DiffEQ 1

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DiffEQ Integration

Differential Equation Basics

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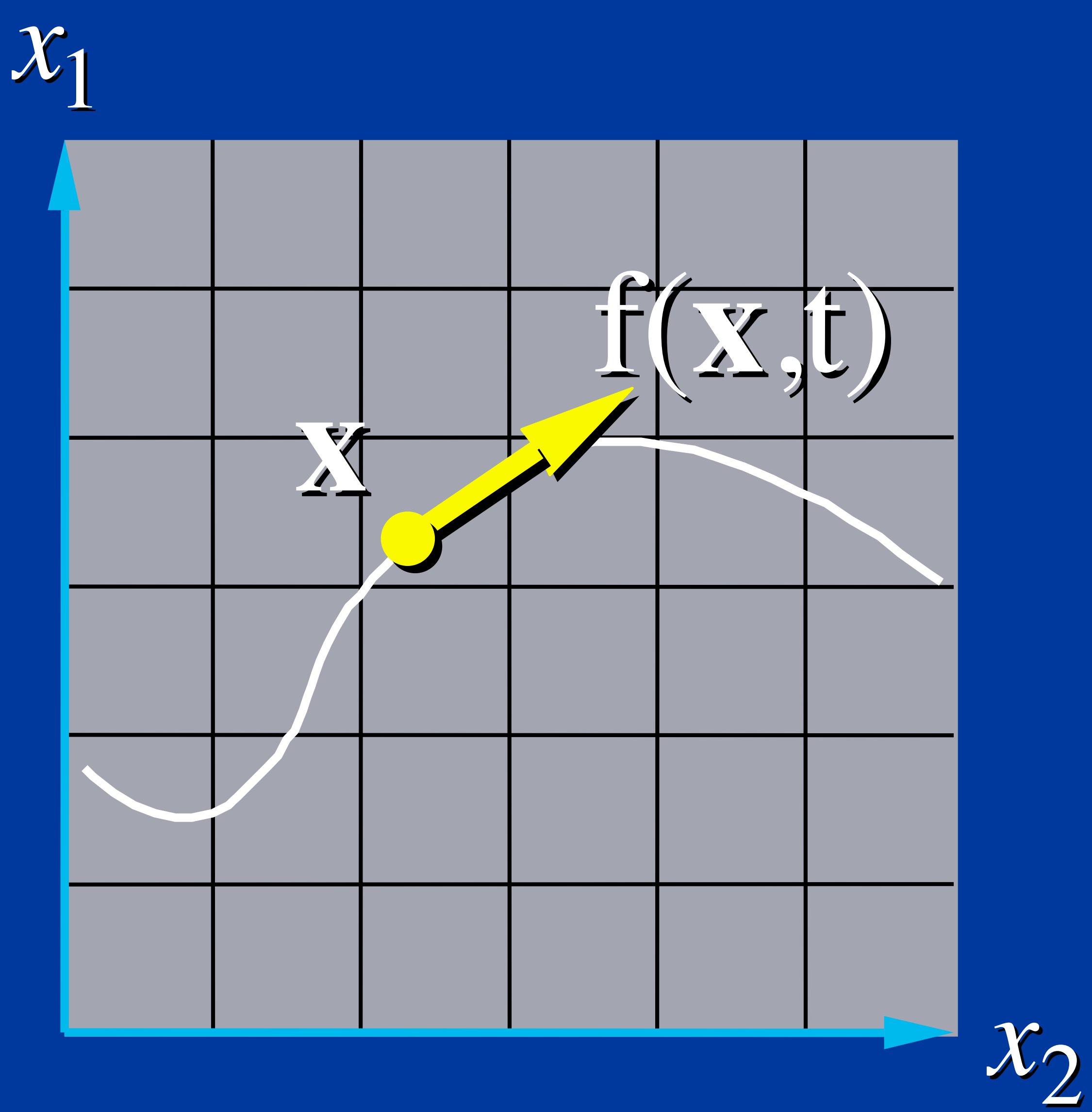
PIXAR
ANIMATION STUDIOS

