Solving the Take-Away Game

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Claim: Let k be the index of the least significant “1”-bit in the binary representation of n, or 0 if there are none. Then $(n, 2^k - 1)$ is a P-position.

Proof: (by induction on n)

- Base Case: $n = 0$

  There are no ”1”s in the binary representation of n, so $k = 0$. It is in fact true that $(n, 2^k - 1) = (0, 0)$ is a P-position, since the next player can not move.

- Inductive Hypothesis: $(\forall l, 0 \leq l < m, \text{where } m \in \mathbb{Z}^+) \quad l$ satisfies the above claim.

- Inductive Step:

  Consider the position $(m, 2^j - 1)$, where j is the index of the least significant “1”-bit in the binary representation of m. Suppose the next player (A) makes the move of taking x chips, where $2^{i-1} \leq x < 2^i$. Then we know that $2^i - x \leq 2^i - 2^{i-1} = 2^{i-1}$. Since player A took $x \geq 2^{i-1}$ chips, player B is permitted to take $2^i - x$ chips, leaving the position $(m' = m - 2^i, 2^i - x)$.

  Note that since $i \leq j$, and j is the least significant “1”-bit of m, all bits of m' to the right of index i must be “0”s. Thus if we let j' be the least significant “1”-bit of m', $j' \geq i$.

  By inductive hypothesis, $(m', 2^{j'} - 1)$ is a P-position, and since $i \leq j'$ and $x > 0$, $(m', 2^i - x)$ is clearly a P-position as well. Thus $(m, 2^j - 1)$ is a P-position. QED