GAME THEORY

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PART IV. Games in Coalitional Form

1. Many-Person TU Games

We now consider many-person **cooperative games**. In such games there are no restrictions on the agreements that may be reached among the players. In addition, we assume that all payoffs are measured in the same units and that there is a **transferrable utility** which allows **side payments** to be made among the players. Side payments may be used as inducements for some players to use certain mutually beneficial strategies. Thus, there will be a tendency for players, whose objectives in the game are close, to form alliances or coalitions. The structure given to the game by coalition formation is conveniently studied by reducing the game to a form in which coalitions play a central role. After defining the coalitional form of a many-person TU game, we shall learn how to transform games from strategic form to coalitional form and vice versa.

1.1 Coalitional Form. Characteristic Functions. Let $n \geq 2$ denote the number of players in the game, numbered from 1 to n, and let N denote the set of players, $N = \{1, 2, \ldots, n\}$. A **coalition**, S, is defined to be a subset of N, $S \subset N$, and the set of all coalitions is denoted by $\mathcal{S}$. By convention, we also speak of the empty set, $\emptyset$, as a coalition, the **empty coalition**. The set N is also a coalition, called the **grand coalition**.

If there are just two players, $n = 2$, then there are four coalition, $\mathcal{S} = \{\emptyset, \{1\}, \{2\}, N\}$. If there are 3 players, there are 8 coalitions, $\mathcal{S} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, N\}$. For n players, the set of coalitions, $\mathcal{S}$, has 2^n elements.

Definition. *The* **coalitional form** *of an n-person game is given by the pair (N, v), where $N = \{1, 2, \ldots, n\}$ is the set of players and v is a real-valued function, called the* **characteristic function** *of the game, defined on the set, $\mathcal{S}$, of all coalitions (subsets of N), and satisfying*

(i) $v(\emptyset) = 0$, and

(ii) (superadditivity) if S and T are disjoint coalitions ($S \cap T = \emptyset$), then $v(S) + v(T) \leq v(S \cup T)$.

Compared to the strategic or extensive forms of n-person games, this is a very simple definition. Naturally, much detail is lost. The quantity $v(S)$ is a real number for each coalition $S \subset N$, which may be considered as the value, or worth, or power, of coalition S when its members act together as a unit. Condition (i) says that the empty set has value zero, and (ii) says that the value of two disjoint coalitions is at least as great when they work together as when they work apart. The assumption of superadditivity is not needed for some of the theory of coalitional games, but as it seems to be a natural condition, we include it in the definition.

1.2 Relation to Strategic Form. Recall that the **strategic form** of an n-person game is given by the $2n$-tuple, $(X_1, X_2, \ldots, X_n, u_1, u_2, \ldots, u_n)$, where

(1) for $i = 1, \ldots, n$, X_i is the set of pure strategies of Player i, and

(2) for $i = 1, \ldots, n$, $u_i(x_1, \ldots, x_n)$ is the payoff function to Player i, if Player 1 uses $x_1 \in X_1$, Player 2 uses $x_2 \in X_2$, ..., and Player n uses $x_n \in X_n$.

Transforming a game from strategic form to coalitional form entails specifying the value, $v(S)$, for each coalition $S \in \mathcal{S}$. The usual way to assign a characteristic function to a strategic form game is to define $v(S)$ for each $S \in \mathcal{S}$ as the **value** of the 2-person zero-sum game obtained when the coalition S acts as one player and the complementary coalition, $\overline{S} = N - S$, acts as the other player, and where the payoff to S is $\sum_{i \in S} u_i(x_1, \ldots, x_n)$, the total of the payoffs to the players in S:

$$v(S) = \text{Val}\big(\sum_{i \in S} u_i(x_1, \ldots, x_n)\big) \tag{1}$$

The value, $v(S)$, is the analogue of the safety level. It represents the total amount that coalition S can guarantee for itself, even if the members of $\overline{S}$ gang up against it, and have as their only object to keep the sum of the payoffs to members of S as small as possible. This is a lower bound to the payoff S should receive because it assumes that the members of $\overline{S}$ ignore what possible payoffs they might receive as a result of their actions. An example of the computations involved in a three person game is given below.

To see that v of equation (1) is a characteristic function, note that Condition (i) holds, since the empty sum is zero. To see that (ii) holds, note that if $\mathbf{s}$ is a set of strategies for S that guarantees them $v(S)$, and $\mathbf{t}$ is a set of strategies for T that guarantees them $v(T)$, then the set of strategies ($\mathbf{s}$,$\mathbf{t}$) guarantees $S \cup T$ at least $v(S) + v(T)$. Perhaps other joint strategies can guarantee even more, so certainly, $v(S \cup T) \geq v(S) + v(T)$.

Every finite n-person game in strategic form can be reduced to coalitional form in this way. Often, *such a reduction to coalitional form loses important features in the game*, such as threats. So for a given characteristic function v, there are usually many games in strategic form whose reduction by the above method has characteristic function v. See Exercise 3 for a two-person game that favors one of the players, yet the the reduction in coalitional form is symmetric in the players.

One way of constructing a strategic form game whose reduction to coalitional form has a given characteristic function, v, is as follows. The strategy space X_i for Player i is taken to be the set of all coalitions that contain i: $X_i = \{S \in \mathcal{S} : i \in S\}$. Then the payoff to player i is the minimum amount, $v(\{i\})$, unless all members of the coalition, S_i, chosen by Player i, choose the same coalition as player i has, in which case the coalition S_i is given its value $v(S_i)$ which it then splits among its members. Thus the payoff function u_i is

$$u_i(S_1, \ldots, S_n) = \begin{cases} v(S_i)/|S_i| & \text{if } S_j = S_i \text{ for all } j \in S_i \\ v(\{i\}) & \text{otherwise} \end{cases} \tag{2}$$

where $|S_i|$ represents the number of members of the coalition S_i. Clearly, a coalition S can guarantee itself $v(S)$ simply by having each member of S select S as his coalition of choice. Moreover, since v is superadditive, the coalition S cannot guarantee more for itself by having its members form subcoalitions.

1.3 Constant-Sum Games. A game in strategic form is said to be **zero-sum** if $\sum_{i\in N} u_i(x_1,\ldots,x_n) = 0$ for all strategy choices $x_1,\ldots,x_n$ of the players. In such a game, we have $\sum_{i\in S} u_i(x_1,\ldots,x_n) = -\sum_{i\in\overline{S}} u_i(x_1,\ldots,x_n)$ for any coalition S, where $\overline{S} = N-S$ is the complement of S. This implies that in the reduction of such a game to coalitional form, the value of the game coalition S plays against $\overline{S}$ is the negative of the value of the game $\overline{S}$ plays against S, so that $v(S)+v(\overline{S}) = 0$ for all coalitions S. We may take this as the definition of a zero-sum game in coalitional form. Similarly, a strategic form game is **constant-sum** if $\sum_{i\in N} u_i(z_1,\ldots,z_n) = c$ for some constant c. By a similar reasoning, the reduction of such a game leads to $v(S)+v(\overline{S}) = c = v(N)$ for all coalitions S in a constant sum game. This may be taken as the definition of a constant-sum game in coalitional form.

Definition. *A game in coalitional form is said to be* **constant-sum**, *if* $v(S)+v(\overline{S}) = v(N)$ *for all coalitions* $S\in\mathcal{S}$. *It is said to be* **zero-sum** *if, in addition,* $v(N) = 0$.

1.4 Example. Consider the three-person game with players I, II, and III with two pure strategies each and with payoff vectors:

If I chooses 1:

		III: 1	III: 2
II:	1	$(0,3,1)$	$(2,1,1)$
	2	$(4,2,3)$	$(1,0,0)$

If I chooses 2

		III: 1	III: 2
II:	1	$(1,0,0)$	$(1,1,1)$
	2	$(0,0,1)$	$(0,1,1)$

Let us find the associated game in coalitional form by finding the characteristic function, v. We automatically have $v(\emptyset) = 0$. It is easy to find $v(N)$. It is the largest sum in the eight cells. This occurs for the cell $(1,2,1)$ and gives total payoff $v(N) = 9$. To find $v(\{1\})$, compute the payoff matrix for the winnings of I against (II,III):

		(II,III) 1,1	1,2	2,1	2,2
I:	1	0	2	4	1
	2	1	1	0	0

The second and third columns are dominated, so $v(\{1\}) = \text{Val}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 1/2$.

