Combinatorial Games

Game 1 Start with n chips. Players A,B alternately take 1,2,3,4 chips until there are none left. The winner is the person who takes the last chip:

Example

	A	B	A	B	A	
$n = 10$	3	2	4	1		B wins
$n = 11$	1	2	3	4	1	A wins

What is the optimal strategy for this game?

Game 2 Chip placed at point (m, n). Players can move chip to (m', n) or (m, n') where $0 \leq m' < m$ and $0 \leq n' < n$. The player who makes the last move and puts the chip onto $(0, 0)$ wins.

What is the optimal strategy for this game?

Game 3 W is a set of words. A and B alternatel remove words $w_1, w_2, \ldots,$ from W. The rule is that the first letter of w_{i+1} must be the same as the last letter of w_i. The player who makes the last legal move wins.

1 Abstraction

Represent each position by a vertex of a digraph $D = (X, A)$. (x, y) is an arc of D iff one can move from position x to position y.

We assume that the digraph is finite and that it is **acyclic** i.e. there are no directed cycles.

The game starts with a chip on vertex x_0 say, and players alternately move the chip to $x_1, x_2, \ldots,$ where $x_{i+1} \in \Gamma^+(x_i)$, the set of out-neighbours of x_i. The game ends when the chip is on a **sink** i.e. a vertex of out-degree zero. The last player to move is the winner.

Example 1: $D = (\{0, 1, \ldots, n\}, A)$ where $(x, y) \in A$ iff $x - y \in \{1, 2, 3, 4\}$.

Example 2: $D = (\{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\}, A)$ where $(x, y) \in \Gamma^+((x', y')))$ iff $x = x'$ and $y > y'$ or $x > x'$ and $y = y'$.

Example 3: $D = (\{(W', w) : W' \subseteq W \setminus \{w\}\}, A)$. w is the last word used and W' is the remaining set of unused words. $(A', w') \in \Gamma^+((A, w))$ iff $w' \in A$ and w' begins with the last letter of w. Also, there is an arc from $(W, \cdot)$ to $(W \setminus \{w\}, w)$ for all w, corresponding to the games start.

We will first argue that such a game must eventually end. A **topological numbering** of digraph D is a map $f : X \to [N]$, $N = |X|$ which satisfies $(x, y) \in A$ implies $f(x) < f(y)$.

Theorem 1. *A finite digraph $D = (X, A)$ is acyclic iff it admits at least one topological numbering.*

Proof Suppose first that D has a topological numbering. We show that it is acyclic. Suppose that $C = (x_1, x_2, \ldots, x_k, x_1)$ is a directed cycle. Then $f(x_1) < f(x_2) < \cdots < f(x_k) < f(x_1)$, contradiction.

Suppose now that D is acyclic. We first argue that D has at least one sink. Thus let $P = (x_1, x_2, \ldots, x_k)$ be a longest simple path in D. We claim that x_k is a sink. If D contains an arc (x_k, y) then either $y = x_i, 1 \leq i \leq k-1$ and this means that D contains the cycle $x_i, x_{i+1}, \ldots, x_k, x_i)$, contradiction or $y \notin \{x_1, x_2, \ldots, x_k\}$ and then (P, y) is a longer simple path than P, contradiction.

We can now prove by induction on N that there is at least one topological numbering. Ifg $N = 1$ and $X = \{x\}$ then $f(x) = 1$ defines a topological numbering.

Now asssume that $N > 1$. Let z be a sink of D and define $f(z) = N$. The digraph $D' = D - z$ is acyclic and by the induction hypothesis it admits a topological numbering, $f : X \setminus \{z\} \to [N-1]$. The function we have defined on X is a topological numbering. If $(x, y) \in A$ then either $x, y \neq z$ and then $f(x) < f(y)$ by our assumption on f, or $y = z$ and then $f(x) < N = f(z)$ ($x \neq z$ because z is a sink). $\square$

The fact that D has a topological numbering implies that the game must end. Each move increases the f value of the current position by at least one and so after at most N moves a sink must be reached.

The positions of a game are partitioned into 2 sets:

- P-positions: The next player cannot win. The previous player can win regardless of the current player's strategy.
- N-positions: The next player has a strategy for winning the game.

The main problem is to determine N and P and what the strategy is for winning from an N-position.

For $x \in X$ let $\Gamma^+(x) = \{y \in X : (x, y) \in A\}$ be the set of pout-neighbours of x.

Labelling procedure

1. Label all sinks with P.
2. Label with N, every position x for which there exists $y \in \Gamma^+(x)$ which is labelled with P.
3. Label with P, every position x for which $\Gamma^+(x)$ is labelled with N.

A position x is an N-position (winning) iff there is a move from x to a losing position for the next player.

The labelling should be carried out in reverse topological order.

Thus there is a unique partition of X into N, P which satisfies the following:

P1 All sinks are in P.

P2 If $x \in N$ then $\Gamma^+(x) \cap P \neq \emptyset$.

P3 If $x \in P$ then $\Gamma^+(x) \subseteq N$.

In Game 1, $P = \{5k : k \geq 0\}$ and in Game 2, $P = \{(x, x) : x \geq 0\}$.