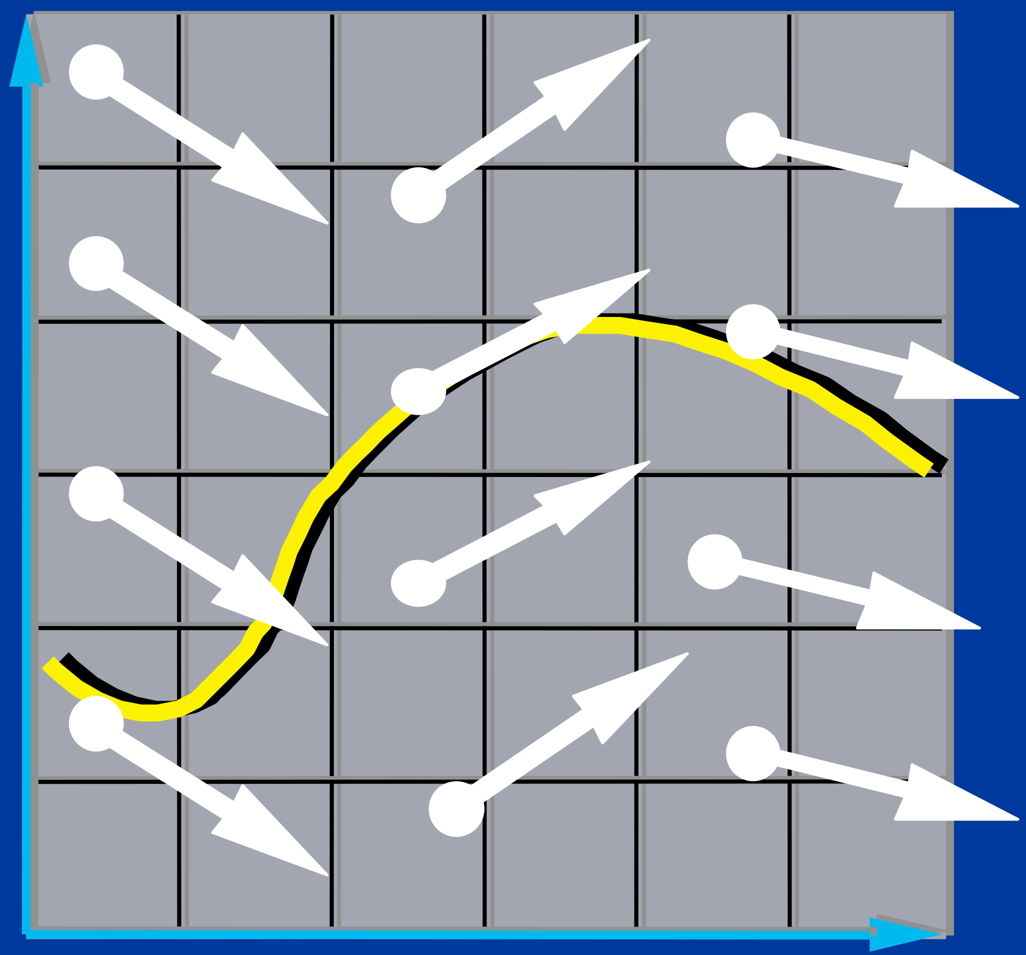
A Canonical Differential Equation



$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

- $\mathbf{x}(t)$: a moving point.
- $\mathbf{f}(\mathbf{x},t)$: $\mathbf{x}$'s velocity.

Vector Field



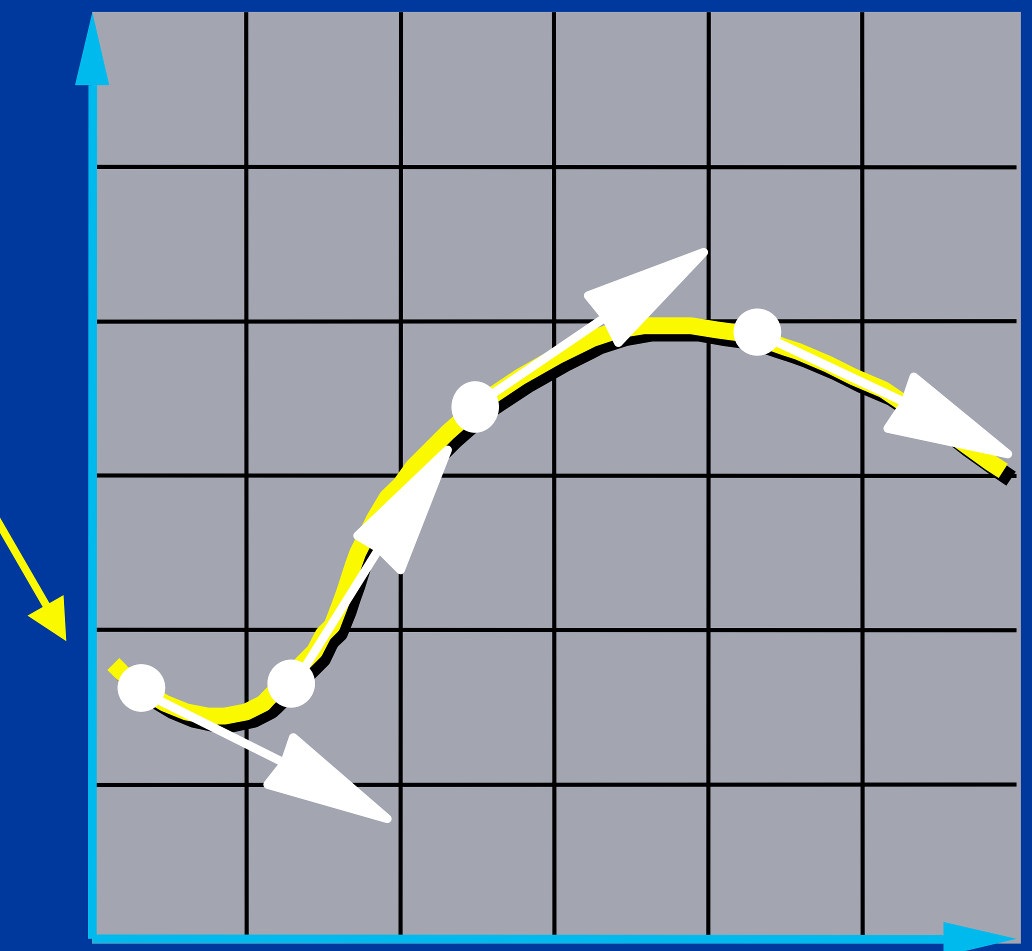
The differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

