15-853:Algorithms in the Real World

Error Correcting Codes II
- Cyclic Codes
- Reed-Solomon Codes

Viewing Messages as Polynomials

A (n, k, n-k+1) code:
Consider the polynomial of degree k-1

$p(x) = a_{k-1} x^{k-1} + \cdots + a_1 x + a_0$

Message: $(a_{k-1}, \ldots, a_1, a_0)$

Codeword: (p(1), p(2), …, p(n))

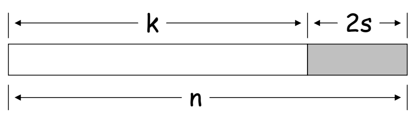
To keep the p(i) fixed size, we use $a_i \in GF(p^r)$

To make the i distinct, $n < p^r$

Unisolvence Theorem: Any subset of size k of (p(1), p(2), …, p(n)) is enough to (uniquely) reconstruct p(x) using polynomial interpolation, e.g., LaGrange's Formula.

Polynomial-Based Code

A (n, k, 2s +1) code:



Can **detect** 2s errors

Can **correct** s errors

Generally can correct α erasures and β errors if $\alpha + 2\beta \leq 2s$

Correcting Errors

Correcting s errors:

1. Find k + s symbols that agree on a polynomial p(x). These must exist since originally k + 2s symbols agreed and only s are in error
2. There are no k + s symbols that agree on the wrong polynomial p'(x)
   - Any subset of k symbols will define p'(x)
   - Since at most s out of the k+s symbols are in error, p'(x) = p(x)

A Systematic Code

Systematic polynomial-based code

$p(x) = a_{k-1} x^{k-1} + \cdots + a_1 x + a_0$

Message: $(a_{k-1}, \ldots, a_1, a_0)$

Codeword: $(a_{k-1}, \ldots, a_1, a_0, p(1), p(2), \ldots, p(2s))$

This has the advantage that if we know there are no errors, it is trivial to decode.

The version of RS used in practice uses something slightly different than p(1), p(2), …

This will allow us to use the "**Parity Check**" ideas from linear codes (i.e., $Hc^T = 0$?) to quickly test for errors.

Reed-Solomon Codes in the Real World

$\mathbf{(204,188,17)_{256}}$: ITU J.83(A)[2]
$\mathbf{(128,122,7)_{256}}$: ITU J.83(B)
$\mathbf{(255,223,33)_{256}}$: Common in Practice
- Note that they are all byte based (i.e., symbols are from $GF(2^8)$).

Decoding rate on 1.8GHz Pentium 4:
- (255,251) = 89Mbps
- (255,223) = 18Mbps

Dozens of companies sell hardware cores that operate 10x faster (or more)
- (204,188) = 320Mbps (Altera decoder)

Applications of Reed-Solomon Codes

- **Storage**: CDs, DVDs, "hard drives",
- **Wireless**: Cell phones, wireless links
- **Sateline and Space**: TV, Mars rover, …
- **Digital Television**: DVD, MPEG2 layover
- **High Speed Modems**: ADSL, DSL, ..

Good at handling burst errors.

Other codes are better for random errors.
- e.g., Gallager codes, Turbo codes

RS and "burst" errors

Let's compare to Hamming Codes (which are "optimal").

	code bits	check bits
RS $\mathbf{(255, 253, 3)_{256}}$	2040	16
Hamming $\mathbf{(2^{11}-1, 2^{11}-11-1, 3)_2}$	2047	11

They can both correct 1 error, but not 2 random errors.
- The Hamming code does this with fewer check bits

However, RS can fix 8 contiguous bit errors in one byte
- Much better than lower bound for 8 arbitrary errors

$$\log\left(1 + \binom{n}{1} + \cdots + \binom{n}{8}\right) > 8\log(n-7) \approx 88 \text{ check bits}$$

Galois Field

$GF(2^3)$ with irreducible polynomial: $x^3 + x + 1$

$\alpha = x$ is a generator

α	x	010	**2**
α^2	x^2	100	**3**
α^3	$x + 1$	011	**4**
α^4	$x^2 + x$	110	**5**
α^5	$x^2 + x + 1$	111	**6**
α^6	$x^2 + 1$	101	**7**
α^7	1	001	**1**

Will use this as an example.