Sprague-Grundy Numbering

For $S \subseteq \{0, 1, 2, \ldots,\}$ let

$$mex(S) = \min\{x \geq 0 : x \notin S\}.$$

Now given an acyclic digraph $D = X, A$ define g recursively by

$$g(x) = \begin{cases} 0 & x \text{ is a sink} \\ mex(\Gamma^+(x)) & \text{otherwise} \end{cases}$$

$g(x)$ can be computed in reverse topological order.

Lemma 1.

$$x \in P \leftrightarrow g(x) = 0.$$

Proof Clearly P1 holds. We check P2 and P3.
P2: If $g(x) > 0$ there must be a $y \in \Gamma^+(x)$ with $g(y) = 0$.
P3: If $g(x) = 0$ there cannot be a $y \in \Gamma^+(x)$ with $g(y) = 0$. □

Sums of games

Suppose that we have n games with digraphs $D_i = (X_i, A_i)$, $i = 1, 2, \ldots, n$. The sum of these games is played as follows. A position is a vector $(x_1, x_2, \ldots, x_n) \in A_1 \times A_2 \times \cdots \times A_n$. To make a move, a player chooses i such that x_i is not a sink of D_i and then replaces x_i by $y \in \Gamma_i^+(x_i)$. The game ends when each x_i is a sink of D_i for $i = 1, 2, \ldots, n$.

Example Nim

In a one pile game, we start with $a \geq 0$ chips and while there is a positive number x of chips, a move consists of deleting $y \leq x$ chips. In this game the N-positions are positive integers and the unique P-position is 0. The Sprague-Grundy numbering is defined by $g(x) = x$.

In general, Nim consists of the sum of n single pile games stsrting with $a_1, a_2, \ldots, a_n > 0$. A move consists of deleting some chips from a non-empty pile.

We now show how to compute the Sprague-Grundy numbering for a sum of games.

For binary integers $a = a_m a_{m-1} \cdots a_1 a_0$ and $b = b_m b_{m-1} \cdots b_1 b_0$ we define $a \oplus b = c_m c_{m-1} \cdots c_1 c_0$ by $c_i = 1$ if $a_i \neq b_i$ and $c_i = 0$ if $a_i = b_i$ for $i = 1, 2, \ldots, m$.

So for example $11 \oplus 5 = 14$.

Theorem 2. *If g_i is the Sprague-Grundy function for game $i = 1, 2, \ldots, n$ then the Sprague-Grundy function g for the sum of the games is defined by*

$$g(x) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_n(x_n)$$

where $x = (x_1, x_2, \ldots, x_n)$.

Proof It is enough to show

1. If $x \in X$ is a sink of D then $g(x) = 0$.
2. If $x \in X$ and $g(x) = b > a \geq 0$ then there exists $x' \in \Gamma^+(x)$ such that $g(x') = a$.
3. If $x \in X$ and $g(x) = b$ and $x' \in \Gamma^+(x)$ then $g(x') \neq g(x)$.

1. If $x = (x_1, x_2, \ldots, x_n)$ is a sink then x_i is a sink of D_i for $i = 1, 2, \ldots, n$. So

$$\begin{aligned} g(x) &= g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_n(x_n) \\ &= 0 \oplus 0 \oplus \cdots \oplus 0 \\ &= 0. \end{aligned}$$

2. Write $d = a \oplus b$. Then

$$\begin{aligned} a &= d \oplus b \\ &= d \oplus g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_n(x_n). \end{aligned} \tag{1}$$

Now suppose that we can show there exists i such that $d \oplus g_i(x_i) < g_i(x_i)$. Then since $g_i(x_i) = mex(\Gamma_i^+(x_i))$ there must exist $x_i' \in \Gamma_i^+(x_i)$ such that $g_i(x_i') = d \oplus g_i(x_i)$. Assume without loss of generality that $i = 1$. Then from (1) we have

$$a = g_1(x_1') \oplus g_2(x_2) \oplus \cdots \oplus g_n(x_n) = g(x_1', x_2, \ldots, x_n).$$

Furthermore, $(x_1', x_2, \ldots, x_n) \in \Gamma^+(x)$ and so we will have verified 2.

Let us prove that such an i exists. Suppose that $2^{k-1} \leq d < 2^k$. Then d has a 1 in position k and no higher. Since $d_k = a_k \oplus b_k$ and $a < b$ we must have $a_k = 0$ and $b_k = 1$. Thus there is at least one i such that $g_i(x_i)$ has a 1 in position k. But then $d \oplus g_i(x_i) < g_i(x_i)$ since d "destroys" the kth bit of $g_i(x_i)$ and does not change any higher bit.

3. Suppose without loss of generality that $g(x_1', x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n)$ where $x_1' \in \Gamma^+(x_1)$. Then $g_1(x_1') \oplus g_2(x_2) \oplus \cdots \oplus g_n(x_n) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_n(x_n)$ implies that $g_1(x_1') = g_1(x_1)$, contradition. □

If we apply this theorem to the game of Nim then if the position x consists of piles of x_i chips for $i = 1, 2, \ldots, n$ then $g(x) = x_1 \oplus x_2 \oplus \cdots \oplus x_n$.

Sums of other subtraction games:

In our first example, $g(x) = x \mod 5$ and so for the sum of n such games we have

$$g(x_1, x_2, \ldots, x_n) = (x_1 \mod 5) \oplus (x_2 \mod 5) \oplus \cdots \oplus (x_n \mod 5).$$

Another subtraction game.
One pile:

- A player can remove any even number of chips, but not the whole pile.
- A player can remove the whole pile if it is odd.