defines a vector field over **x**.

Integral Curves

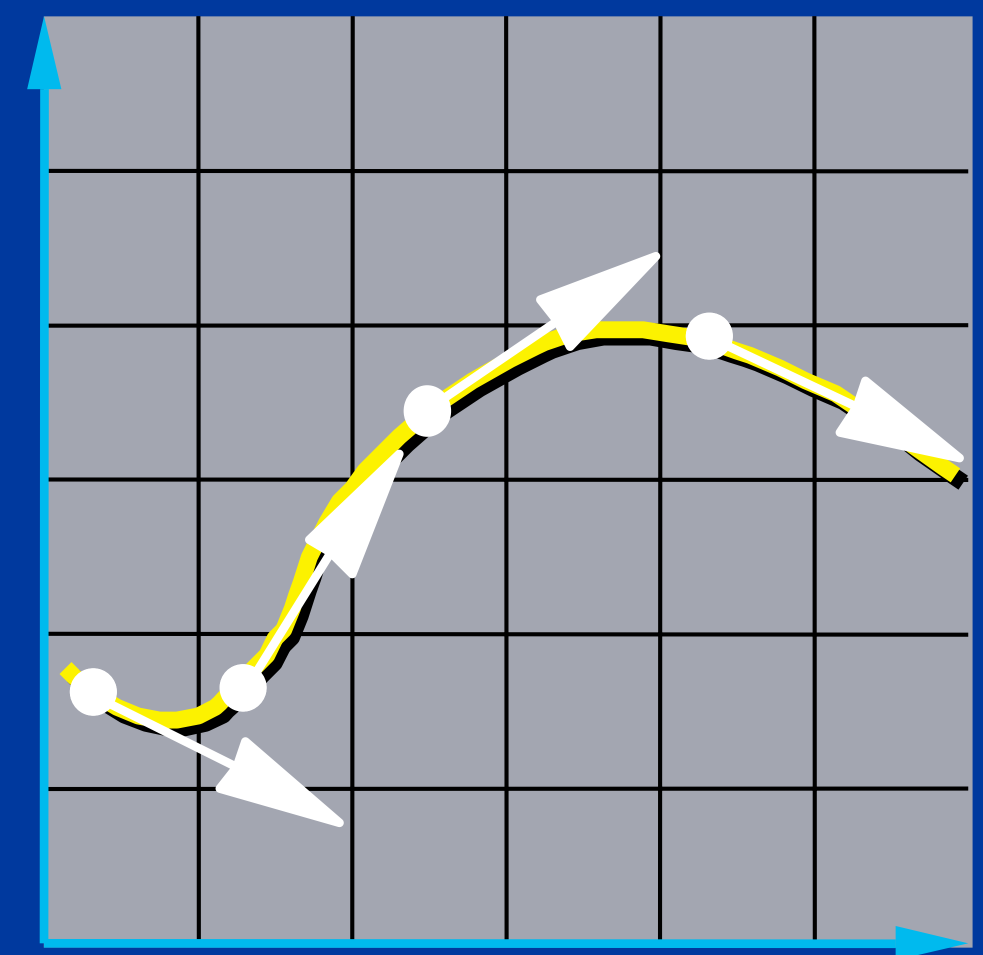
Start Here



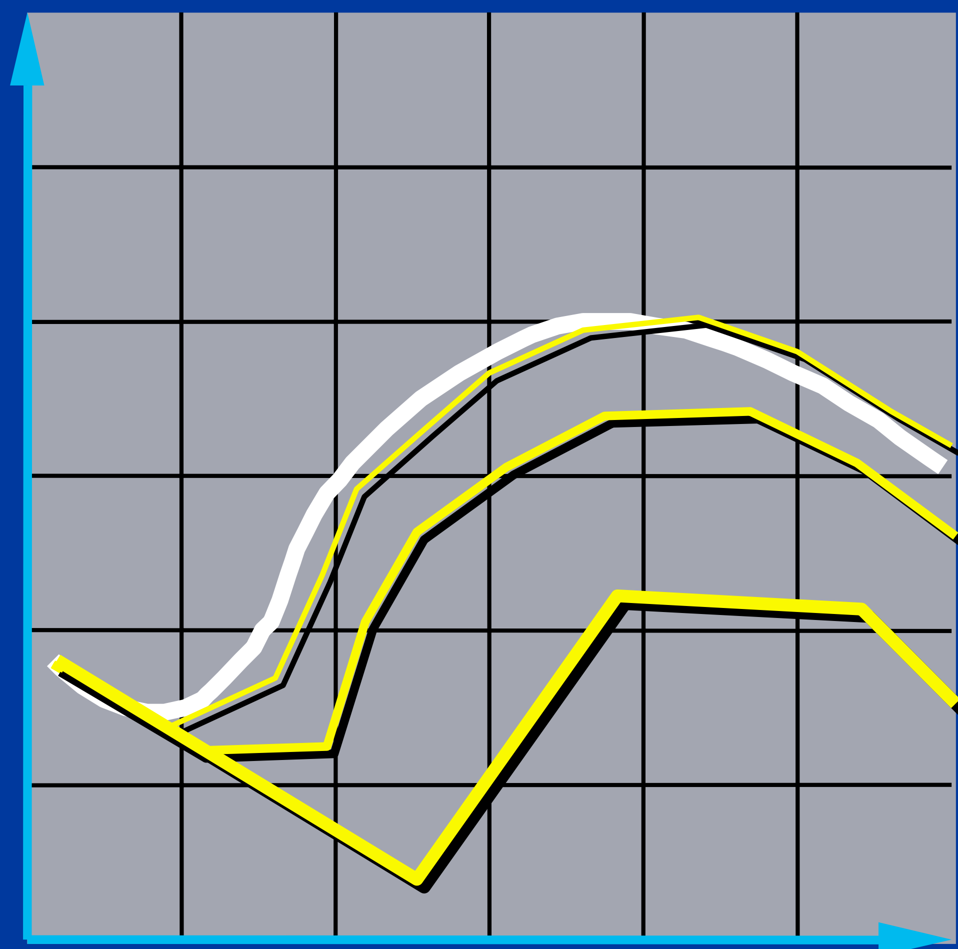
Pick any starting point, and follow the vectors.

Initial Value Problems

Given the starting point, follow the integral curve.



Euler's Method



- Simplest numerical solution method
- Discrete time steps
- Bigger steps, bigger errors.

$$\mathbf{x}(t + \Delta t) = \mathbf{x}(t) + \Delta t\, \mathbf{f}(\mathbf{x}, t)$$

