Midterm Exam
Mathematical Games, Spring 2005
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Due Tuesday March 22, 2005, in class

Do four of the following five problems. Please work alone. Please do not consult any material outside of that supplied in the class. Please feel free to contact the instructors for clarification.

Problem 1: Game Chromatic Numbers

The random bipartite graph $G = G_{n,n,1/2}$ is constructed by randomly deleting each edge of $K_{n,n}$ with probability $1/2$. Let V_1, V_2 denote the bipartition. It has the following property:

- Every set $S \subseteq V_1$ of size $k \leq \alpha \log_2 n$ has $(1 \pm \epsilon)n/2^k$ non-neigbours in V_2 (and vice-versa). Here α is any contant less than 1 and ϵ is any positive constant.
- Every set $S \subseteq V_1$ of size $2\log_2 n$ has no non-neighbours in V_2, and vice-versa.

This all happens with probability close to one, if n is large.

Show that the game chromatic $\gamma_g(G)$ number of G satisfies

$$n^{1-\beta} \leq \gamma_g(G) \leq C\frac{n}{\log n}$$

for any $\beta > 0$ and for a suitably large constant $C > 0$.

(Fix β and then let α, ϵ be sufficiiently close to 1,0 repsectively.)

Problem 2: Erdős and Selfridge Revisited

Let $\mathcal{A} = (A_1, A_2, \ldots, A_N\}$ be a collection of sets. There are two players, P1 and P2. P1 goes first and colors p_1 elements of $A = \bigcup_{i=1}^{N} A_i$. Then P2 colors p_2 more elements of A and then P1 colors p_1 elements and so on. Each element is coloured once. The last move may involve coloring fewer than p_1

or p_2 colors. P1 wins if at the end of the game, after all elements of A have been colored, there is a set A_i, all of whose elements are colored with P1's color. Otherwise P2 wins.

Show that P2 can win if

$$\sum_{i=1}^{N}(1+p_2)^{-|A_i|/p_1} < \frac{1}{1+p_2}.$$

Hint: Let X_i, Y_i denote the set of elements that have been colored by players P1,P2 repsctively after P1 has made i moves and P2 has made $i-1$ moves. Let $\mu = (1+p_2)^{1/p_1} - 1$ and define the potential function

$$\Psi(\mathcal{A}_i) = \sum_{A \in \mathcal{A}_i} \frac{1}{(1+\mu)^{|A|}}$$

where $\mathcal{A}_i = \{A_j \setminus X_i : \ A_j \cap Y_i = \emptyset\}$. Now use a suitable modification of the argument of Erdős and Selfridge.

Use the argument to prove the following result: Two playes Breaker and Maker, with Breaker going first play a game on the complete graph K_n. A move of Breaker involves the deletion of b edges and a move of maker involves the fortification of one undeleted edge, so that Breaker cannot delete it in subsequent rounds. Maker wins if at the end of the game, the set of fortified edges forms a *connected* spanning subgraph of K_n, otherwise Breaker wins.

Show that if ϵ is a positive constant and n is sufficiently large and if $b < n^{\frac{\ln 2 - \epsilon}{\ln n}}$ then Maker can win this game.

Problem 3: A Decomposition Theorem

Here's an excerpt from Ferguson's book:

> In this class of games, called Coin Turning Games, we are given a finite number of coins in a row, each showing either heads or tails. A move consists of turning over, from heads to tails or from tails to heads, all coins in a set of coins allowed by the rules of the game. The rules always specify that the rightmost coin turned over must go from heads to tails. The purpose of this is to guarantee that the game will end in a finite number of moves no matter how it is played (the Ending Condition).

> The rules also specify that the set of coins that may be turned over depends only on the position of the rightmost coin turned over and not otherwise on the position of the other heads and tails, or on the previous moves of the game, or on the time, etc. Moreover, the games are impartial and the last player to move wins.
>
> Under these conditions, the same decomposition that worked for Turning Turtles also works for all these games, namely: A position with k heads in positions $x_1, \ldots, x_k$ is the (disjunctive) sum of k games each with exactly one head, where for $j = 1, \ldots, k$ the head in game j is at x_j. For example, the game THHTTH, is the sum of three games TH, TTH, and TTTTTH. This implies that to find the Sprague-Grundy value of a position, we only need to know the Sptrgue-Grundy values of positions with exactly one head. In the example, $g(\text{THHTTH}) = g(\text{TH}) \oplus g(\text{TTH}) \oplus g(\text{TTTTTH} \ldots$

Prove that the decomposition property described in the last paragraph is correct.

Problem 4: Forbidden Numbers

Consider the following two-player impartial misère game. Players alternate picking non-negative numbers. A number is forbidden if it can be expressed as a sum (with repetitions) of previous numbers. The last player (the one who exhausts all the numbers) loses. For example if the first player picks 3, the the second player cannot pick 3, 6, 9, etc. If the second player picks 2, then the only number left for the first player is 1, which he takes and loses.

(a) Prove that the game must terminate.

(b) It's easy to see that if the first player plays 1, 2, or 3, she loses. Evaluate the natural outcome of the game if the first player plays 4. (Who wins, and what's the strategy?)

Problem 5: All Red-Blue Hackenbush Positions are Numbers

Recall that a game $G = \{G^L | G^R\}$ is a number if all of the left options are numbers and all of the right options are numbers, and all of the left options are less than all of the right options. Recall that in this case the value of the game is the "simplest" number strictly between the left numbers and the right numbers.

Prove that all Red-Blue Hackenbush positions are numbers.