The terminal positions are 0 or 2.

Lemma 2. *$g(0) = 0$, $g(2k) = k - 1$ and $g(2k - 1) = k$ for $k \geq 1$.*

Proof 0,2 are terminal postions and so $g(0) = g(2) = 0$. $g(1) = 1$ because the only position one can move to from 1 is 0. We prove the remainder by induction on k.

Assume that $k > 1$.

$$\begin{aligned} g(2k) &= mex\{g(2k-2), g(2k-4), \ldots, g(2)\} \\ &= mex\{k-2, k-3, \ldots, 0\} \\ &= k - 1. \\ g(2k-1) &= mex\{g(2k-3), g(2k-5), \ldots, g(1), g(0)\} \\ &= mex\{k-1, k-2, \ldots, 0\} \\ &= k. \end{aligned}$$

□

A more complicated one pile game

Start with n chips. First player can remove up to $n - 1$ chips.

In general, if the previous player took x chips, then the next player can take $y \leq x$ chips.

Thus a games position can be represented by (n, x) where n is the current size of the pile and x is the maximum number of chips that can be removed in this round.

Theorem 3. *Suppose that the position is (n, x) where $n = m2^k$ and m is odd. Then,*

(a) *This is an N-position if $x \geq 2^k$.*

(b) *This is a P-position if $m = 1$ and $x < n$.*

Proof For a non-negative integer $n = m2^k$, let $\langle n \rangle$ denote the number of bits in the binary expansion of n and let $k = \rho(n)$ determine the position of the right-most one in this expansion. We claim that the following strategy is a win for the player in a postion described in (a): Remove $y = 2^k$ chips. Suppose this player is A.

If $m = 1$ then $x \geq n$ and A wins. Otherwise, after such a move the position is (n', y) where where $\rho(n') > \rho(n)$. Note first that $\langle n' \rangle = \langle n \rangle - 1 > 0$. Thus B cannot win at this point. Second, B cannot remove more than 2^k chips and so if B moves the position to (n'', x'') then $\langle n'' \rangle \geq \langle n' \rangle$ and furthermore, $x'' \geq 2^{\rho(n'')}$, since x'' must have a 1 in position $\rho(n'')$. Thus, by induction, A is in an N-position and wins the game.

To prove (b), note that after the first move, the position satisfies the conditions of (a). $\square$.

Let us next consider a generalisation of this game. There are 2 players A and B and A goes first. We have a non-decreasing function f from $\boldsymbol{N} \to \boldsymbol{N}$ where $\boldsymbol{N} = \{1, 2, \dots\}$ which satisfies $f(x) \geq x$. At the first move A takes any number less than h from the pile, where h is the size of the initial pile. Then on a subsequent move, if a player takes x chips then the next player is constrained to take at most $f(x)$ chips. Thus the above considered the cases $f(x) = x$.

There is a set $\mathcal{H} = \{H_1 = 1 < H_2 < \dots\}$ of initial pile sizes for which the first player will lose, assuming that the second player plays optimally. Also, if the initial pile size $h \notin \mathcal{H}$ then the first player has a winning strategy. It will turn out that the sequence satisfies the recurrence:

$$H_{j+1} = H_j + H_\ell \text{ where } H_\ell = \min_{i \leq j}\{H_i \mid f(H_i) \geq H_j\}, \qquad \text{for } j \geq 0. \tag{2}$$

Note that

$$H_{j+1} \leq 2H_j. \tag{3}$$

[The reader should check that if $f(x) = x$ then $H_i = 2^i$. Another case to check is $f(x) = 2x$. This gives $\mathcal{H} = \{1, 2, 3, 5, 8, \dots, \}$ i.e. the Fibonacci sequence.]

The key to the game is the following result.

Theorem 4. *Every positive integer n can be* **uniquely** *written as the sum*

$$n = H_{j_1} + H_{j_2} + \cdots + H_{j_p} \tag{4}$$

where $f(H_{j_i}) < H_{j_{i+1}}$ for $1 \leq i < p$.

Proof We prove this by induction on n. If $n = 1$ then $n = H_1$ is the unique decomposition.

Existence
Assume that any $n < H_k$ can be represented as a sum of distinct H_{j_i}'s with $f(H_{j_i}) < H_{j_{i+1}}$ and suppose that $H_k \leq n < H_{k+1}$. Inequality (3) implies that $n - H_k < H_k$.

It follows by induction that

$$n - H_k = H_{j_1} + \cdots + H_{j_p}, \tag{5}$$

where $f(H_{j_i}) < H_{j_{i+1}}$ for $i = 1, 2, ..., p-1$. To establish existence we need only show that $f(H_{j_p}) < H_k$. Assume to the contrary that $f(H_{j_p}) \geq H_k$. But then for some $m \leq j_p$ we have

$$H_{k+1} = H_k + H_m \leq H_k + H_{j_p} \leq n,$$

contradicting the choice of n.