Two Problems

- Accuracy
- Instability

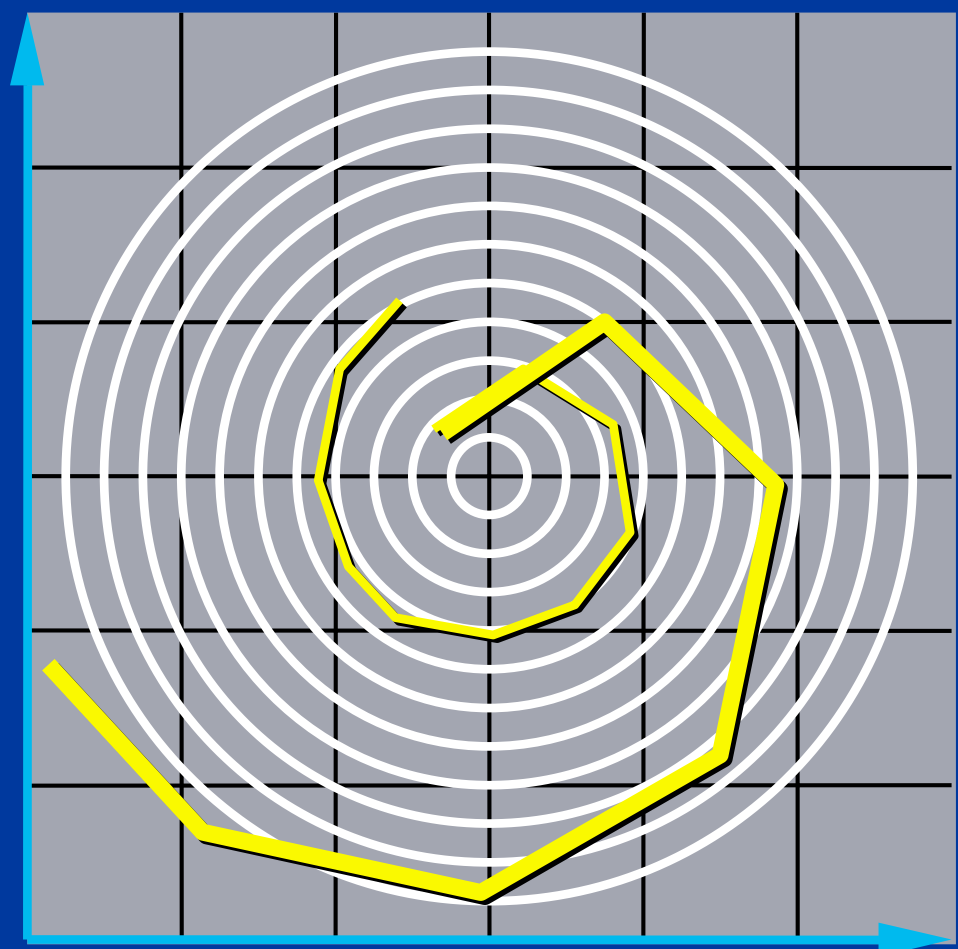
Accuracy

Consider the equation:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}$$