Discrete Fourier Transform (DFT)

Another View of polynomial-based codes

α is a primitive n^{th} root of unity ($\alpha^n = 1$) – a generator

$$T = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \\ 1 & \alpha^2 & \alpha^4 & \cdots & \alpha^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^{n-1} & \alpha^{2(n-1)} & \cdots & \alpha^{(n-1)(n-1)} \end{pmatrix} \qquad \begin{pmatrix} c_0 \\ \vdots \\ c_{k-1} \\ c_k \\ \vdots \\ c_{n-1} \end{pmatrix} = T \cdot \begin{pmatrix} m_0 \\ \vdots \\ m_{k-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Evaluate polynomial $m_{k-1}x^{k-1} + \cdots + m_1x + m_0$ at n distinct roots of unity, 1, α, α^2, α^3, $\cdots$, α^{n-1}

Inverse DFT: $m = T^{-1}c$

DFT Example

$\alpha = x$ is 7^{th} root of unity in $GF(2^3)/x^3 + x + 1$

(i.e., multiplicative group, which excludes additive inverse)

Recall α = "2", α^2 = "3", … , α^7 = 1 = "1"

$$T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 & \alpha^6 \\ 1 & \alpha^2 & \alpha^4 & \alpha^6 & & & \\ 1 & \alpha^3 & \alpha^6 & & & & \\ 1 & \alpha^4 & & \ddots & & & \\ 1 & \alpha^5 & & & & & \\ 1 & \alpha^6 & & & & & \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 \\ 1 & 3 & 3^2 & 3^3 & & & \\ 1 & 4 & 4^2 & & & & \\ 1 & 5 & & \ddots & & & \\ 1 & 6 & & & & & \\ 1 & 7 & & & & & 7^6 \end{pmatrix}$$

Should be clear that $c = T \bullet (m_0, m_1, \ldots, m_{k-1}, 0, \ldots)^T$ is the same as evaluating $p(x) = m_0 + m_1x + \ldots + m_{k-1}x^{k-1}$ at n points.

Decoding

Why is it hard?

Brute Force: try k+2s choose k + s possibilities and solve for each.

Cyclic Codes

A linear code is cyclic if:

$(c_0, c_1, \ldots, c_{n-1}) \in C \Rightarrow (c_{n-1}, c_0, \ldots, c_{n-2}) \in C$

Both **Hamming** and **Reed-Solomon** codes are cyclic.

Note: we might have to reorder the columns to make the code "cyclic".

Motivation: They are more efficient to decode than general codes.

Generator and Parity Check Matrices

Generator Matrix:

A k x n matrix $\mathbf{G}$ such that:

$C = \{m \bullet \mathbf{G} \mid m \in \Sigma^k\}$

Made from stacking the basis vectors

Parity Check Matrix:

A (n – k) x n matrix $\mathbf{H}$ such that:

$C = \{v \in \Sigma^n \mid H \bullet v^T = 0\}$

Codewords are the nullspace of H

These **always exist for linear codes**

$H \bullet G^T = 0$

Generator and Parity Check Polynomials

Generator Polynomial:

A degree (n-k) polynomial **g** such that:

$C = \{m \bullet \mathbf{g} \mid m \in m_0 + m_1x + \ldots + m_{k-1}x^{k-1}\}$

such that $\mathbf{g} \mid x^n - 1$

Parity Check Polynomial:

A degree k polynomial **h** such that:

$C = \{v \in \Sigma^n[x] \mid h \bullet v = 0 \pmod{x^n - 1}\}$

such that $\mathbf{h} \mid x^n - 1$

These **always exist for linear cyclic codes**

$h \bullet g = x^n - 1$

Viewing g as a matrix

If $g(x) = g_0 + g_1x + \ldots + g_{n-k-1}x^{n-k-1}$

We can put this generator in matrix form:

$$G = \begin{pmatrix} g_0 & g_1 & \cdots & \cdots & g_{n-k-1} & 0 & \cdots & 0 \\ 0 & g_0 & \cdots & \cdots & g_{n-k-2} & g_{n-k-1} & \cdots & 0 \\ \vdots & & \ddots & & & & \ddots & \vdots \\ 0 & 0 & \cdots & g_0 & g_1 & \cdots & \cdots & g_{n-k-1} \end{pmatrix}$$

Write $m = m_0 + m_1x + \ldots + m_{k-1}x^{k-1}$ as $(m_0, m_1, \ldots, m_{k-1})$

Then c = mG

g generates cyclic codes

$$G = \begin{pmatrix} g_0 & g_1 & \cdots & g_{n-k-1} & 0 & \cdots & 0 \\ 0 & g_0 & \cdots & g_{n-k-2} & g_{n-k-1} & \cdots & 0 \\ \vdots & & \ddots & & & \ddots & \vdots \\ 0 & 0 & \cdots & g_0 & g_1 & \cdots & g_{n-k-1} \end{pmatrix} = \begin{pmatrix} g \\ xg \\ \vdots \\ x^{k-1}g \end{pmatrix}$$

Codes are linear combinations of the rows.
All but last row is clearly cyclic (based on next row)
Shift of last row is $x^k g \bmod (x^n - 1) = g_{n-k-1}, 0, \ldots, g_0, g_1, \ldots, g_{n-k-2}$
Consider $h = h_0 + h_1 x + \ldots + h_{k-1}x^{k-1}$ ($gh = x^n - 1$)

$h_0 g + (h_1 x)g + \ldots + (h_{k-2}x^{k-2})g + (h_{k-1}x^{k-1})g = x^n - 1$

$x^k g = -h_{k-1}^{-1}(h_0 g + h_1(xg) + \ldots + h_{k-1}(x^{k-1}g)) \bmod (x^n - 1)$

This is a linear combination of the rows.

Viewing h as a matrix

If $h = h_0 + h_1 x + \ldots + h_{k-1}x^{k-1}$
we can put this parity check poly. in matrix form:

$$H = \begin{pmatrix} 0 & \cdots & 0 & h_{k-1} & \cdots & h_1 & h_0 \\ 0 & \cdots & h_{k-1} & h_{k-2} & \cdots & h_0 & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & & & \cdot^{\cdot^{\cdot}} & & \vdots \\ h_{k-1} & \cdots & h_1 & h_0 & 0 & \cdots & 0 \end{pmatrix}$$

$Hc^T = 0$

Hamming Codes Revisited

The Hamming $(7,4,3)_2$ code.

$g = 1 + x + x^3$ $\quad$ $h = x^4 + x^2 + x + 1$

$$G = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix} \quad H = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

$gh = x^7 - 1, \quad GH^T = 0$

The columns are not identical to the previous example Hamming code.

Factors of $x^n - 1$

Intentionally left blank

Another way to write g

Let α be a **generator** of $GF(p^r)$.
Let $n = p^r - 1$ (the size of the multiplicative group)
Then we can write a generator polynomial as
$g(x) = (x-\alpha)(x-\alpha^2) \dots (x - \alpha^{n-k})$, $h = (x - \alpha^{n-k+1})\dots(x-\alpha^n)$
Lemma: $g \mid x^n - 1$, $h \mid x^n - 1$, $gh \mid x^n - 1$
($a \mid b$ means a divides b)
Proof:

- $\alpha^n = 1$ (because of the size of the group)
  $\Rightarrow \alpha^n - 1 = 0$
  $\Rightarrow \alpha$ root of $x^n - 1$
  $\Rightarrow (x - \alpha) \mid x^n - 1$
- similarly for $\alpha^2, \alpha^3, \dots, \alpha^n$
- therefore $x^n - 1$ is divisible by $(x - \alpha)(x - \alpha^2) \dots$

Back to Reed-Solomon

Consider a generator polynomial $g \in GF(p^r)[x]$, s.t. $g \mid (x^n - 1)$
Recall that $n - k = 2s$ (the degree of g is n-k-1, n-k coefficients)
Encode:

- $m' = m\, x^{2s}$ (basically shift by 2s)
- $b = m' \pmod g$
- $c = m' - b = (m_{k-1}, \dots, m_0, -b_{2s-1}, \dots, -b_0)$
- Note that c is a **cyclic code** based on g
  - $m' = qg + b$
  - $c = m' - b = qg$

Parity check:

- $h\, c = 0$?