Uniqueness

We will first prove by induction on p that if $f(H_{j_i}) < H_{j_{i+1}}$ for $1 \leq i < p$ then

$$H_{j_1} + H_{j_2} + \cdots + H_{j_p} < H_{j_p+1}. \tag{6}$$

If $p = 2$ then we are saying that if $f(H_{j_1}) < H_{j_2}$ then $H_{j_1} + H_{j_2} < H_{j_2+1}$. But this follows directly from $H_{j_2+1} = H_{j_2} + H_m$ where $f(H_m) \geq H_{j_2}$ i.e. $H_m > H_{j_1}$.

So assume that (6) is true for $p \geq 2$. Now

$$H_{j_{p+1}+1} = H_{j_{p+1}} + H_m \text{ and } f(H_{j_p}) < H_{j_{p+1}}$$

implies that $m \geq j_p + 1$.

Thus

$$\begin{array}{rcl} H_{j_{p+1}+1} & \geq & H_{j_{p+1}} + H_{j_p+1} \\ & > & H_{j_{p+1}} + H_{j_p} + H_{j_{p-1}} + \cdots + H_{j_1} \end{array}$$

after applying induction to get the second inequality.

This completes the induction for (6).

Now assume by induction on k that $n < H_k$ has a unique decomposition (4). This is true for $k = 2$ and so now assume that $k \geq 2$ and $H_k \leq n < H_{k+1}$. Consider a decomposition

$$n = H_{j_1} + H_{j_2} + \cdots + H_{j_p}.$$

It follows from (6) that $j_p = k$. Indeed, $j_p \leq k$ since $n < H_{k+1}$ and if $j_p < k$ then $H_{j_1} + H_{j_2} + \cdots + H_{j_p} < H_{j_p+1} \leq H_k$, contradicting our choice of n. So H_k appears in every decomposition of n.

Now (3) and $n < H_{k+1}$ implies $n - H_k < H_k$ and so, by induction, $n - H_k$ has a unique decompositon. But then if n had two distinct decompositions, H_k would appear in each, implying that $n - H_k$ also had two distinct decompositions, contradiction. □

One simple consequence of the uniqueness of the decomposition is that

$$H_k \neq H_{j_1} + H_{j_2} + \cdots + H_{j_p} \tag{7}$$

for all k and sequences $j_1, j_2, \ldots, j_p$ where $f(H_{j_i}) < H_{j_{i+1}}$ for $i = 1, 2, ..., p-1$.

It follows from the above Lemma that the integers n can be given unique "binary" representations by representing $n = H_{j_1} + H_{j_2} + \cdots + H_{j_p}$ by the 0-1 string with a 1 in posiitons $j_1, j_2, \ldots, j_p$ and 0 everywhere else. We call this the H-representation of n. This then leads to the following

Theorem 5. *Suppose that the position is* $(n, *)$. *Then,*

(a) *This is an N-position if* $n \notin \mathcal{H} = \{H_1, H_2, \ldots, \}$.

(b) *This is a P-position if* $n \in \mathcal{H}$.

Proof
(a) The winning strategy is to delete a number of chips equal to H_{j_1} where j_1 is the index of the rightmost 1 in the H-representation of n.

All we have to do is verify that this strategy is possible. Note that if A deletes H_{j_1} chips, then B cannot respond by deleting H_{j_2} chips, because $H_{j_2} > f(H_{j_1})$ and so it is only A that can reduce the number of 1's in the H-representation of n.

The thing to check is that if A starts in $(n, *)$ then A can always delete H_{j_1} chips i.e. the positions (m, x) that A will face satisfy $f(x) \geq H_{k_1}$ where $m = H_{k_1} + H_{k_2} + \cdots + H_{k_q}$. We do this by induction on the number of plays in the game so far. It is true in the first move and suppose it is true for (m, x) and A removes H_{k_1} and B removes y where $y \leq \min\{m - H_{k_1}, f(H_{k_1})\} < H_{k_2}$. Now

$$\begin{aligned} m - H_{k_1} - y &= H_{k_2} - y + H_{k_3} + \cdots + H_{k_q} \\ &= H_{\ell_1} + H_{\ell_2} + \cdots + H_{\ell_r} + H_{k_2} + \cdots + H_{k_q} \end{aligned}$$

and we need to argue that $H_{\ell_1} \leq f(y)$. But if $f(y) < H_{\ell_1}$ then we have

$$\begin{aligned} H_{k_2} &= y + H_{\ell_1} + H_{\ell_2} + \cdots + H_{\ell_r} \\ &= H_{a_1} + \cdots + H_{a_s} + H_{\ell_1} + H_{\ell_2} + \cdots + H_{\ell_r} \end{aligned}$$

where $f(H_{a_s}) \leq f(y) < H_{\ell_1}$, contradicting (7). Thus A can remove H_{ℓ_1} in the next round, as required.

(b) Assume that $n = H_k$. After A removes x chips we have

$$H_k - x = H_{j_1} + H_{j_2} + \cdots + H_{j_p}$$

chips left.

All we have to show is that B can now remove H_{j_1} chips i.e. $H_{j_1} \leq f(x)$. But if this is not the case then we argue as above that $H_k = H_{a_1} + \cdots + H_{a_s} + H_{j_1} + H_{j_2} + \cdots + H_{j_p}$, where $x = H_{a_1} + \cdots + H_{a_s}$ and $f(H_{j_1}) \leq f(x) < H_{j_1}$, contradicting (7). □

Geography

Start with a chip sitting on a vertex v of a graph or digraph G.

