CS 15-851: Algorithms for Big Data	Spring 2025
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1 KVW Protocol

In this section, we study KVW Protocol, motivated by problems associated with the FSS Protocol.

1.1 Problems with FSS Protocol

There are some problems with FSS protocol that remain unsolved.

- sdk / ϵ real numbers of communication
- bit complexity can be large
- running time for SVDs
- doesn't work in arbitrary partition model

We handle the second, third, and fourth problem with the KVW Protocol.

1.2 Arbitrary Partition Model Protocol

Apart from FSS, we want a new protocol that could work on arbitrary partition model. Inspired by the sketching algorithms presented earlier, let S be one of the $k/\epsilon * n$ random matrices discussed: S can be generated pseudorandomly from small seed, where the Coordinator sends small seed for S to all servers. Each Server t computes SA^t and sends it to Coordinator; and the Coordinator sends $\sum_{t=1}^{s} SA^t = SA$ to all servers.

There is a good k-dimensional subspace inside of SA. If we knew it, t-th server could output projection of A^t onto it.

However, there are some problem here. We cannot output projection of A^t onto SA since the rank is too large. We could communicate this projection to the coordinator who could find a k-dimensional space, but communication depends on n.

To fix these problems, instead of projecting A onto SA, recall we can solve

$$min_{rank-kX}|A(SA)^T XSA - A|_F^2.$$

Let T_1 and T_2 be affine embeddings, solve

$$min_{rank-kX}|T_1A(SA)^T XSAT_2 - T_1AT_2|_F^2.$$

This optimization problem is small and has a closed form solution. Everyone can then compute XSA and then output k directions.

In Phase 1, We would like to learn the row space of SA by finding the optimal k-dimensional space in SA. Then $cost \leq (1+\epsilon)|A - A_k|_F$.

In Phase 2, we would like to find an approximately optimal space W inside of SA that achieve

$$cost \leq (1+\epsilon)^2 |A - A_k|_F$$

2 BWZ Protocol

2.1 Main Problem with KVW

In KVW protocol, communication is $O(skd/\epsilon) + poly(sk/\epsilon)$, but we want $O(skd) + poly(sk/\epsilon)$ communication. Therefore, we proceed by deriving a new protocol BWZ protocol that has $O(skd) + poly(sk/\epsilon)$ communication cost.

2.2 Protocol

The main idea here is to use projection-cost preserving sketches(CEMMP). The BWZ protocol is as follows:

Let A be an $n * d$ matrix. If S is a random $k/\epsilon^2 * n$ matrix, then there is a scalar $c \geq 0$ so that for all k-dimensional projection matrices P:

$$|A(I-P)|_F^2 \leq |SA(I-P)|_F^2 + c \leq |(1+\epsilon)A(I-P)|_F^2$$

Note: The implication here is that if $I - \tilde(P)$ is the minimizer of $|SA(I-P)|_F^2$, and $I - P^*$ is the minimizer of $|A(I-P)|_F^2$, then

$$|A(I-\tilde{P})|_F^2 \leq (1+\epsilon)|A - A_k|_F^2$$

so

$$|SA - [SA]_k|_F^2 + c \leq |(1+\epsilon)A(I-\tilde{P})|_F^2 \leq (1+O(\epsilon))|A - A_k|_F^2$$

Let S be a $k/\epsilon^2 * n$ projection-cost preserving sketch. Let T be a $d * k/\epsilon^2$ projection-cost preserving sketch. Server t sends SA^tT to Coordinator, and Coordinator sends back $SAT = \sum_t SA^tT$ to servers. Each server computes $k/\epsilon^2 * k$ matrix U of top k left singular vectors of SAT.

Intuitively, U looks like top k left singular vectors of SA, thus $U^T SA$ looks like top k scaled right singular vectors of SA.

Server t sends $U^T SA^t$ to Coordinator, and the Coordinator returns the space $U^T SA = \sum_t U^T SA^t$ to output.

Note: Top k right singular vectors of SA work because S is a projection-cost preserving sketch.

2.3 Analysis

Let W be the row span of $U^T SA$ and P be the projection onto W.

Then, we want to show

$$|A - AP|_F^2 \leq (1+\epsilon)|A - A_k|_F^2$$

Since T is a projection-cost preserving sketch,

$$(*)|SA - SAP|_F^2 \leq |SA - UU^T SA|_F^2 \leq (1+\epsilon)|SA - [SA]_k|_F^2$$

Since S is a projection-cost preserving sketch, there is a scalar $c \geq 0$, so that for all k-dimensional projection matrices Q,

$$|SA - SAQ|_F^2 + c = (1+\epsilon)|A - AQ|_F^2$$

$$|SA - SAP|_F^2 + c \leq (1+\epsilon)|SA - [SA]_k|_F^2 + c$$

Add c to both sides of the inequality $(*)$ to conclude that

$$|A - AP|_F^2 \leq (1 + O(\epsilon))|A - A_k|_F^2$$

2.4 Conclusion

With BWZ protocol, we achieve the optimal $O(sdk) + poly(sk/\epsilon)$ communication protocol for low rank approximation in arbitrary partition model. It has the following properties:

- Handle bit complexity by adding noise (omitted)
- Input sparsity time
- 2 rounds, which is optimal.

3 Communication of other optimization problems

We also discussed other potential optimization problems and their communications. For example,

- computing the rank of an $n * n$ matrix over the reals.
- Linear Programming
- Graph problems: Matching