To find $v(\{2\})$ and $v(\{3\})$, we make similar constructions of the matrices of II's winnings vs I and III, and III's winnings vs I and II and find the values of the resulting games. In the matrix of II's winnings, the choice 2 by II and (2,1) by (I,III) is a saddlepoint with value $v(\{2\}) = 0$. In the matrix of III's winnings, the value is $v(\{3\}) = 3/4$.

To find $v(\{1,3\})$ say, we first construct the matrix of the sum of the winnings of I and III playing against II. This is

		II: 1	2
(I,III):	1,1	1	7
	1,2	3	1
	2,1	1	1
	2,2	2	1

The lower two rows are dominated by the second row, so that the value is $v(\{1,3\}) = \text{Val}\begin{pmatrix} 1 & 7 \\ 3 & 1 \end{pmatrix} = 5/2$. Similarly, we may compute the matrix of I and II playing against III, and the matrix of II and III playing against I. Both these matrices have saddle points. We find $v(\{1,2\}) = 3$ and $v\{2,3\}) = 2$. This completes the specification of the characteristic function.

1.5 Exercises.

1. Find the characteristic function of the 3-person game with players I, II, and III with two pure strategies each and with the following payoff vectors. Note that this is a zero-sum game. Hence, $v(\{1,3\}) = -v(\{2\})$, etc.

If I chooses 1:

	III: 1	2
II: 1	$(-2,1,1)$	$(1,-4,3)$
2	$(1,3,-4)$	$(10,-5,-5)$

If I chooses 2

	III: 1	2
II: 1	$(-1,-2,3)$	$(-4,2,2)$
2	$(12,-6,-6)$	$(-1,3,-2)$

2. Find the characteristic function of the 3-person game in strategic form when the payoff vectors are:

If I chooses 1:

	III: 1	2
II: 1	$(1,2,1)$	$(3,0,1)$
2	$(-1,6,-3)$	$(3,2,1)$

If I chooses 2

	III: 1	2
II: 1	$(-1,2,4)$	$(1,0,3)$
2	$(7,5,4)$	$(3,2,1)$

3. Consider the two-person game with bimatrix

$$\begin{pmatrix} (0,2) & (4,1) \\ (2,4) & (5,4) \end{pmatrix}$$

(a) Find the associated game in coalitional form. Note that in coalitional form the game is symmetric in the two players.

(b) Argue that the above game is actually favorable to Player 2.

(c) Find the TU-value as a game in strategic form. Note that this value gives more to Player 2 than to Player 1.

(d) If the game in coalitional form found in (a) is transformed to strategic form by the method of Equation (2), what is the bimatrix that arises?

2. Imputations and the Core

In cooperative games, it is to the joint benefit of the players to form the grand coalition, N, since by superadditivity the amount received, $v(N)$, is as large as the total amount received by any disjoint set of coalitions they could form. As in the study of 2-person TU games, it is reasonable to suppose that "rational" players will agree to form the grand coalition and receive $v(N)$. The problem is then to agree on how this amount should be split among the players. In this section, we discuss one of the possible properties of an agreement on a fair division, that it be stable in the sense that no coalition should have the desire and power to upset the agreement. Such divisions of the total return are called points of the core, a central notion of game theory in economics.

2.1 Imputations. A payoff vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ of proposed amounts to be received by the players, with the understanding that player i is to receive x_i, is sometimes called an **imputation**. The first desirable property of an imputation is that the total amount received by the players should be $v(N)$.

Definition. *A payoff vector, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, is said to be* **group rational** *or* **efficient** *if $\sum_{i=1}^n x_i = v(N)$.*

No player could be expected to agree to receive less than that player could obtain acting alone. Therefore, a second natural condition to expect of an imputation, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, is that $x_i \ge v(\{i\})$ for all players, i.

Definition. *A payoff vector, $\boldsymbol{x}$, is said to be* **individually rational** *if $x_i \ge v(\{i\})$ for all $i = 1, \ldots, n$.*

Imputations are defined to be those payoff vectors that satisfy both these conditions.

Definition. *An* **imputation** *is a payoff vector that is group rational and individually rational. The set of imputations may be written*

$$\{\boldsymbol{x} = (x_1, \ldots, x_n) : \textstyle\sum_{i \in N} x_i = v(N), \text{ and } x_i \ge v(\{i\}) \text{ for all } i \in N\}. \tag{1}$$

Thus, an imputation is an n-vector, $\boldsymbol{x} = (x_1, \ldots, x_n)$, such that $x_i \ge v(\{i\})$ for all i and $\sum_{i=1}^n x_i = v(N)$. The set of imputations is never empty, since from the superadditivity of v, we have $\sum_{i=1}^n v(\{i\}) \le v(N)$. For example, one imputation is given by $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, where $x_i = v(\{i\})$ for $i = 1, \ldots, n-1$, and $x_n = v(N) - \sum_1^{n-1} v(\{i\})$. This is the imputation most preferred by player n. In fact the set of imputations is exactly the simplex consisting of the convex hull of the n points obtained by letting $x_i = v(\{i\})$ for all x_i except one, which is then chosen to satisfy $\sum_1^n x_i = v(N)$.

In Example 1.4, $v(\{1\}) = 1/2$, $v(\{2\}) = 0$, $v(\{3\}) = 1$, and $v(N) = 9$. The set of imputations is

$$\{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 9,\ x_1 \ge 1/2,\ x_2 \ge 0,\ x_3 \ge 1\}.$$

This is a triangle each of whose vertices satisfy two of the three inequalities with equality, namely, $(8, 0, 1)$, $(1/2, 15/2, 1)$, and $(1/2, 0, 17/2)$. These are the imputations most preferred by players 1, 2, and 3 respectively.

2.2 Essential Games. There is one trivial case in which the set of imputations consists of one point. Such a game is called inessential.

Definition. *A game in coalitional form is said to be* **inessential** *if* $\sum_{i=1}^{n} v(\{i\}) = v(N)$, *and* **essential** *if* $\sum_{i=1}^{n} v(\{i\}) < v(N)$.

If a game is inessential, then the unique imputation is $\boldsymbol{x} = (v(\{1\}), \ldots, v(\{n\}))$, which may be considered the "solution" of the game. Every player can expect to receive his safety level. Two-person zero-sum games are all inessential. (Exercise 1.)

From the game-theoretic viewpoint, inessential games are very simple. For every coalition S, $v(S)$ is determined by $v(S) = \sum_{i \in S} v(\{i\})$. There is no tendency for the players to form coalitions.

In Example 1.4, $v(\{1\}) + v(\{2\}) + v(\{3\}) = 1/2 + 0 + 1 < 9 = v(N)$, so the game is essential.

2.3 The Core. Suppose some imputation, $\boldsymbol{x}$, is being proposed as a division of $v(N)$ among the players. If there exists a coalition, S, whose total return from $\boldsymbol{x}$ is less than what that coalition can achieve acting by itself, that is, if $\sum_{i \in S} x_i < v(S)$, then there will be a tendency for coalition S to form and upset the proposed $\boldsymbol{x}$ because such a coalition could guarantee each of its members more than they would receive from $\boldsymbol{x}$. Such an imputation has an inherent instability.