A move consists of moving the chip to a neighbouring vertex. In edge geography, moving the chip from x to y deletes the edge (x, y). In vertex geography, moving the chip from x to y deletes the vertex x.

The problem is given a position (G, v), to determine whether this is a P or N position.

Complexity Both edge and vertex geography are Pspace-hard on digraphs. Edge geography is Pspace-hard on an undirected graph. Only vertex geography on a graph is polynomial time solvable.

2 Undirected Vertex Geography – UVG

Theorem 6. *(G, v) is an N-position in UVG iff every maximum matching of G covers v.*

Proof (i) Suppose that M is a maximum matching of G which covers v. Player 1's strategy is now: Move along M-edge that contains current vertex.

If Player 1 were to lose, then there would exist a sequence of edges $e_1, f_1, \dots, e_k, f_k$ such that $v \in e_1$, $e_1, e_2, \dots, e_k \in M$, $f_1, f_2, \dots, f_k \notin M$ and $f_k = (x, y)$ where y is the current vertex for Player 1 and y is not covered by M. But then if $A = \{e_1, e_2, \dots, e_k\}$ and $B = \{f_1, f_2, \dots, f_k\}$ then $(M \setminus A) \cup B$ is a maximum matching (same size as M) which does not cover v, contradiction.

(ii) Suppose now that there is some maximum matching M which does not cover v. Then if (v, w) is Player 1's move, w must be covered by M, else M is not a maximum matching. Player 2's strategy is now: Move along M-edge that contains current vertex. If Player 2 were to lose then there exists $e_1 = (v, w), f_1, \dots, e_k, f_k, e_{k+1} = (x, y)$ where y is the current vertex for Player 2 and y is not covered by M. But then we have defined an augmenting path from v to y and so M is not a maximum matching, contradiction. □

Note that we can determine whether or not v is covered by all maximum matchings as follows: Find the size σ of the maximum matching G. This can be done in $O(n^3)$ time on an n-vertex graph. Then find the size σ' of a maximum matching in $G - v$. Then v is covered by all maximum matchings of G iff $\sigma \neq \sigma'$.

3 Undirected Edge Geography – UEG on a bipartite graph

An *even kernel* of G is a non-empty set $S \subseteq V$ such that (i) S is an independent set and (ii) $v \notin S$ implies that $deg_S(v)$ is even, (possibly zero). ($deg_S(v)$ is the number of neighbours of v in S.)

Lemma 3. *If S is an even kernel and $v \in S$ then (G, v) is a P-position in UEG.*

Proof Any move at a vertex in S takes the chip outside S and then Player 2 can immediately put the chip back in S. After a move from $x \in S$ to $y \notin S$, $deg_S(y)$ will become odd and so there is an edge back to S. making this move, makes $deg_S(y)$ even again. Eventually, there will be no $S : \bar{S}$ edges and Player 1 will be stuck in S. □

We now discuss Bipartite UEG i.e. we assume that G is bipartite, G has bipartion consisting of a copy of $[m]$ and a disjoint copy of $[n]$ and edges set E. Now consider the $m \times n$ 0-1 matrix A with $A(i, j) = 1$ iff $(i, j) \in E$.

We can play our game on this matrix: We are either positioned at row i or we are positioned at column j. If say, we are positioned at row i, then we choose a j such that $A(i, j) = 1$ and (i) make $A(i, j) = 0$ and (ii) move the position to column j. An analogous move is taken when we positioned at column j.

Lemma 4. *Suppose the current position is row i. This is a P-position iff row i is in the span of the remaining rows (is the sum (mod 2) of a subset of the other rows) or row i is a zero row. A similar statement can be made if the position is column j.*

Proof If row i is a zero row then vertex i is isolated and this is clearly a P-position. Otherwise, assume the position is row 1 and there exists $I \subseteq [m]$ such that $1 \in I$ and

$$r_1 = \sum_{i \in I \setminus \{1\}} r_i (mod\ 2) \text{ or } \sum_{i \in I} r_i = 0 (mod\ 2) \tag{8}$$

where r_i denotes row i.

I is an even kernel: If $x \notin I$ then either (i) x corresponds to a row and there are no x, I edges or (ii) x corresponds to a column and then $\sum_{i \in I} A(i, x) = 0 (mod\ 2)$ from (8) and then x has an even number of neighbours in I.

Now suppose that (8) does not hold for any I. We show that there exists a ℓ such that $A(1,\ell)=1$ and putting $A(1,\ell)=0$ makes column ℓ dependent on the remaining columns. Then we will be in a P-position, by the first part.

Let e_1 be the m-vector with a 1 in row 1 and a 0 everywhere else. Let A^* be obtained by adding e_1 to A as an $(n+1)$th column. Now the row-rank of A^* is the same as the row-rank of A (here we are doing all arithmetic modulo 2). Suppose not, then if r_i^* is the ith row of A^* then there exists a set J such that

$$\sum_{i\in J} r_i = 0(mod\ 2) \neq \sum_{i\in J} r_i^*(mod\ 2).$$

Now $1 \notin J$ because r_1 is independent of the remaining rows of A, but then $\sum_{i\in J} r_i = 0(mod\ 2)$ implies $\sum_{i\in J} r_i^* = 0(mod\ 2)$ since the last column has all zeros, except in row 1.

