Algorithm Design and Analysis

Victor Adamchik — CS 15-451 — Spring 2014

Lecture 14 — Feb 14, 2014 — Carnegie Mellon University

Graph Algorithms - 4

Plan:

Min-cost Spanning Tree Algorithms:

- Prim's (review)
- Arborescence problem

Kleinberg-Tardos, Ch. 4

The Minimum Spanning Tree for <u>Undirected</u> Graphs

Find a spanning tree of minimum total weight.

The weight of a spanning tree is the sum of the weights on all the edges which comprise the spanning tree.

The Minimum Spanning Tree

Joseph Kruskal (1929-2010)

Boruvka's Algorithm (1926)
Kruskal's Algorithm (1956)
Prim's Algorithm (1957)

Robert Prim (1921-)

Prim's Algorithm

Greedy algorithm that builds a tree one VERTEX at a time.

First described by Jarnık in a 1929 letter to Boruvka.
Rediscovered by Kruskal in 1956, by Prim in 1957, by Loberman and Weinberger in 1957, and finally by Dijkstra in 1958.

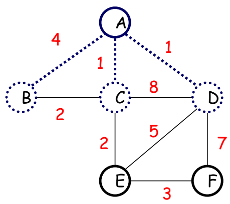
Prim's Algorithm

algorithm builds a tree one VERTEX at a time.

- Start with an arbitrary vertex as component *C*
- Expand *C* by adding a new vertex having the minimum weight edge with exactly one end point in *C*.
- Continue to grow the tree until *C* gets all vertices.

Prim's Algorithm

C={a}



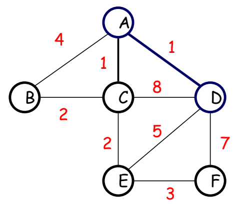
heap

d-1 c-1 b-4 e-oo f-oo

decleteMin

Prim's Algorithm

C={a,d}

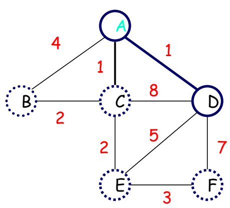


heap

c-1 b-4 e-oo f-oo

Prim's Algorithm

C={a,d}



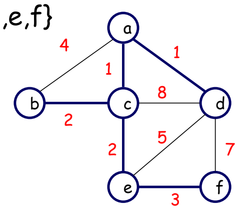
heap

c-1 b-4 e-5 f-7

decreaseKey

Prim's Algorithm

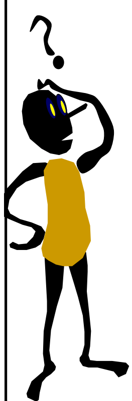
C={a,d,c,b,e,f}



Weight = 1+1+2+2+3 = 9

Property of the MST

Lemma: Let X be any subset of the vertices of G, and let edge e be the smallest edge connecting X to G-X. Then e is part of the minimum spanning tree.



What is the worst-case runtime complexity of Prim's Algorithm?

We run deleteMin V times
We update the queue E times

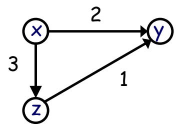
O(V*log V + E*log V)

deleteMin

decreaseKey

O(1) – Fibonacci heap

The Minimum Spanning Tree for Directed Graphs



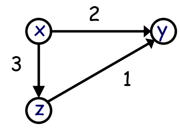
Start at X and follow the greedy approach

We will get a tree of size 5, though the min is 4.

However there is even a smaller subset of edges - 3

The Minimum Spanning Tree for Directed Graphs

This example exhibits two problems



What is the meaning of MST for directed graphs?

Clearly, we want to have a rooted tree, in which we can reach any vertex staring at the root

How would you find it?

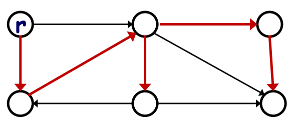
Clearly, the greedy approach of Prim's does not work

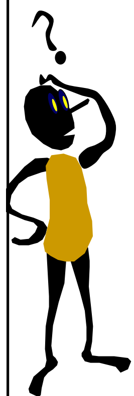
Arborescences

Def. Given a digraph G = (V, E) and a vertex r∈V, an arborescence (rooted at r) is a treeT s.t.

T is a spanning tree of G if we ignore the direction of edges.

There is a directed unique path in T from r to each other node v ∈ V.





Given a digraph G, find an arborescence rooted at r (if one exists)

Run DFS or BFS

Arborescences

Theorem. A subgraph T of G is an arborescence rooted at r iff T has no directed cycles and each node v ≠ r has exactly one entering edge.

Proof.

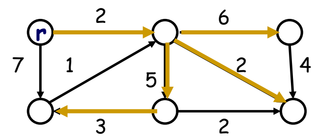
⇒) Trivial.

⇐) Start a vertex v and follow edges in backward direction.

Since no cycles you eventually reach r.

Min-cost Arborescences

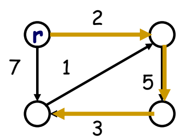
Given a digraph G with a root node r and with a nonnegative cost on each edge, compute an arborescence rooted at r of minimum cost.



We assume that all vertices are reachable from r.

Min-cost Arborescences

Observation 1. This is not a min-cost spanning tree. It does not necessarily include the cheapest edge.

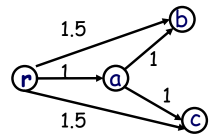


Running Prim's on undirected graph won't help.

Running an analogue of Prim's for directed graph won't help either

Min-cost Arborescences

Observation 2. This is not a shortest-path tree



Edges rb and rc won't be in the min-cost arborescence tree

Edge reweighting

For each $v \neq r$, let $\delta(v)$ denote the min cost of any edge entering v.

