simplified proof for modified one-pile game
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Claim. $(2^k, *) \in P$ for all $k \geq 1$.

Proof. We prove the stronger claim that $(n, x) \in P$ when $n = 2^k m$, with $2 \nmid m$, and $0 < x < 2^k$, by induction on n. Base case $n = 2$ is easy.

In general, if player A takes y,

$$1.\ A : (n, x)$$
$$2.\ B : (n - y, y)$$

then

Case 1: $2^{k-1} \leq y \leq x$. Then player B takes $2^k - y$, leaving

$$3.\ A : (n - 2^k, 2^k - y)$$

so the result follows by induction on n, since $n - 2^k = 2^{k+j}(\frac{m-1}{2^j})$ for some $j \geq 1$, and $2^k - y < 2^k < 2^{k+j}$.

Case 2: $0 < y < 2^{k-1}$. Then there is some $0 < \ell < k$ such that $2^{\ell-1} \leq y < 2^\ell$. So player B may take $2^\ell - y$, leaving

$$3.\ A : (n - 2^\ell, 2^\ell - y)$$

and again the result follows by induction on n, since $n - 2^\ell = 2^\ell(2^{k-\ell}m - 1)$, and $2^\ell - y < 2^\ell$. □