Thus rank A^* = rank A and so there exists $K \subseteq [n]$ such that

$$e_1 = \sum_{k\in K} c_k(mod\ 2) \text{ or } e_1 + \sum_{k\in K} c_k = 0(mod\ 2) \tag{9}$$

where c_k denotes column k of A. Thus there exists $\ell \in K$ such that $A(1,\ell)=1$. Now let $c_j' = c_j$ for $j \neq \ell$ and c_ℓ' be obtained from c_ℓ by putting $A(1,\ell)=0$ i.e. $c_\ell' = c_\ell + e_1$. But then (9) implies that $\sum_{k\in K} c_k' = 0(mod\ 2)$ ($K=\{k\}$ is a possibility here).. □

Tic Tac Toe and extensions

We consider the following multi-dimensional version of Tic Tac Toe (Noughts and Crosses to the English). The *board* consists of $[n]^d$. A point on the board is therefore a vector $(x_1,x_2,\ldots,x_d)$ where $1 \leq x_i \leq n$ for $1 \leq i \leq d$.

A *line* is a set points $(x_j^{(1)},x_j^{(2)},\ldots,x_j^{(d)})$, $j=1,2,\ldots,n$ where each sequence $x^{(i)}$ is either (i) of the form $k,k,\ldots,k$ for some $k\in[n]$ or is (ii) $1,2,\ldots,n$ or is (iii) $n,n-1,\ldots,1$. Finally, we cannot have Case (i) for all i.

Thus in the (familiar) 3×3 case, the top row is defined by $x^{(1)}=1,1,1$ and $x^{(2)}=1,2,3$ and the diagonal from the bottom left to the top right is defined by $x^{(1)}=3,2,1$ and $x^{(2)}=1,2,3$

Lemma 5. *The number of winning lines in the (n,d) game is $\frac{(n+2)^d-n^d}{2}$.*

Proof In the definition of a line there are n choices for k in (i) and then (ii), (iii) make it up to $n+2$. There are d independent choices for each i making $(n+2)^d$. Now delete n^d choices where only Case (i) is used. Then divide by 2 because replacing (ii) by (iii) and vice-versa whenever Case (i) does not hold produces the same set of points (traversing the line in the other direction). □

The game is played by 2 players. The Red player (X player) goes first and colours a point red. Then the Blue player (0 player) colours a different point blue and so on. A player wins if there is a line, all of whose points are that players colour. If neither player wins then the game is a draw. The second player does not have a wnning strategy:

Lemma 6. *Player 1 can always get at least a draw.*

Proof We prove this by considering *strategy stealing.* Suppose that Player 2 did have a winning strategy. Then Player 1 can make an arbitrary first move x_1. Player 2 will then move with y_1. Player 1 will now win playing the winning strategy for Player 2 against a first move of y_1. This can be carried out until the strategy calls for move x_1 (if at all). But then Player 1 can make an arbitrary move and continue, since x_1 has already been made. □

3.1 Pairing Strategy

$$\begin{bmatrix} 11 & 1 & 8 & 1 & 12 \\ 6 & 2 & 2 & 9 & 10 \\ 3 & 7 & * & 9 & 3 \\ 6 & 7 & 4 & 4 & 10 \\ 12 & 5 & 8 & 5 & 11 \end{bmatrix}$$

The above array gives a strategy for Player 2 the 5×5 game ($d = 2, n = 5$). For each of the 12 lines there is an associated pair of positions. If Player 1 chooses a position with a number i, then Player 2 responds by choosing the other cell with the number i. This ensures that Player 1 cannot take line i. If Player 1 chooses the * then Player 2 can choose any cell with an unused number. So, later in the game if Player 1 chooses a cell with j and Player 2 already has the other j, then Player 1 can choose an arbitrary cell. Player 2's strategy is to ensure that after all cells have been chosen, he/she will have chosen one of the numbered cells asociatded with each line. This prevents Player 1 from taking a whole line. This is called a *pairing* strategy.

We now generalise the game to the following: We have a family $\mathcal{F} = A_1, A_2, \ldots, A_N \subseteq A$. A move consists of one player, taking an uncoloured member of A and giving it his colour. A player wins if one of the sets A_i is completely coloured with his colour.

A pairing strategy is a collection of distinct elements $X = \{x_1, x_2, \ldots, x_{2N-1}, x_{2N}\}$ such that $x_{2i-1}, x_{2i} \in A_i$ for $i \geq 1$. This is called a *draw forcing pairing.* Player 2 responds to Player 1's choice of $x_{2i+\delta}, \delta = 0, 1$ by choosing $x_{2i+3-\delta}$. If Player 1 does not choose from X, then Player 2 can choose any uncoloured element of X. In this way, Player 2 avoids defeat, because at the end of the game Player 2 will have coloured at least one of each of the pairs x_{2i-1}, x_{2i} and so Player 1 cannot have completely coloured A_i for $i = 1, 2, \ldots, N$.

Theorem 7. *If*

$$\left| \bigcup_{A \in \mathcal{G}} A \right| \geq 2|\mathcal{G}| \qquad \forall \mathcal{G} \subseteq \mathcal{F} \tag{10}$$

then there is a draw forcing pairing.