Definition. *An imputation* $\boldsymbol{x}$ *is said to be* **unstable through a coalition** S *if* $v(S) > \sum_{i \in S} x_i$. *We say* $\boldsymbol{x}$ *is* **unstable** *if there is a coalition* S *such that* $\boldsymbol{x}$ *is unstable through* S, *and we say* $\boldsymbol{x}$ *is* **stable** *otherwise.*

Definition. *The set,* C, *of stable imputations is called the* **core**,

$$C = \{\boldsymbol{x} = (x_1, \ldots, x_n) : \textstyle\sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S), \text{ for all } S \in \mathcal{S}\}. \tag{2}$$

The core can consist of many points as in the examples below; but the core can also be empty. It may be impossible to satisfy all the coalitions at the same time. One may take the size of the core as a measure of stability, or of how likely it is that a negotiated agreement is prone to be upset. One may take the essential constant-sum games as examples of games with empty cores:

Theorem 1. *The core of an essential* n*-person constant-sum game is empty.*

Proof. Let $\boldsymbol{x}$ be an imputation. Since the game is essential, we have $\sum_{i \in N} v(\{i\}) < v(N)$. Then there must be a player, k, such that $x_k > v(\{k\})$, for otherwise $v(N) = \sum_{i \in N} x_i \leq \sum_{i \in N} v(\{i\}) < v(N)$. Since the game is constant-sum, we have $v(N - \{k\}) + v(\{k\}) =$

$v(N)$. But then, $\boldsymbol{x}$ must be unstable through the coalition $N - \{k\}$, because $\sum_{i \neq k} x_i = \sum_{i \in N} x_i - x_k < V(N) - v(\{k\}) = v(N - \{k\})$. ∎

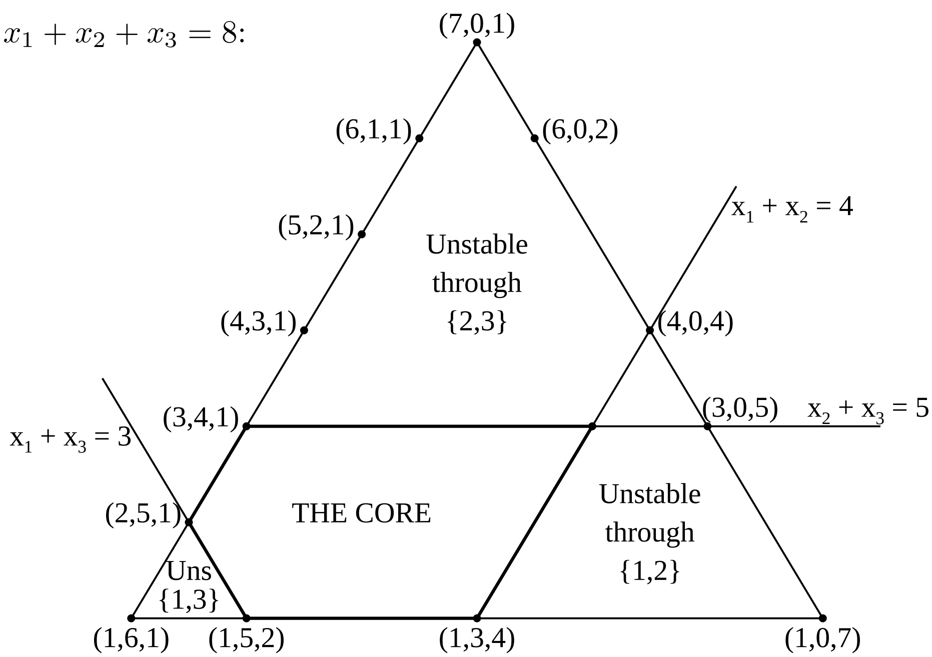
2.4 Examples. *Example 1.* Consider the game with characteristic function v given by

$$\begin{array}{llll} & v(\{1\}) = 1 & v(\{1,2\}) = 4 & \\ v(\emptyset) = 0 & v(\{2\}) = 0 & v(\{1,3\}) = 3 & v(\{1,2,3\}) = 8 \\ & v(\{3\}) = 1 & v(\{2,3\}) = 5 & \end{array}$$

The imputations are the points (x_1, x_2, x_3) such that $x_1 + x_2 + x_3 = 8$ and $x_1 \geq 1$, $x_2 \geq 0$, $x_3 \geq 1$. This set is the triangle with vertices $(7, 0, 1)$, $(1, 6, 1)$ and $(1, 0, 7)$.

It is useful to plot this triangle in **barycentric coordinates**. This is done by pretending that the plane of the plot is the plane $x_1 + x_2 + x_3 = 8$, and giving each point on the plane three coordinates which add to 8. Then it is easy to draw the lines $x_1 = 1$ or the line $x_1 + x_3 = 3$ (which is the same as the line $x_2 = 5$), etc. It then becomes apparent that the set of imputations is an equilateral triangle.

On the plane $x_1 + x_2 + x_3 = 8$:



Let us find which imputations are unstable. The coalition $\{2, 3\}$ can guarantee itself $v(\{2, 3\}) = 5$, so all points (x_1, x_2, x_3) with $x_2 + x_3 < 5$ are unstable through $\{2, 3\}$. These are the points below the line $x_2 + x_3 = 5$ in the diagram. Since $\{1, 2\}$ can guarantee itself $v(\{1, 2\}) = 4$, all points below and to the right of the line $x_1 + x_2 = 4$ are unstable. Finally, since $\{1, 3\}$ can guarantee itself $v(\{1, 3\}) = 3$, all points below the line $x_1 + x_3 = 3$ are unstable. The core is the remaining set of points in the set of imputations given by the 5-sided figure in the diagram, including the boundary.

Example 2. A certain *objet d'art* is worth a_i dollars to Player i for $i = 1, 2, 3$. We assume $a_1 < a_2 < a_3$, so Player 3 values the object most. But Player 1 owns this object so $v(\{1\}) = a_1$. Player 2 and 3 by themselves can do nothing, so $v(\{2\}) = 0$, $v(\{3\}) = 0$, and

$v(\{2,3\}) = 0$. If Players 1 and 2 come together, the joint worth is a_2, so $v(\{1,2\}) = a_2$. Similarly, $v(\{1,3\}) = a_3$. If all three get together, the object is still only worth a_3, so $v(N) = a_3$. Let us find the core of this game.

The core consists of all vectors (x_1, x_2, x_3) satisfying

$$\begin{array}{lll} x_1 \geq a_1 & x_1 + x_2 \geq a_2 & \\ x_2 \geq 0 & x_1 + x_3 \geq a_3 & x_1 + x_2 + x_3 = a_3 \\ x_3 \geq 0 & x_2 + x_3 \geq 0 & \end{array}$$

It follows from $x_2 = a_3 - x_1 - x_3 \leq 0$ and $x_2 \geq 0$ that $x_2 = 0$ for all points of the core. Then we find that $x_1 \geq a_2$ and $x_3 = a_3 - x_1$. Hence the core is $C = \{(x, 0, a_3 - x) : a_2 \leq x \leq a_3\}$.

This indicates that the object will be purchased by Player 3 at some purchase price x between a_2 and a_3. Player 1 ends up with x dollars and player 3 ends up with the object minus x dollars. Player 2 plays no active role in this, but without her around Player 3 might hope to get the object for less than a_2.

2.5 Exercises.

1. Show that every 2-person constant-sum game is inessential.

2. Find the set of imputations and the core of the battle of the sexes with bimatrix:

$$\begin{pmatrix} (4,2) & (1,1) \\ (0,0) & (2,4) \end{pmatrix}.$$

3. Graph the core for the 3-person game with characteristic function: $v(\emptyset) = 0$, $v(\{1\}) = 0$, $v(\{2\}) = -1$, $v(\{3\}) = 1$, $v(\{1,2\}) = 3$, $v(\{1,3\}) = 2$, $v(\{2,3\}) = 4$, and $v(N) = 5$.

Definition. *A game with characteristic function v is said to be* **symmetric** *if $v(S)$ depends only on the number of elements of S, say $v(S) = f(|S|)$ for some function f.*

4. (a) In a symmetric 3-player game with $v(\{i\}) = 0$, $v(\{i,j\}) = a$ and $v(\{1,2,3\}) = 3$, for what values of a is the core non-empty?
(b) In a symmetric 4-player game with $v(\{i\}) = 0$, $v(\{i,j\}) = a$, $v(\{i,j,k\}) = b$, and $v(N) = 4$, for what values of a and b is the core non-empty?
(c) Generalize. Find necessary and sufficient conditions on the values of $f(|S|) = v(S)$ for a symmetric game to have a non-empty core.

5. Let $\delta_i = v(N) - v(N - \{i\})$ for $i = 1, \ldots, n$. Show that the core is empty if $\sum_1^n \delta_i < v(N)$.

6. We say that Player i is a **dummy** in a game (N, v), if $v(\{i\} \cup S) = v(S)$ for all coalitions, S. In particular, $v(\{i\}) = 0$. Thus, a dummy cannot help (or harm) any coalition. Show that if Player 1 is a dummy and if $(x_1, x_2, \ldots, x_n)$ is in the core, then $x_1 = 0$.