What do the integral curves look like?

Problem I: Inaccuracy



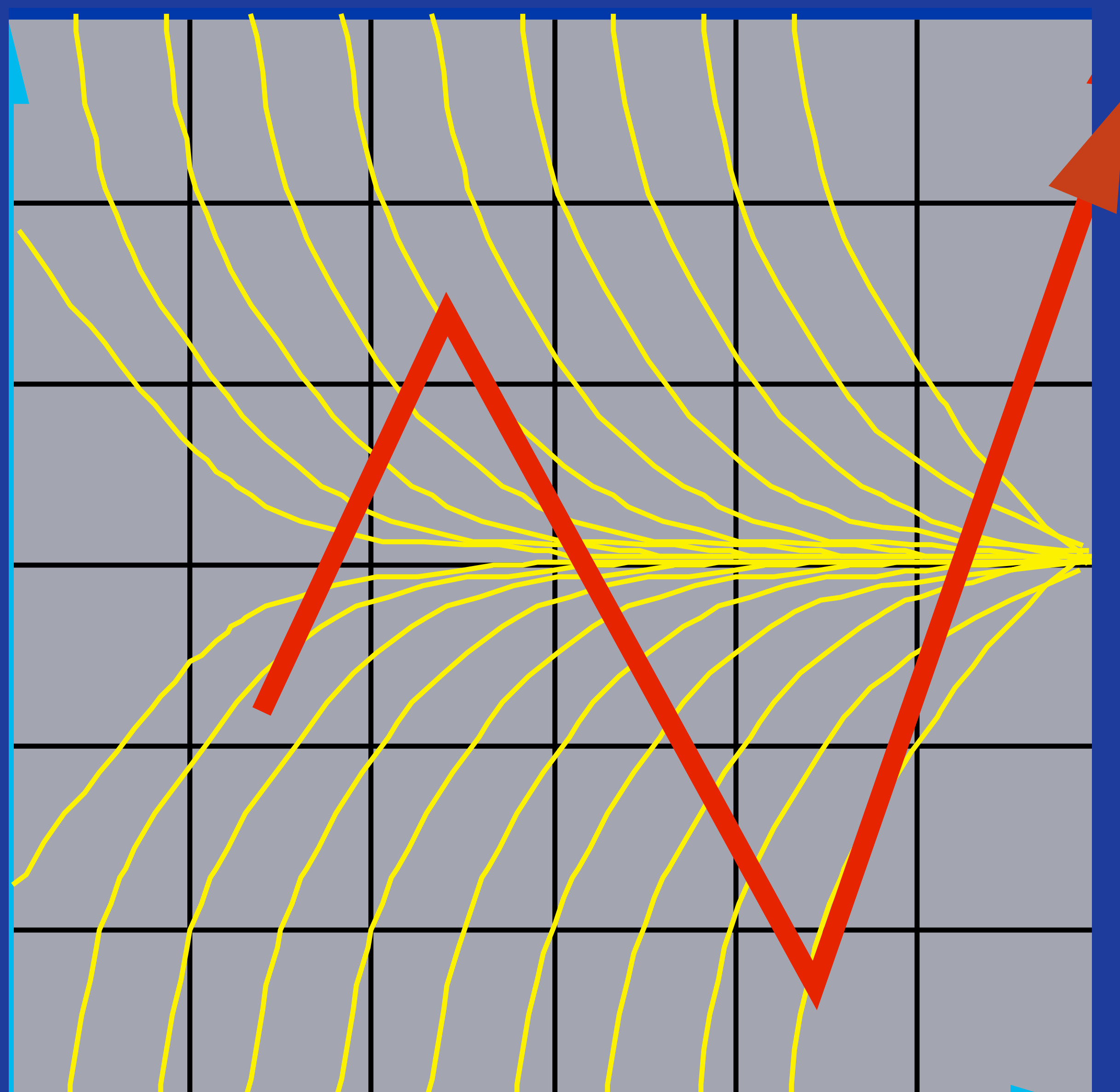
Error turns x(t) from a circle into the spiral of your choice.