Proof We define a bipartite graph Γ. A will be one side of the bipartition and $B = \{b_1, b_2, \ldots, b_{2N}\}$. Here b_{2i-1} and b_{2i} both represent A_i in the sense that if $a \in A_i$ then there is an edge (a, b_{2i-1}) and an edge (a, b_{2i}). A draw forcing pairing corresponds to a complete matching of B into A and the condition (10) implies that Hall's condition is satisfied. □

Corollary 8. *If $|A_i| \geq n$ for $i = 1, 2, \ldots, n$ and every $x \in A$ is contained in at most $n/2$ sets of $\mathcal{F}$ then there is a draw forcing pairing.*

Proof The degree of $a \in A$ is at most $2(n/2)$ in Γ and the degree of each $b \in B$ is at least n. This implies (via Hall's condition) that there is a complete matching of B into A. □

Consider Tic tac Toe when case $d = 2$. If n is even then every array element is in at most 3 lines (one row, one column and at most one diagonal) and if n is odd then every array element is in at most 4 lines (one row, one column and at most two diagonals). Thus there is a draw forcing pairing if $n \geq 6$, n even and if $n \geq 9$, n odd. (The cases $n = 4, 7$ have been settled as draws. $n = 7$ required the use of a computer to examine all possible strategies.

In general we have

Lemma 7. *If $n \geq 3^d - 1$ and n is odd or if $n \geq 2^d - 1$ and n is even, then there is a draw forcing pairing of (n, d) Tic tac Toe.*

Proof We only have to estimate the number of lines through a fixed point $\mathbf{c} = (c_1, c_2, \ldots, c_d)$. If n is odd then to choose a line L through $\mathbf{c}$ we specify, for each index i whether L is (i) constant on i, (ii) increasing on i or (iii) decreasing on i. This gives 3^d choices. Subtract 1 to avoid the all constant case and divide by 2 because each line gets counted twice this way.

When n is even, we observe that once we have chosen in which positions L is constant, L is determined. Suppose $c_1 = x$ and 1 is not a fixed position. Then every other non-fixed position is x or $n-x+1$. Asuning w.l.o.g. that $x \leq n/2$ we see that $x < n-x = 1$ and the positions with x increase together at the same time as the positions with $n-x+1$ decrease together. Thus the number of lines through $\mathbf{c}$ in this case is bounded by $\sum_{i=0}^{d-1} \binom{d}{i} = 2^d - 1$. □

3.2 Quasi-probabilistic method

We now prove a theorem of Erdős and Selfridge.

Theorem 9. *If $|A_i| \geq n$ for $i \in [N]$ and $N < 2^{n-1}$, then Player 2 can get a draw in the game defined by $\mathcal{F}$.*

Proof At any point in the game, let C_j denote the set of elements in A which have been coloured with Player j's colour, $j = 1, 2$ and $U = A \setminus C_1 \cup C_2$. Let

$$\Phi = \sum_{i: A_i \cap C_2 = \emptyset} 2^{-|A_i \cap U|}.$$

Suppose that the players choices are $x_1, y_1, x_2, y_2, \ldots,$. Then we observe that immediately after Player 1's first move, $\Phi < N 2^{-(n-1)} < 1$.

We will show that Player 2 can keep $\Phi < 1$ through out. Then at the end, when $U = \emptyset$, $\Phi = \sum_{i: A_i \cap C_2 = \emptyset} 1 < 1$ implies that $A_i \cap C_2 \neq \emptyset$ for all $i \in [N]$.

So, now let Φ_j be the value of Φ after the choice of $x_1, y_1, \ldots, x_j$. then if U, C_1, C_2 are defined at precisely this time,

$$\begin{aligned}
\Phi_{j+1} - \Phi_j &= -\sum_{\substack{i: A_i \cap C_2 = \emptyset \\ y_j \in A_i}} 2^{-|A_i \cap U|} + \sum_{\substack{i: A_i \cap C_2 = \emptyset \\ y_j \notin A_i, x_{j+1} \in A_i}} 2^{-|A_i \cap U|} \\
&\leq -\sum_{\substack{i: A_i \cap C_2 = \emptyset \\ y_j \in A_i}} 2^{-|A_i \cap U|} + \sum_{\substack{i: A_i \cap C_2 = \emptyset \\ x_{j+1} \in A_i}} 2^{-|A_i \cap U|}
\end{aligned}$$

We deduce that $\Phi_{j+1} - \Phi_j \leq 0$ if Player 2 chooses y_j to maximise over y, $\sum_{\substack{i: A_i \cap C_2 = \emptyset \\ y \in A_i}} 2^{-|A_i \cap U|}$.

In this way, Player 2 keeps $\Phi < 1$ and obtains a draw. □

In the case of (n, d) Tic Tac Toe, we see that Player 2 can force a draw if (see Lemma 5)

$$\frac{(n+2)^d - n^d}{2} < 2^{n-1}$$

which is implied, for n large, by

$$n \geq (1 + \epsilon) d \log_2 d$$

where $\epsilon > 0$ is a small positive constsnt.

Shannon Switching Game Start with a connected multi-graph $G = (V, E)$.
Two players: Player A goes first and deletes edges and player B fortifies edges making them invulnerable to deletion by B. Player B wins iff the fortified edges contain a spanning tree of G.

