chip allows you one guess that's too high. If you guess too high, and you have no chips, you lose. So, for example, if you start with no chips, then you can win in N guesses simply guessing the sequence $1, 2, 3, \ldots N$. In each of the following parts, try to find the best strategy you can.

(a) What if you have 1 chip? Describe a strategy that makes $o(N)$ guesses. Find a function $g_1(N) = o(N)$ which is an upper bound on the number of guesses your strategy needs.

(b) What if you have 2 chips? Describe a strategy and a function $g_2(N) = o(g_1(N))$ which is an upper bound on the number of guesses your strategy needs.

(c) What if you have an unlimited number of chips? Describe a strategy and a function $g_\infty(N) = o(g_2(N))$ which is an upper bound on the number of guesses your strategy needs.

3 Getting Your Gun

It's the GI Joe ball, and everyone's invited. However, everyone must leave their gun at the door, in a big basket. There are n people at the ball, and each brings one gun. Suddenly, Cobra attacks. Each GI Joe soldier runs to the gun basket, grabs a random gun, and returns fire. In expectation, how many GI Joe soldiers are holding their own guns when the dust settles? (Hint: use linearity of expectations.)

4 Permutations

Let P be the following permutation of three elements:

$$P(abc) \longrightarrow bca$$

Starting with the sequence $1, 2, \ldots, n$, permutation P is repeatedly applied (at arbitrary starting points) to consecutive three-tuples of the sequence. Answer the following questions, and prove your answers.

(a) Is it possible to achieve any permutation of $1, 2, \ldots, n$ by such applications of P?

(b) What fraction of the $n!$ permutations are achievable in this manner?

5 Mutant Vampire Bats

There are k hikers trekking through a cave after dark. Unfortunately for them, they are about to stumble upon n angry mutant vampire bats. Each bat randomly attacks a hiker, independently of the others. Since these are really angry bats, they each want a victim to themselves, and will be mad at other bats that attack the same hiker they do.

(a) In expectation, how many (unordered) pairs of bats get angry at each other? Don't worry if your answer doesn't simplify very much. Now, use your formula to find this expectation in the special case that $n = k$, using $\Theta(\cdot)$ notation (i.e. your answer should be of the form $\Theta(f(k))$).

(b) In expectation, how many bats are angry at some other bat? Again, use your formula to find this expectation in the special case that $n = k$, using $\Theta(\cdot)$ notation.

For this problem, you may find the following useful. For $1 \leq k \leq n$

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq e^k \left(\frac{n}{k}\right)^k$$

For all real numbers a, b

$$(1-a)^b \leq e^{-ba}$$

and $(1 - 1/n)^n \geq 1/4$ for all $n \geq 2$.