Problem 2: Instability

- **Consider the following system:**

$$\begin{cases} \dot{x} = -x \\ x(0) = 1 \end{cases}$$

Problem 2: Instsability



To Neptune!

Accuracy of Euler Method

$$\dot{x} = f(x)$$

Accuracy of Euler Method

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Consider Taylor Expansion about x(t)...

Accuracy of Euler Method

$$\dot{x} = f(x)$$

Consider Taylor Expansion about x(t)...

$$x(t + h) =$$

Accuracy of Euler Method

$$\dot{x} = f(x)$$

Consider Taylor Expansion about x(t)...

$$x(t+h) = \underline{\qquad\qquad}$$

constant

Accuracy of Euler Method

$$\dot{x} = f(x)$$

Consider Taylor Expansion about x(t)...

$$x(t+h) = \underbrace{x(t)}_{\text{constant}} + \dots$$

Accuracy of Euler Method

$$\dot{x} = f(x)$$

Consider Taylor Expansion about x(t)...

$$x(t+h) = \underbrace{x(t)}_{\text{constant}} + \underbrace{\ldots}_{\text{linear}}$$

Accuracy of Euler Method

$$\dot{x} = f(x)$$

Consider Taylor Expansion about x(t)...

$$x(t+h) = \underbrace{x(t)}_{\text{constant}} + \underbrace{hf(x(t))}_{\text{linear}} + \ldots$$

Accuracy of Euler Method

$$\dot{x} = f(x)$$

Consider Taylor Expansion about x(t)...

$$x(t+h) = \underbrace{x(t)}_{\text{constant}} + \underbrace{hf(x(t))}_{\text{linear}} + \underbrace{\ldots}_{\text{everything else}}$$

Accuracy of Euler Method

$$\dot{x} = f(x)$$

Consider Taylor Expansion about x(t)...

$$x(t+h) = \underbrace{x(t)}_{\text{constant}} + \underbrace{hf(x(t))}_{\text{linear}} + \underbrace{O(h^2)}_{\text{everything else}}$$

Accuracy of Euler Method

$$\dot{x} = f(x)$$

Consider Taylor Expansion about x(t)...

Euler step

$$x(t+h) = \underbrace{x(t)}_{\text{constant}} + \underbrace{hf(x(t))}_{\text{linear}} + \underbrace{O(h^2)}_{\text{everything else}}$$

Accuracy of Euler Method

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Therefore, Euler's method has error $O(h^2)$... it is *first order.*

Accuracy of Euler Method

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Therefore, Euler's method has error $O(h^2)$... it is *first order.*

How can we get to $O(h^3)$ error?