7. **The Glove Market.** Let N consist of two types of players, $N = P \cup Q$, where $P \cap Q = \emptyset$. Let the characteristic function be defined by

$$v(S) = \min\{|S \cap P|, |S \cap Q|\}.$$

The game (N, v) is called the glove market because of the following interpretation. Each player of P owns a right-hand glove and each player of Q owns a left-hand glove. If j members of P and k members of Q form a coalition, they have $\min\{j, k\}$ complete pairs of gloves, each being worth 1. Unmatched gloves are worth nothing.

(a) Suppose $|P| = 2$ and $|Q| = 2$. Find the core.

(b) Suppose $|P| = 2$ and $|Q| = 3$. Show the core consists of a single point.

(c) Generalize to arbitrary $|P|$ and $|Q|$.

8. There are two machine owners and three workers. Each machine owner owns two machines. Each worker can produce 1 unit on any machine. Thus,

$v(\{i, k\}) = 1$ for $i = 1, 2$ and $k = 3, 4, 5$,

$v(\{i, j, k\}) = 2$ for $i = 1, 2$ and $j, k = 3, 4, 5$.

$v(\{1, 2, 3, 4, 5\}) = 3$.

Find the core.

3. The Shapley Value

We now treat another approach to n-person games in characteristic function form. The core concept is useful as a measure of stability. As a solution concept, it presents a set of imputations without distinguishing one point of the set as preferable to another. Indeed, the core may be empty.

Here we deal with the concept of a value. In this approach, one tries to assign to each game in coalitional form a unique vector of payoffs, called the value. The ith entry of the value vector may be considered as a measure of the value or power of the ith player in the game. Alternatively, the value vector may be thought of as an arbitration outcome of the game decided upon by some fair and impartial arbiter. The central "value concept" in game theory is the one proposed by Shapley in 1953. We define the Shapley value in this section and discuss its application to measuring power in voting systems where it is called the Shapley-Shubik power index. In Section 4, we treat another value concept, the nucleolus.

3.1 Value Functions. The Shapley Axioms. As an example of the type of reasoning involved in arbitrating a game, consider Example 1 of Section 2.4. Certainly the arbiter should require the players to form the grand coalition to receive 8, but how should this be split among the players? Player 2 can get nothing by himself, yet he is more valuable than 1 or 3 in forming coalitions. Which is more important? We approach this problem by axiomatizing the concept of fairness.

A **value function**, $\boldsymbol{\phi}$, is function that assigns to each possible characteristic function of an n-person game, v, an n-tuple, $\boldsymbol{\phi}(v) = (\phi_1(v), \phi_2(v), \ldots, \phi_n(v))$ of real numbers. Here $\phi_i(v)$ represents the worth or value of player i in the game with characteristic function v. The axioms of fairness are placed on the function, $\boldsymbol{\phi}$.

The Shapley Axioms for $\boldsymbol{\phi}(v)$:

1. **Efficiency.** $\sum_{i \in N} \phi_i(v) = v(N)$.
2. **Symmetry.** If i and j are such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for every coalition S not containing i and j, then $\phi_i(v) = \phi_j(v)$.
3. **Dummy Axiom.** If i is such that $v(S) = v(S \cup \{i\})$ for every coalition S not containing i, then $\phi_i(v) = 0$.
4. **Additivity.** If u and v are characteristic functions, then $\boldsymbol{\phi}(u + v) = \boldsymbol{\phi}(u) + \boldsymbol{\phi}(v)$.

Axiom 1 is group rationality, that the total value of the players is the value of the grand coalition. The second axiom says that if the characteristic function is symmetric in players i and j, then the values assigned to i and j should be equal. The third axiom says that if player i is a dummy in the sense that he neither helps nor harms any coalition he may join, then his value should be zero. The strongest axiom is number 4. It reflects the feeling that the arbitrated value of two games played at the same time should be the sum of the arbitrated values of the games if they are played at different times. It should be noted that if u and v are characteristic functions, then so is $u + v$.

Theorem 1. *There exists a unique function $\boldsymbol{\phi}$ satisfying the Shapley axioms.*

Proof. For a given set $S \subset N$, let w_S represent the special characteristic function,

$$w_S(T) = \begin{cases} 1 & \text{if } S \subset T \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

From axiom 3, $\phi_i(w_S) = 0$ if $i \notin S$. From axiom 2, if both i and j are in S, then $\phi_i(w_S) = \phi_j(w_S)$. From axiom 1, $\sum_{i \in N} \phi_i(w_S) = w_S(N) = 1$, so that $\phi_i(w_S) = 1/|S|$ for all $i \in S$. Applying similar analysis to the characteristic function cw_S for a given number, c, we find

$$\phi_i(cw_S) = \begin{cases} c/|S| & \text{for } i \in S \\ 0 & \text{for } i \notin S. \end{cases} \tag{2}$$

In the next paragraph, we show that any characteristic function, v, is representable as a weighted sum of characteristic functions of the form (1), $v = \sum_{S \subset N} c_S w_S$, for some appropriate, easily computable, constants c_S. Then axiom 4 may be applied to show that if a value function exists, it must be

$$\phi_i(v) = \sum_{\substack{S \subset N \\ i \in S}} \frac{c_S}{|S|}. \tag{3}$$

where this sum is taken over all coalitions S containing i. This works even if some of the c_S are negative, since axiom 4 also implies that $\boldsymbol{\phi}(u - v) = \boldsymbol{\phi}(u) - \boldsymbol{\phi}(v)$, provided u, v, and $u - v$ are characteristic functions. (Just write $u = (u - v) + v$.) To complete the proof, one must show existence, namely that (3) with the c_S defined below, satisfies the Shapley axioms. This is not difficult but we defer the proof to Theorem 2, where we show existence by showing another value function satisfies the Shapley axioms (and therefore must be the same as (3)).

Now let us show that any v may be written as $v = \sum_{S \subset N} c_S w_S$ by finding the constants c_S. Let $c_\emptyset = 0$, and define inductively on the number of elements in T, for all $T \subset N$,

$$c_T = v(T) - \sum_{\substack{S \subset T \\ S \neq T}} c_S. \tag{4}$$

Note that each c_T is defined in terms of c_S where S has a smaller number of elements than T. Then,

$$\sum_{S \subset N} c_S w_S(T) = \sum_{S \subset T} c_S = c_T + \sum_{\substack{S \subset T \\ S \neq T}} c_S = v(T). \tag{5}$$

Hence, $v = \sum_{S \subset N} c_S w_S$ as was to be shown. ∎

It is interesting to note that the superadditivity of v is not needed in this proof.

3.2 Computation of the Shapley Value. The proof of Theorem 1 provides a method of computing the Shapley value: First find the numbers c_S inductively using Equation (4). Then form the Shapley value using Equation (3).

As an example of this method, consider the characteristic function of Example 1 of Section 2.4.

$$\begin{array}{llll} & v(\{1\}) = 1 & v(\{1,2\}) = 4 & \\ v(\emptyset) = 0 & v(\{2\}) = 0 & v(\{1,3\}) = 3 & v(\{1,2,3\}) = 8 \\ & v(\{3\}) = 1 & v(\{2,3\}) = 5 & \end{array}$$

We find inductively, $c_{\{1\}} = v(\{1\}) = 1$, $c_{\{2\}} = 0$ and $c_{\{3\}} = 1$. Then, $c_{\{1,2\}} = v(\{1,2\}) - c_{\{1\}} - c_{\{2\}} = 4 - 1 - 0 = 3$, $c_{\{1,3\}} = 3 - 1 - 1 = 1$, and $c_{\{2,3\}} = 5 - 0 - 1 = 4$. Finally,

$$c_N = v(N) - c_{\{1,2\}} - c_{\{1,3\}} - c_{\{2,3\}} - c_{\{1\}} - c_{\{2\}} - c_{\{3\}} = 8 - 3 - 1 - 4 - 1 - 0 - 1 = -2.$$

Hence, we know we can write v as

$$v = w_{\{1\}} + w_{\{3\}} + 3w_{\{1,2\}} + w_{\{1,3\}} + 4w_{\{2,3\}} - 2w_{\{1,2,3\}}$$

From this we find

$$\begin{aligned} \phi_1(v) &= 1 + \frac{3}{2} + \frac{1}{2} - \frac{2}{3} = 2 + \frac{1}{3} \\ \phi_2(v) &= \frac{4}{2} + \frac{3}{2} - \frac{2}{3} = 2 + \frac{5}{6} \\ \phi_3(v) &= 1 + \frac{1}{2} + \frac{4}{2} - \frac{2}{3} = 2 + \frac{5}{6} \end{aligned}$$

The Shapley value is $\boldsymbol{\phi} = (14/6, 17/6, 17/6)$. This point is in the core (see the diagram in Section 2). It could happen that the core is empty, so the Shapley value is not always in the core. But even if the core is not empty, the Shapley value is not necessarily in the core. (See Exercise 1, for example.)