Theorem 10. *Player B wins iff G contains two edge disjoint spanning trees.*

Proof (a) Here we assume that G has two edge disjoint spanning trees T_1, T_2 We prove this by induction on $|V|$. If $|V| = 2$ then G must contain at least two parallel edges joining the two vertices and so B can win. Suppose next that $|V| > 2$. Suppose that A deletes an edge $e = (x, y)$ of T_2 red. This breaks T_2 into two sub-trees T_2', T_2''. B will choose an edge $f = (u, v) \in T_1$ with one end in $V(T_2')$ and the other end in $V(T_2'')$. Now contract the edge f. In the new graph G^*, both T_1 and T_2 become spanning trees T_1^* and T_2^* and they are edge disjoint. It follows by induction that B can win the game on G^* and then wins the game on G by uncontracting the edge f. Of course f is chosen first of all still!

If A chooses an edge x in neither of the trees then B can choose an arbitrary edge f of T_1. Now let e be any edge of the unique cycle contained in $T_2 + e$. B can continue playing on $G - x$ as though e was the deleted edge. We can contract f as before and apply the above inductive argument.

(b) For this part we use a Theorem due to Nash-Williams:

Theorem 11. *Let k be a positive integer. Then G contains k edge disjoint spanning trees iff for every partition $\mathcal{P} = (V_1, V_2, \dots, V_\ell)$ of V we have*

$$e(\mathcal{P}) = |E(\mathcal{P})| = \sum_{1 \le i < j \le \ell} e(V_i, V_j) \ge k(\ell - 1). \tag{11}$$

Here $E(\mathcal{P})$ is the set of edges joining different parts of the partition and $e(V_i, V_j)$ is the number of edges joining V_i and V_j.

Let us apply Theorem 11 with $k = 2$. If G does not contain two edge disjoint spanning trees, then it contains a partition $\mathcal{P} = (V_1, V_2, \dots, V_\ell)$ with $e(\mathcal{P}) \le 2\ell - 3$. A starts by deleting an edge $e \in E(\mathcal{P})$. B will fortify an edge $f = (u, v)$. If u, v join different sets in the partition $\mathcal{P}$ then we can merge them and consider $\mathcal{P}'$ which has one less part and satisfies $e(\mathcal{P}') \le e(\mathcal{P}) - 2$ (edegs e, f have gone from the count). Otherwise B chooses an edge entirely inside a part of $\mathcal{P}$ and the number of parts does not change, but $e(\mathcal{P})$ goes down by one. Eventually, we come to a point where one part is joined to the rest of the graph by a single edge ($2\ell - 3 = 1$ when $\ell = 2$) and A wins by deleting this edge. □

Sketch of proof of Theorem 11

If $\mathcal{P} = (V_1, V_2, \dots, V_\ell)$ is a partition and T is a spanning tree then T contains at least $\ell - 1$ edges of $E(\mathcal{P})$ and the only if part is straightforward.

Suppose now that (11) holds for all partitions. Let $\mathcal{F}$ be the set of edge disjoint forests containing the maximum number of edges. If $F = (F_1, F_2, \dots, F_k) \in \mathcal{F}$ and $e \in E \setminus E[\mathcal{F}]$ then every $F_i + e$ contains a cycle. If e' belongs to this cycle then $F' \in \mathcal{F}$ where $F_j' = F_j$ for $j \ne i$ and $F_i' = F_i + e' - e$. We say that F' is obtained from F by a *replacement*.

Consider now a fixed $F^0 = (F_1^0, F_2^0, \dots, F_k^0) \in \mathcal{F}$ and let $\mathcal{F}^0$ be the set of k-tples in $\mathcal{F}$ that can be obtained from F^0 by a sequence of replacements. Then let

$$E^0 = \bigcup_{F \in \mathcal{F}^0} (E \setminus E([F]).$$

Claim 1. *For every $e^0 \in E \setminus E([F^0]$ there exists a set $U \subseteq V$ that contains the endpoints of e^0 and induces a connected tree in F_i^0 for $1 \le i \le k$.*

Assume the claim for the moment. Suppose that not every F_i^0 is a spanning tree. Then G contains at least $k(|V|-1)$ edges (from (11) applied to the parttion of V into singletons) and so there exists $e^0 \in E \setminus E[F^0]$. Shrink the vertices of the set U in the claim to a single vertex v_U to obtain a graph G'. Apply induction to G' to get a set of k disjoint spanning trees $T_1', T_2', \ldots, T_k'$ of G'. Now expand v_U back to U. Each T_i' expands to a spanning tree of G. In this way we get k edge-disjoint spanning trees of G.

Proof of Claim 1

Let $G^0 = (V, E^0)$ and let C_0 be the component of G^0 that contains e^0. Let $U = V(C^0)$. First verify that if $F = (F_1, F_2, \ldots, F_k) \in \mathcal{F}^0$ and F' is obtained from F by a replacement and x, y are the ends of a path in $F_i' \cap U$ then x, y are joined by a path $xF_iy \subseteq U$. (Exercise).

We now show that $F_i^0 \cap U$ is connected. Let (x, y) be an edge of C^0. Since C^0 is connected, we only have to show that F_i^0 contains a path from x to y, all of whose vertices belong to U. But this follows by using the exercise and backwards induction starting from some $F \in \mathcal{F}^0$ for which F_i contains the edge (x, y). $\square$