Example

Lets consider the $(7,3,5)_8$ Reed-Solomon code.
We use $GF(2^3)/x^3 + x + 1$

α	x	010	**2**
α^2	x^2	100	**3**
α^3	$x + 1$	011	**4**
α^4	$x^2 + x$	110	**5**
α^5	$x^2 + x + 1$	111	**6**
α^6	$x^2 + 1$	101	**7**
α^7	1	001	**1**

Example RS $(7,3,5)_8$

$n = 7$, $k = 3$, $n-k = 2s = 4$, $d = 2s+1 = 5$

$g = (x - \alpha)(x - \alpha^2)(x - \alpha^3)(x - \alpha^4)$
$= x^4 + \alpha^3x^3 + x^2 + \alpha x + \alpha^3$
$h = (x - \alpha^5)(x - \alpha^6)(x - \alpha^7)$
$= x^3 + a^3x^3 + a^2x + a^4$
$gh = x^7 - 1$
Consider the message: 110 000 110
$m = (\alpha^4, 0, \alpha^4) = \alpha^4x^2 + \alpha^4$
$m' = x^4m = \alpha^4x^6 + \alpha^4x^4$
$= (\alpha^4 x^2 + x + \alpha^3)g + (\alpha^3x^3 + \alpha^6x + \alpha^6)$
$c = (\alpha^4, 0, \alpha^4, \alpha^3, 0, \alpha^6, \alpha^6)$
= 110 000 110 011 000 101 101

α	010
α^2	100
α^3	011
α^4	110
α^5	111
α^6	101
α^7	001

$ch = 0 \pmod{x^7 - 1}$

A useful theorem

Theorem: For any β, if $g(\beta) = 0$ then $\beta^{2s}m(\beta) = b(\beta)$

Proof:

$x^{2s}m(x) = m'(x) = g(x)q(x) + b(x)$

$\beta^{2s}m(\beta) = g(\beta)q(\beta) + b(\beta) = b(\beta)$

Corollary: $\beta^{2s}m(\beta) = b(\beta)$ for $\beta \in \{\alpha, \alpha^2, \alpha^3, \ldots, \alpha^{2s=n-k}\}$

Proof:

$\{\alpha, \alpha^2, \ldots, \alpha^{2s}\}$ are the roots of g by definition.

Fixing errors

Theorem: Any k symbols from c can reconstruct c and hence m

Proof:

We can write 2s equations involving m ($c_{n-1}, \ldots, c_{2s}$) and b ($c_{2s-1}, \ldots, c_0$). These are

$\alpha^{2s}\ m(\alpha) = b(\alpha)$

$\alpha^{4s}\ m(\alpha^2) = b(\alpha^2)$

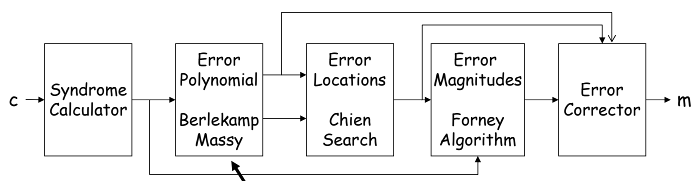
…

$\alpha^{2s(2s)}\ m(\alpha^{2s}) = b(\alpha^{2s})$

We have at most 2s unknowns, so we can solve for them. (I'm skipping showing that the equations are linearly independent).

Efficient Decoding

I don't plan to go into the Reed-Solomon decoding algorithm, other than to mention the steps.



This is the hard part. CD players use this algorithm.
(Can also use Euclid's algorithm.)