3.3 An Alternative Form of the Shapley Value. There is an alternate way of arriving at the Shapley value that gives additional insight into its properties. Suppose we form the grand coalition by entering the players into this coalition one at a time. As each player enters the coalition, he receives the amount by which his entry increases the value of the coalition he enters. The amount a player receives by this scheme depends on the order in which the players are entered. The Shapley value is just the average payoff to the players if the players are entered in completely random order.

Theorem 2. The Shapley value is given by $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)$, where for $i = 1, \ldots, n$,

$$\phi_i(v) = \sum_{\substack{S \subset N \\ i \in S}} \frac{(|S|-1)!(n-|S|)!}{n!}[v(S) - v(S - \{i\})]. \tag{6}$$

The summation in this formula is the summation over all coalitions S that contain i. The quantity, $v(S) - v(S - \{i\})$, is the amount by which the value of coalition $S - \{i\}$ increases when player i joins it. Thus to find $\phi_i(v)$, merely list all coalitions containing

i, compute the value of player i's contribution to that coalition, multiply this by $(|S|-1)!(n-|S|)!/n!$, and take the sum.

The interpretation of this formula is as follows. Suppose we choose a random order of the players with all $n!$ orders (permutations) of the players equally likely. Then we enter the players according to this order. If, when player i is enters, he forms coalition S (that is, if he finds $S-\{i\}$ there already), he receives the amount $[v(S)-v(S-\{i\})]$.

The probability that when i enters he will find coalition $S-\{i\}$ there already is $(|S|-1)!(n-|S|)!/n!$. The denominator is the total number of permutations of the n players. The numerator is number of these permutations in which the $|S|-1$ members of $S-\{i\}$ come first ($(|S|-1)!$ ways), then player i, and then the remaining $n-|S|$ players ($(n-|S|)!$ ways). So this formula shows that $\phi_i(v)$ is just the average amount player i contributes to the grand coalition if the players sequentially form this coalition in a random order.

As an illustration of the use of this formula, let us compute $\phi_1(v)$ again for Example 1 of Section 2.4. The probability that player 1 enters first is $2!0!/3! = 1/3$, and then his payoff is $v(\{1\}) = 1$. The probability that 1 enters second and finds 2 there is 1/6, and his payoff is $v(\{1,2\})-v(\{2\}) = 4-0 = 4$. The probability that 1 enters second and finds 3 there is 1/6, and the expected payoff is $v(\{1,3\})-v(\{3\}) = 3-1 = 2$. The probability that 1 enters last is 1/3, and then his payoff is $v(\{1,2,3\})-v(\{2,3\}) = 8-5 = 3$. Player 1's average payoff is therefore

$$\phi_1(v) = \frac{1}{3}\cdot 1 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 2 + \frac{1}{3}\cdot 3 = 14/6$$

as found earlier.

The following table shows the computations for all three players simultaneously. The 6 different orders of the players are listed along with the payoffs to the players. In the first row, the players enter in the order 1, 2, 3. Player 1 receives $v(1) = 1$ upon entry; then Player 2 receives $v(1,2)-v(1) = 4-1 = 3$; finally Player 3 receives $v(N)-v(1,2) = 8-4 = 4$. Each of the six rows is equally likely, probability 1/6 each. The Shapley value is the average of the six numbers in each column.

	Player			
Order of Entry	1	2	3	Total
1 2 3	1	3	4	8
1 3 2	1	5	2	8
2 1 3	4	0	4	8
2 3 1	3	0	5	8
3 1 2	2	5	1	8
3 2 1	3	4	1	8
Average	14/6	17/6	17/6	8

Proof of Theorem 2. To see that formula (6) gives the Shapley value, we have only to see that it satisfies axioms 1 through 4, since we have already shown that there is at most one such function, $\boldsymbol{\phi}$. Axioms 2, 3, and 4 are easy to check directly from the formula of Theorem 2. Axiom 1 follows from the above interpretation of the formula, since in each realization of forming the grand coalition, exactly $v(N)$ is given to the players. Hence, the average amount given to the players is also $v(N)$. ∎

3.4 Simple Games. The Shapley-Shubik Power Index. The Shapley value has an important application in modeling the power of members of voting games. This application was developed by Shapley and Shubik in 1954 and the measure is now known as the Shapley-Shubik Power Index.

Players are members of legislature or members of the board of directors of a corporation, etc. In such games, a proposed bill or decision is either passed or rejected. Those subsets of the players that can pass bills without outside help are called winning coalitions while those that cannot are called losing coalitions. In all such games, we may take the value of a winning coalition to be 1 and the value of a losing coalition to be 0. Such games are called simple games.

Definition. *A game* (N, v) *is* **simple** *if for every coalition* $S \subset N$*, either* $v(S) = 0$ *or* $v(S) = 1$.

In a simple game, a coalition S is said to be a **winning** coalition if $v(S) = 1$ and a **losing** coalition if $v(S) = 0$. So in a simple game every coalition is either winning or losing. It follows from superadditivity of v that in simple games every subset of a losing coalition is losing, and every superset of a winning coalition is winning.

Typical examples of simple games are
(1) the **majority rule game** where $v(S) = 1$ if $|S| > n/2$, and $v(S) = 0$ otherwise;
(2) the **unanimity game** where $v(S) = 1$ if $S = N$ and $v(S) = 0$ otherwise; and
(3) the **dictator game** where $v(S) = 1$ if $1 \in S$ and $v(S) = 0$ otherwise.

For simple games, formula (6) for the Shapley value simplifies because the difference $[v(S) - v(S - \{i\})]$ is always zero or one. It is zero if $v(S)$ and $v(S - \{i\})$ are both zero or both one, and it is one otherwise. Therefore we may remove $[v(S) - v(S - \{i\})]$ from formula (6) provided we sum only over those coalitions S that are winning with i and losing without i. Formula (6) for the Shapley value (the Shapley-Shubik Index) becomes

$$\phi_i(v) = \sum_{\substack{S \text{ winning} \\ S - \{i\} \text{ losing}}} \frac{(|S| - 1)!(n - |S|)!}{n!}. \tag{7}$$

Example. There is a large class of simple games called **weighted voting games**. They are defined by a characteristic function of the form

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i > q \\ 0 & \text{if } \sum_{i \in S} w_i \le q \end{cases},$$

for some non-negative numbers w_i, called the weights, and some positive number q, called the quota. If $q = (1/2) \sum_{i \in N} w_i$, this is called a weighted majority game.

As an example, consider the game with players 1, 2, 3, and 4, having 10, 20, 30, and 40 shares of stock respectively, in a corporation. Decisions require approval by a majority (more than 50%) of the shares. This is a weighted majority game with weights $w_1 = 10$, $w_2 = 20$, $w_3 = 30$ and $w_4 = 40$ and with quota $q = 50$.

Let us find the Shapley value of this game. The winning coalitions are $\{2,4\}$, $\{3,4\}$, $\{1,2,3\}$, and all supersets (sets containing one of these). For $i = 1$, $v(S) - v(S - \{1\}) = 0$ unless $S = \{1,2,3\}$. So

$$\phi_1(v) = \frac{2!1!}{4!} = \frac{1}{12}.$$

For $i = 2$, $v(S) - v(S - \{2\}) = 0$ unless $S = \{2,4\}$, $\{1,2,3\}$, or $\{1,2,4\}$, so that

$$\phi_2(v) = \frac{1!2!}{4!} + 2\frac{2!1!}{4!} = \frac{1}{4}.$$

Similarly, $\phi_3(v) = 1/4$. For $i = 4$, $v(S) - v(S - \{4\}) = 0$ unless $S = \{2,4\}$, $\{3,4\}$, $\{1,2,4\}$, $\{1,3,4\}$ or $\{2,3,4\}$. So

$$\phi_4(v) = 2\frac{1!2!}{4!} + 3\frac{2!1!}{4!} = \frac{5}{12}.$$

The Shapley value is $\phi = (1/12, 3/12, 3/12, 5/12)$. Note that the value is the same for players 2 and 3 although player 3 has more shares.