The Midpoint Method

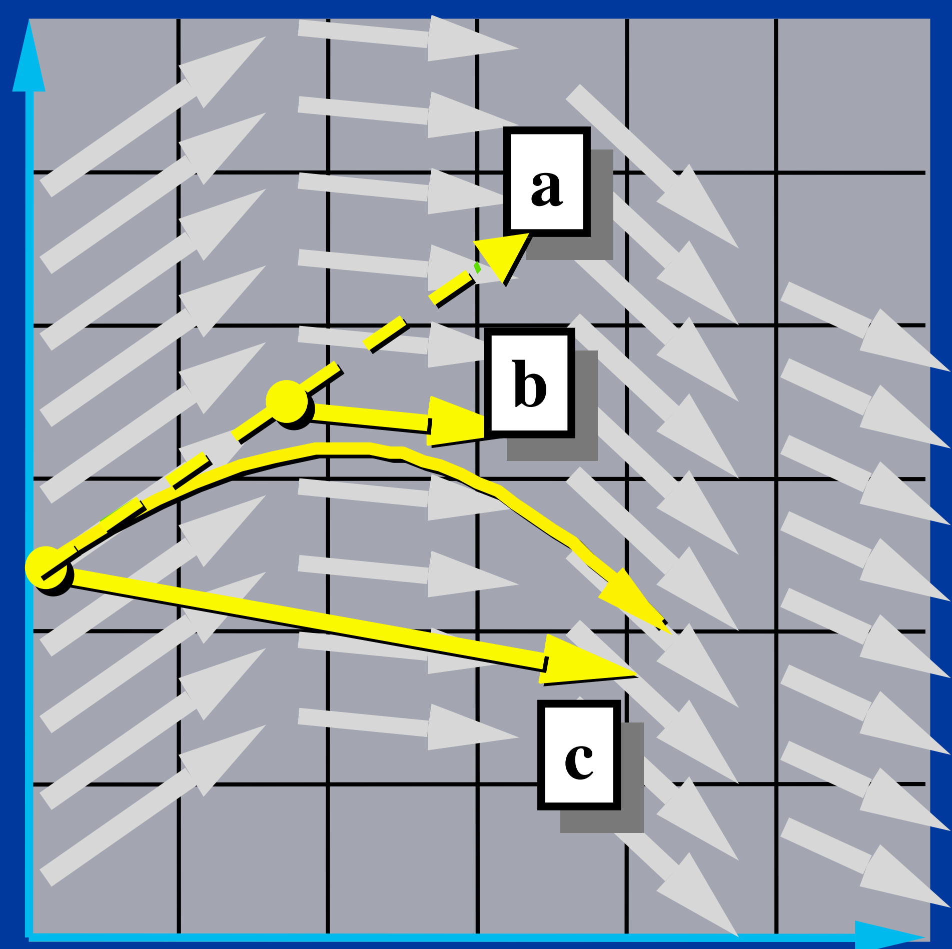
- **Also known as second order Runge-Kutte:**

$$k_1 = h(f(x_0, t_0)$$

$$k_2 = hf(x_0 + \frac{k_1}{2}, t_0 + \frac{h}{2})$$

$$x(t_0 + h) = x_0 + k_2 + O(h^3)$$

The Midpoint Method



a. Compute an Euler step

$$\Delta \mathbf{x} = \Delta t\, \mathbf{f}(\mathbf{x}, t)$$

b. Evaluate f at the midpoint

$$\mathbf{f}_{mid} = \mathbf{f}\left(\frac{\mathbf{x} + \Delta \mathbf{x}}{2}, \frac{t + \Delta t}{2}\right)$$

c. Take a step using the midpoint value

$$\mathbf{x}(t + \Delta t) = \mathbf{x}(t) + \Delta t\, \mathbf{f}_{mid}$$

q-Stage Runge-Kutta

General Form:

$$x(t_0 + h) = x_0 + h \sum_{i=1}^{q} w_i k_i$$

where:

$$k_i = f\left(x_0 + h \sum_{j=1}^{i-1} \beta_{ij} k_j\right)$$

Find the constant that ensure accuracty $O(h^n)$.

4th-Order Runge-Kutta

$$k_1 = hf(x_0, t_0)$$

$$k_2 = hf(x_0 + \frac{k_1}{2}, t_0 + \frac{h}{2})$$

$$k_3 = hf(x_0 + \frac{k_2}{2}, t_0 + \frac{h}{2})$$

$$k_4 = hf(x_0 + k_3, t_0 + h)$$

$$x(t_0 + h) = x_0 + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4 + O(h^5)$$

4th-Order Runge-Kutta

$$k_1 = hf(x_0, t_0)$$

$$k_2 = hf(x_0 + \frac{k_1}{2}, t_0 + \frac{h}{2})$$

$$k_3 = hf(x_0 + \frac{k_2}{2}, t_0 + \frac{h}{2})$$

$$k_4 = hf(x_0 + k_3, t_0 + h)$$

$$x(t_0 + h) = x_0 + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4 + O(h^5)$$

Why so popular?