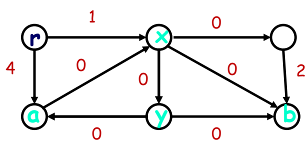
In the picture, $\delta(x)$ is 1.

The reduced cost $w^*(u, v) = w(u, v) - \delta(v) \geq 0$

$\delta(y)$ is 5.

$\delta(a)$ is 3.

$\delta(b)$ is 3.



$w^*(u, v) = w(u, v) - \delta(v)$

Lemma. An arborescence in a digraph has the min-cost with respect to w iff it has the min-cost with respect to w*.

Proof. Let T be an arborescence in G(V,E).

Compute w(T) – w*(T)

$\delta(v)$ - min cost of any edge entering v

$$w(T) - w^*(T) = \sum_{e \in T} w(e) - w^*(e) = \sum_{v \in V \setminus r} \delta(v)$$

The last term does not depend on T. QED

Algorithm: intuition

Let G* denote a new graph after reweighting.

For every v≠r in G* pick 0-weight edge entering v.

Let B denote the set of such edges.

If B is an arborescence, we are done.

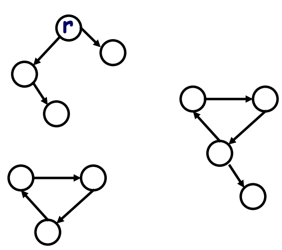
Note B is the min-cost since all edges have 0 cost.

If B is NOT an arborescence...

When B is not an arborescence?

How can it happen B is not an arborescence?

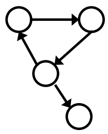
Note, only a single edge can enter a vertex



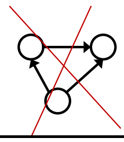
when it has a directed cycle or several cycles...

How can it happen B is not an arborescence?

It must be a cycle

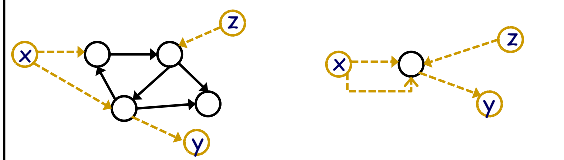


a directed cycle…



Vertex contraction

We contract every cycle into a supernode
Dashed edges and nodes are from the original graph G.



Recursively solve the problem in contracted graph

The Algorithm

For each v≠r compute δ(v) – the mincost of edges entering *v.*

For each v≠r compute w*(u, v) = w(u, v) – δ(v).

For each v≠r choose 0-cost edge entering *v.*

Let us call this subset of edges – B.

If B forms an arborescence, we are done.

else

Contract every cycle *C* to a supernode

Repeat the algorithm

Extend an arborescence by adding all but one edge of *C*.

Return

Complexity

At most V contractions (since each one reduces the number of nodes).

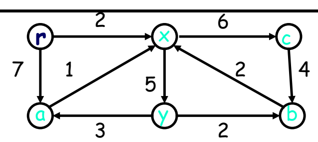
Finding and contracting the cycle *C* takes O(E).

Transforming T' into T takes O(E) time.

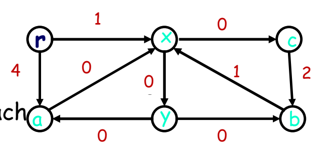
Total - O(V E).

Faster for Fibonacci heaps.

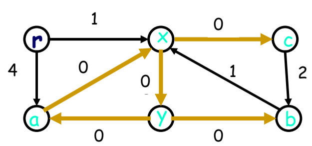
reweight
δ(a) is 3



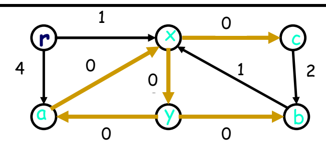
Take
0-weight
edge for each
vertex



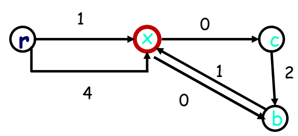
Cycle
AXY



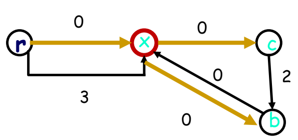
Contract
AXY



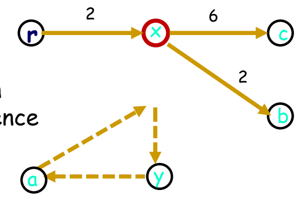
reweight
δ(x) is 1



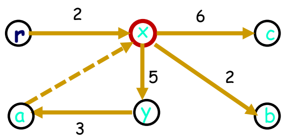
Take
0-weight
edges.
break ties
arbitrarily



Extend an arborescence

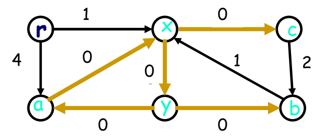


By removing an edge from a cycle



Correctness

Lemma. Let C be a cycle in G consisting of 0-cost edges. There exists a mincost arborescence rooted at r that has exactly one edge entering C.



Correctness

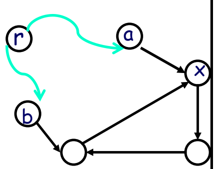
Lemma. Let C be a cycle in G consisting of 0-cost edges. There exists a mincost arborescence rooted at r that has exactly one edge entering C.

Proof. Let T be a min-cost arborescence that has more than one edge enters C

Let (a,x) lies on a shortest path from r.

We delete all edges in T that enters C except (a,b)

We add all edges in C except the one that enters x.



Correctness

Lemma. Let C be a cycle in G consisting of 0-cost edges. There exists a mincost arborescence rooted at r that has exactly one edge entering C.

Claim: that new tree T* is a mincost arborescence

1. cost(T*) ≤ cost(T) since we add 0-cost edges

2. T* has exactly one edge entering each vertex

3. T* has no cycles.