3.5 Exercises.

1. (Market with one seller and two buyers) Player 1 owns an art object of no intrinsic worth to him. Therefore he wishes to sell it. The object is worth \$30 to player 2 and \$40 to player 3. Set this up as a game in characteristic function form. Find the Shapley value. Is the Shapley value in the core? (Refer to Example 2 of Section 2.4.)

2. Find the Shapley value of the game with characteristic function

$$\begin{array}{llll} & v(\{1\}) = 1 & v(\{1,2\}) = 2 & \\ v(\emptyset) = 0 & v(\{2\}) = 0 & v(\{1,3\}) = -1 & v(\{1,2,3\}) = 6 \\ & v(\{3\}) = -4 & v(\{2,3\}) = 3 & \end{array}$$

3. Using the superadditivity of v, show that the Shapley value is an imputation.

4. Find the Shapley value of the n-person game with characteristic function,

(a) $v(S) = \begin{cases} |S| & \text{if } 1 \in S \\ 0 & \text{otherwise.} \end{cases}$ (b) $v(S) = \begin{cases} |S| & \text{if } 1 \in S \text{ or } 2 \in S \\ 0 & \text{otherwise.} \end{cases}$

(c) $v(S) = \begin{cases} |S| & \text{if } 1 \in S \text{ and } 2 \in S \\ 0 & \text{otherwise.} \end{cases}$.

5. Is every simple game a weighted voting game? Prove or give a counterexample.

6. Find the Shapley value of the weighted majority game with 4 players having 10, 30, 30, and 40 shares.

7. Find the Shapley value of the weighted majority game with $n \geq 3$ players in which player 1 has $2n - 3$ shares and players 2 to n have 2 shares each.

8. Modify the example of Section 3.4 so that the chairman of the board may decide tie votes. (The chairman of the board is a fifth player who has no shares.) Find the Shapley value.

9. (a) (One large political party and three smaller ones.) Consider the weighted majority game with one large party consisting of 1/3 of the votes and three equal sized smaller parties with 2/9 of the vote each. Find the Shapley value. Is the power of the large party greater or less than its proportionate size?

(b) (Two large political parties and three smaller ones.) Consider the weighted majority game with two large parties with 1/3 of the votes each and three smaller parties with 1/9 of the votes each. Find the Shapley value. Is the combined power of the two larger parties greater or less than its proportionate size?

10. (L. S. Shapley (1981) "Measurement of Power in Political Systems" in *Game Theory and its Applications* Proceedings in Applied Mathematics vol. 24, Amer. Math. Soc., Providence RI.) "County governments in New York are headed by Boards of Supervisors. Typically each municipality in a county has one seat, though a larger city may have two or more. But the supervisorial districts are usually quite unequal in population, and an effort is made to equalize citizen representation throughout the county by giving individual supervisors different numbers of votes in council. Table 1 shows the situation in Nassau County in 1964.

Table 1

District	Population	%	No. of votes	%
Hempstead 1 }	728,625	57.1	{ 31	27.0
Hempstead 2 }			{ 31	27.0
Oyster Bay	285,545	22.4	28	24.3
North Hempstead	213,335	16.7	21	18.3
Long Beach	25,654	2.0	2	1.7
Glen Cove	22,752	1.8	2	1.7
Totals	1,275,801 (sic)		115	

Under this system, a majority of 58 out of 115 votes is needed to pass a measure. But an inspection of the numerical possibilities reveals that the three weakest members of the board actually have no voting power at all. Indeed, their combined total of 25 votes is never enough to tip the scales. The assigned voting weights might just as well have been (31, 31, 28, 0, 0, 0) — or (1, 1, 1, 0, 0, 0) for that matter."

The Shapley value is obviously (1/3,1/3,1/3,0,0,0). This is just as obviously unsatisfactory. In 1971, the law was changed, setting the threshold required to pass legislation to 63 votes rather than 58. Find the Shapley value under the changed rules, and compare to the above table.

11. In the United Nations Security Council, there are 15 voting nations, including the "big five". To pass a resolution, 9 out of the 15 votes are needed, but each of the big five has veto power. One way of viewing this situation is as a weighted voting game in which each of the big five gets 7 votes and each of the other 10 nations gets 1 vote, and 39 votes are required to pass a resolution. Find the Shapley value.

12. **Cost Allocation.** A scientist has been invited for consultation at three distant cities. In addition to her consultation fees, she expects travel compensation. But since these three cities are relatively close, travel expenses can be greatly reduced if she accommodates them all in one trip. The problem is to decide how the travel expenses should be split among her hosts in the three cities. The one-way travel expenses among these three cities, A, B, and C, and her home base, H, are given in the accompanying table (measured in some unspecified units).

Between H and A, cost $= 7$.	Between A and B, cost $= 2$.
Between H and B, cost $= 8$.	Between A and C, cost $= 4$.
Between H and C, cost $= 6$.	Between B and C, cost $= 4$.

Set the problem up as a three-person game in coalitional form by specifying the characteristic function. Assume that the value of the visit is the same for each of the hosts, say 20 units each. Find the Shapley value. How much is the trip going to cost and how much should each host contribute to travel expenses?

13. **A one-product balanced market.** (Vorob'ev) Consider a market with one completely divisible commodity where the set N of players is divided into two disjoint sets, the buyers B and the sellers C, $N = B \cup C$. Each seller owns a certain amount of the commodity, say seller $k \in C$ owns y_k, and each buyer demands a certain amount, say buyer $j \in B$ demands x_j. We assume that the market is balanced, that is that the supply is equal to demand, $\sum_{k\in C} y_k = \sum_{j\in B} x_j$. We may set up such a game in characteristic function form by letting

$$v(S) = \min\left\{ \sum_{j\in S\cap B} x_j \ , \sum_{k\in S\cap C} y_k \right\}$$

Thus, the value of a coalition is the total amount of trading that can be done among members of the coalition. Find the Shapley value of the game. (Hint: for each permutation of the players, consider also the reverse permutation in which the players enter the grand coalition in reverse order.)

14. Consider the n-person game with players $1, 2, \ldots, n$, whose characteristic function satisfies

$$v(S) = k \quad \text{if} \quad \{1, \ldots, k\} \subset S \quad \text{but} \quad k+1 \notin S.$$

For example, $v(\{1, 3, 4, 6\}) = 1$ and $v(\{1, 2, 3, 5, 6, 7\}) = 3$. Find the Shapley value.

15. **The Airport Game.** (Littlechild and Owen (1973).) Consider the following cost allocation problem. Building an airfield will benefit n players. Player j requires an airfield that costs c_j to build, so to accommodate all the players, the field will be built at a cost of $\max_{1\le j\le n} c_j$. How should this cost be split among the players? Suppose all the costs are distinct and let $c_1 < c_2 < \cdots < c_n$. Take the characteristic function of the game to be

$$v(S) = -\max_{j\in S} c_j.$$

(a) Let $R_k = \{k, k+1, \ldots, n\}$ for $k = 1, 2, \ldots, n$, and define the characteristic function v_k through the equation

$$v_k(S) = \begin{cases} -(c_k - c_{k-1}) & \text{if } S \cap R_k \neq \emptyset \\ 0 & \text{if } S \cap R_k = \emptyset. \end{cases}$$

Show that $v = \sum_{k=1}^{n} v_k$.

(b) Find the Shapley value.

16. **A Market with 1 Seller and m Buyers.** Player 0 owns an object of no intrinsic worth to himself. Buyer j values the object at a_j dollars. Suppose $a_1 > a_2 > \cdots > a_m > 0$. Set up the characteristic function and find the Shapley value.
(Answer: For the seller, $\phi_0(v) = \sum_{k=1}^{m} a_k/(k(k+1))$, and for the buyers,

$$\phi_j(v) = \frac{a_j}{j(j+1)} - 2 \sum_{k=j+1}^{m} \frac{a_k}{(k-1)k(k+1)}.)$$

17. **The Core of a Simple Game.** In a simple game, (N, v), a player, i, is said to be a veto player, if $v(N - \{i\}) = 0$.

(a) Show that the core is empty if there are no veto players.

(b) Show, conversely, that the core is not empty if there is at least one veto player.

(c) Characterize the core.