Order vs. Stages

Order	1	2	3	4	5	6	7	8
Stages	1	2	3	4	6	7	9	11

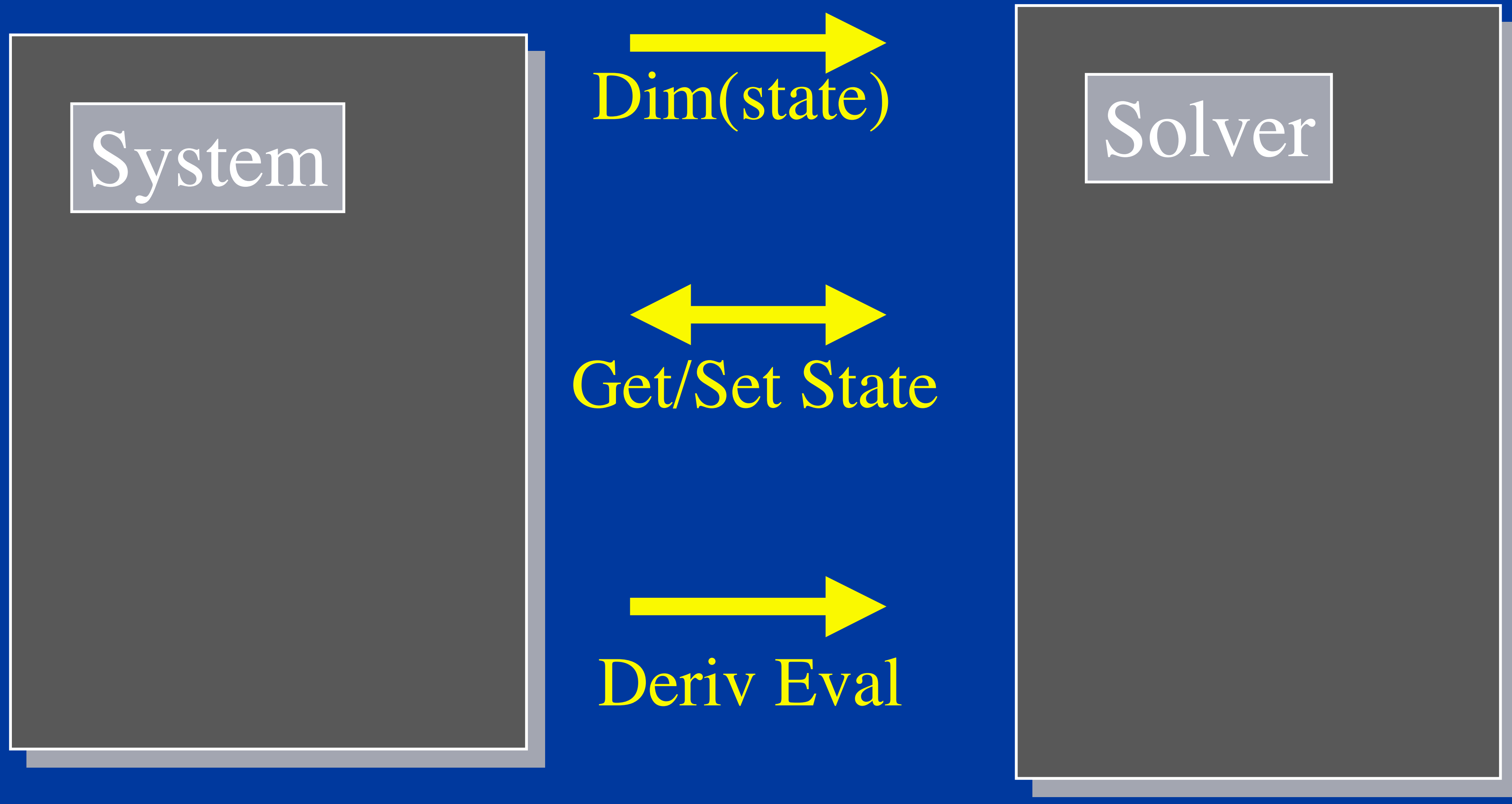
More methods…

- **Euler's method is *1st Order.***
- **The midpoint method is *2nd Order.***
- **Just the tip of the iceberg. See *Numerical Recipes* for more.**
- **Helpful hints:**
  - ***Don't* use Euler's method (you will anyway.)**
  - ***Do* use adaptive step size.**

Modular Implementation

- Generic operations:
  - Get dim(x)
  - Get/set x and t
  - Deriv Eval at current (x,t)
- Write solvers in terms of these.
  - Re-usable solver code.
  - Simplifies model implementation.

Solver Interface



A Code Fragment

```
void eulerStep(Sys sys, float h) {
   float t = getTime(sys);
   vector<float> x0, deltaX;

   t = getTime(sys);
   x0 = getState(sys);
   deltaX = derivEval(sys,x0, t);
   setState(sys, x0 + h*deltaX, t+h);
}
```

Question

- **What sorts of common physical phenomena are *not* well modeled by differential equations?**

Student Answers

What sorts of phenomena are not well modeled?

electrons - probabilistic WRONG
theory of relativity WRONG
sudden forces - not continuous
interactions between multiple particles WRONG
anything nondeterministic WRONG
shattering