4. The Nucleolus

Another interesting value function for n-person cooperative games may be found in the nucleolus, a concept introduced by Schmeidler (SIAM J. Appl. Math, 1969). Instead of applying a general axiomatization of fairness to a value function defined on the set of all characteristic functions, we look at a fixed characteristic function, v, and try to find an imputation $\boldsymbol{x} = (x_1, \ldots, x_n)$ that minimizes the worst inequity. That is, we ask each coalition S how dissatisfied it is with the proposed imputation $\boldsymbol{x}$ and we try to minimize the maximum dissatisfaction.

4.1 Definition of the Nucleolus. As a measure of the inequity of an imputation $\boldsymbol{x}$ for a coalition S is defined as the *excess*,

$$e(\boldsymbol{x}, S) = v(S) - \sum_{j \in S} x_j,$$

which measures the amount (the size of the inequity) by which coalition S falls short of its potential $v(S)$ in the allocation $\boldsymbol{x}$. Since the core is defined as the set of imputations such that $\sum_{i \in S} x_i \geq v(S)$ for all coalitions S, we immediately have that *an imputation $\boldsymbol{x}$ is in the core if and only if all its excesses are negative or zero.*

On the principle that the one who yells loudest gets served first, we look first at those coalitions S whose excess, for a fixed allocation $\boldsymbol{x}$, is the largest. Then we adjust $\boldsymbol{x}$, if possible, to make this largest excess smaller. When the largest excess has been made as small as possible, we concentrate on the next largest excess, and adjust $\boldsymbol{x}$ to make it as small as possible, and so on. An example should clarify this procedure.

Example 1. The Bankruptcy Game. (O'Niell (1982)) A small company goes bankrupt owing money to three creditors. The company owes creditor A \$10,000 , creditor B \$20,000 , and creditor C \$30,000. If the company has only \$36,000 to cover these debts, how should the money be divided among the creditors? A **pro rata** split of the money would lead to the allocation of \$6000 for A, \$12,000 for B, and \$18,000 for C, denoted by $\boldsymbol{x} = (6, 12, 18)$ in thousands of dollars. We shall compare this allocation with those suggested by the Shapley value and the nucleolus.

First, we must decide on a characteristic function to represent this game. Of course we will have $v(\emptyset) = 0$ from the definition of characteristic function, and $v(ABC) = 36$ measured in thousands of dollars, By himself, A is not guaranteed to receive anything since the other two could receive the whole amount; thus we take $v(A) = 0$. Similarly, $v(B) = 0$. Creditor C is assured of receiving at least \$6000, since even if A and B receive the total amount of their claim, namely \$30,000, that will leave \$36,000 - \$30,000 = \$6000 for C. Thus we take $v(C) = 6$. Similarly, we find $v(AB) = 6$, $v(AC) = 16$, and $v(BC) = 26$.

To find the nucleolus of this game, let $\boldsymbol{x} = (x_1, x_2, x_3)$ be an efficient allocation (that is, let $x_1 + x_2 + x_3 = 36$), and look at the excesses as found in the table below. We may omit the empty set and the grand coalition from consideration since their excesses are always zero.

To get an idea of how to proceed, consider first an arbitrary point, say the pro rata point $(6, 12, 18)$. As seen in the table, the vector of excesses is $\boldsymbol{e} = (-6, -12, -12, -12, -8, -4)$. The largest of these numbers is -4 corresponding to the coalition BC. This coalition will claim that every other coalition is doing better than it is. So we try to improve on things for this coalition by making x_2+x_3 larger, or, equivalently, x_1 smaller (since $x_1 = 36-x_2-x_3$). But as we decrease the excess for BC, the excess for A will increase at the same rate and so these excesses will meet at -5, when $x_1 = 5$. It is clear that no choice of $\boldsymbol{x}$ can make the maximum excess smaller than -5 since at least one of the coalitions A or BC will have excess at least -5. Hence, $x_1 = 5$ is the first component of the nucleolus.

S	$v(S)$	$e(x, S)$	$(6, 12, 18)$	$(5, 12, 19)$	$(5, 10.5, 20.5)$	$(6, 11, 19)$
A	0	$-x_1$	-6	-5	-5	-6
B	0	$-x_2$	-12	-12	-10.5	-11
C	6	$6 - x_3$	-12	-13	-14.5	-13
AB	6	$6 - x_1 - x_2$	-12	-11	-9.5	-11
AC	16	$16 - x_1 - x_3$	-8	-8	-9.5	-9
BC	26	$26 - x_2 - x_3$	-4	-5	-5	-4

Though x_1 is fixed, we still have x_2 and x_3 to vary subject to $x_2 + x_3 = 36 - 5 = 31$, and we choose them to make the next largest excess smaller. If we choose the point $\boldsymbol{x} = (5, 12, 19)$ as the next guess, we see that the next largest excess after the -5's is the -8 corresponding to coalition AC. To make this smaller, we must increase x_3 (decrease x_2). But as we do so, the excesses for coalitions B and AB increase at the same rate. Since the excess for coalition AB starts closer to -8 we find x_2 and x_3 so that $e(\boldsymbol{x}, AB) = e(\boldsymbol{x}, AC)$. This occurs at $x_2 = 10.5$ and $x_3 = 20.5$. The nucleolus is therefore $(5, 10.5, 20.5)$.

It is of interest to compare this solution to the Shapley value. We may compute the Shapley value by any of the methods given in Section 3. Using the formula, we find

$$\phi_A = (1/3)(0) + (1/6)(6) + (1/6)(10) + (1/3)(10) = 6$$
$$\phi_B = (1/3)(0) + (1/6)(6) + (1/6)(20) + (1/3)(20) = 11$$
$$\phi_C = (1/3)(6) + (1/6)(16) + (1/6)(26) + (1/3)(30) = 19$$

The last column in the table shows the excesses for the Shapley value. ■

It is time to define more precisely the concept of the nucleolus of a game with characteristic function v. First we define an ordering on vectors that reflects the notion of smaller maximum excess as given in the above example.

Define $\boldsymbol{O}(\boldsymbol{x})$ as the vector of excesses arranged in decreasing (nonincreasing) order. In the example, if $\boldsymbol{x} = (6, 12, 18)$ then $\boldsymbol{O}(\boldsymbol{x}) = (-4, -6, -8, -12, -12, -12)$. On the vectors $\boldsymbol{O}(\boldsymbol{x})$ we use the lexographic order. We say a vector $\boldsymbol{y} = (y_1, \ldots, y_k)$ is lexographically less than a vector $\boldsymbol{z} = (z_1, \ldots, z_k)$, and write $\boldsymbol{y} <_L \boldsymbol{z}$, if $y_1 < z_1$, or $y_1 = z_1$ and $y_2 < z_2$, or $y_1 = z_1$, $y_2 = z_2$ and $y_3 < z_3$,…, or $y_1 = z_1$, …, $y_{k-1} = z_{k-1}$ and $y_k < z_k$.

That is, $\boldsymbol{y} <_L \boldsymbol{z}$ if in the first component in which $\boldsymbol{y}$ and $\boldsymbol{z}$ differ, that component of $\boldsymbol{y}$ is less than the corresponding component of $\boldsymbol{z}$. Similarly, we write $\boldsymbol{y} \leq_L \boldsymbol{z}$ if either $\boldsymbol{y} <_L \boldsymbol{z}$

or $\boldsymbol{y} = \boldsymbol{z}$. The nucleolus is an efficient allocation that minimizes $\boldsymbol{O}(\boldsymbol{x})$ in the lexographic ordering.

Definition. *Let $X = \{\boldsymbol{x} : \sum_{j=1}^{n} x_j = v(N)\}$ be the set of efficient allocations. We say that a vector $\boldsymbol{\nu} \in X$ is a nucleolus if for every $\boldsymbol{x} \in X$ we have $\boldsymbol{O}(\boldsymbol{\nu}) \leq_L \boldsymbol{O}(\boldsymbol{x})$.*

4.2 Properties of the Nucleolus. The main properties of the nucleous are stated without proof in the following theorem.

Theorem 1. *The nucleolus of a game in coalitional form exists and is unique. The nucleolus is group rational, individually rational, and satisfies the symmetry axiom and the dummy axiom. If the core is not empty, the nucleolus is in the core.*

The most difficult part to prove is the uniqueness of the nucleolus. See the book of Owen for a discussion. Since the nucleolus always exists and is unique, we may speak of *the* nucleolus of a game. Like the Shapley value, the nucleolus will satisfy individual rationality if the characteristic function is super-additive or, more generally, if it is monotone in the sense that for all players i and for all coalitions S not containing i, we have $v(S)+v(\{i\}) \leq v(S \cup \{i\})$. In contrast to the Shapley value, the nucleolus will be in the core provided the core is not empty. (Exercise 1.)

Since the nucleolus satisfies the first three axioms of the Shapley value, it does not satisfy the linearity axiom.

It is interesting to see how the nucleolus and the Shapley value change in the bankrupt company example as the total remaining assets of the company change from \$0 to \$60,000, that is, as $v(N)$ changes from 0 to 60. Consider the nucleolus. If $v(N)$ is between 0 and 15, the nucleolus divides this amount equally among the players. For $v(N)$ between 15 and 25, the nucleolus splits the excess above 15 equally between B and C, while for $v(N)$ between 25 and 35, all the excess above 25 goes to C. For $v(N)$ between 35 and 45, the excess above 35 is split between B and C, and for $v(N)$ between 45 and 60, the excess above 45 is divided equally among the three players.

Nucleolus

Amount of $v(N)$ between	0 and 15	share equally
	15 and 25	B and C share
	25 and 35	C gets it all
	35 and 45	B and C share
	45 and 60	share equally

Shapley Value

Amount of $v(N)$ between	0 and 10	share equally
	10 and 20	B and C share
	20 and 40	C gets 2/3rds and A and B get 1/6th
	40 and 50	B and C share
	50 and 60	share equally

One notes that at $v(N) = 30$, the nucleolus and the Shapley value coincide with the pro rata point. Compared to the pro rata point, both the Shapley value and the nucleolus favor the weaker players if $v(N)$ is small, and favor the stronger players if $v(N)$ is large, more so for the Shapley value than the nucleolus.

4.3 Computation of the Nucleolus. The nucleolus is more difficult to compute than the Shapley value. In fact, the first step of finding the nucleolus is to find a vector $\boldsymbol{x} = (x_1, \ldots, x_n)$ that minimizes the maximum of the excesses $e(\boldsymbol{x}, S)$ over all S subject to $\sum x_j = v(N)$. This problem of minimizing the maximum of a collection of linear functions subject to a linear constraint is a linear programming problem and can thus be solved easily by the simplex method, for example. After this is done, one may have to solve a second linear programming problem to minimize the next largest excess, and so on.

For $n = 3$, these problems are not hard, but they may be more difficult than the example of the bankrupt company. It may be useful to work out another example. Suppose

$$\begin{array}{llll} & v(\{A\}) = -1 & v(\{AB\}) = 3 & \\ v(\emptyset) = 0 & v(\{B\}) = 0 & v(\{AC\}) = 4 & v(\{ABC\}) = 5 \\ & v(\{C\}) = 1 & v(\{BC\}) = 2 & \end{array}$$

Alone, A is in the worst position, but in forming coalitions he is more valuable. The Shapley value is $\phi = (10/6, 7/6, 13/6)$. Let us find the nucleolus.

As an initial guess, try $(1, 1, 3)$. In the table below, we see that the maximum excess occurs at the coalition AB. To improve on this, we must decrease x_3. Since the next largest excess is for coalition AC, we keep x_2 fixed (increase x_1) and choose $x_3 = 2$ to make the excess for AB equal to the excess for AC. This leads to the point $(2, 1, 2)$ whose largest excess is 0, occurring at coalitions AB and AC. To make this smaller, we must decrease both x_2 and x_3. This involves increasing x_1, and will increase the excess for BC. We can see that the best we can do will occur when the excesses for AB and AC and BC are all equal. Solving the three equations,

$$x_3 - 2 = x_2 - 1 = x_1 - 3, \quad \text{and} \quad x_1 + x_2 + x_3 = 5,$$

we find $x_3 = x_2 + 1$ and $x_1 = x_2 + 2$ so that the solution is $\boldsymbol{x} = (8/3, 2/3, 5/3)$. This is the nucleolus.

S	$v(S)$	$e(x, S)$	$(1,1,3)$	$(2,1,2)$	$(8/3,2/3,5/3)$
A	-1	$-1 - x_1$	-2	-3	$-11/3$
B	0	$-x_2$	-1	-1	$-2/3$
C	1	$1 - x_3$	-2	-1	$-2/3$
AB	3	$3 - x_1 - x_2 = x_3 - 2$	1	0	$-1/3$
AC	4	$4 - x_1 - x_3 = x_2 - 1$	0	0	$-1/3$
BC	2	$2 - x_2 - x_3 = x_1 - 3$	-2	-1	$-1/3$

4.4 Exercises.

1. Show that if the core is not empty, then the nucleolus is in the core.

2. Show that for a constant-sum three-person game, the nucleolus is the same as the Shapley value.

3. **The Cattle Drive.** Rancher A has some cattle ready for market, and he foresees a profit of $1200 on the sale. But two other ranchers lie between his ranch and the market town. The owners of these ranches, B and C, can deny passage through their land or require payment of a suitable fee. The question is: What constitutes a suitable fee? The characteristic function may be taken to be: $v(A) = v(B) = v(C) = v(BC) = 0$ and $v(AB) = v(AC) = v(ABC) = 1200$.
(a) Find the core, and note that it consists of one point. This point must then be the nucleolus. (Why?)
(b) Find the Shapley value.
(c) Which do you think is more suitable for settling the question of a fee, the nucleolus or the Shapley value, and why?

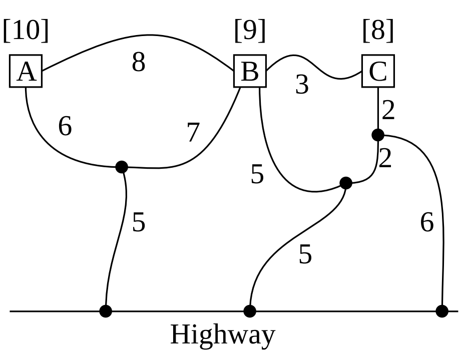
4. Find the nucleolus for Exercise 3.5.1. Compare to the Shapley value. How could you tell before computing it that the nucleolus was not the same as the Shapley value?

5. Find the nucleolus for Exercise 3.5.2.

6. Find the nucleolus for Exercise 3.5.4(a). You may assume that the nucleolus satisfies the symmetry axiom.

7. Find the nucleolus for Exercise 3.5.6. You may assume that the nucleolus satisfies the dummy axiom.

8. **Cost Allocation.** Three farms are connected to each other and to the main highway by a series of rough trails as shown in the figure. Each farmer would benefit by having a paved road connecting his farm to the highway. The amounts of these benefits are indicated in square brackets [...]. The costs of paving each section of the trails are also indicated on the diagram.



It is clear that no single farmer would find it profitable to build his own road, but a cooperative project would obviously be worthwhile.
(a) Determine the characteristic function.

(b) Find the Shapley value.
(c) Find the nucleolus.

9. **The Landowner and the Peasants.** Here is a generalization of symmetric games allowing one special player. The game is played with one landowner and m peasants, $n = m + 1$ players. The peasants can produce nothing by themselves, but neither can the landowner. All peasants are interchangeable. If k peasants and the landowner cooperate, they can jointly receive the amount $f(k)$, where $0 = f(0) < f(1) < f(2) < \cdots < f(m)$. We denote the landowner as player number 1 and the peasants as players 2 through n. Thus,

$$v(S) = \begin{cases} f(|S| - 1) & \text{if } 1 \in S \\ 0 & \text{if } 1 \notin S. \end{cases}$$

(a) Suppose $m = 3$ and $f(x) = x$. Find the Shapley value.
(b) Suppose $m = 3$ and $f(x) = x$. Find the nucleolus.
(c) See if you can find a simple formula for the Shapley value for general m and $f(x)$.
(d) See if you can find a general formula for the nucleolus of this game.

10. **An Assignment Game.** Two house owners, A and B, are expecting to sell their houses to two potential buyers, C and D, each wanting to buy one house at most. Players A and B value their houses at 10 and 20 respectively, in some unspecified units. In the same units, Player C values A's house at 14 and B's house at 23, while Player D values A's house at 18 and B's house at 25.
(a) Determine a characteristic function for the game.
(b) Find the Shapley value.
(c) Find the nucleolus.