Sketching Theorem

Theorem:

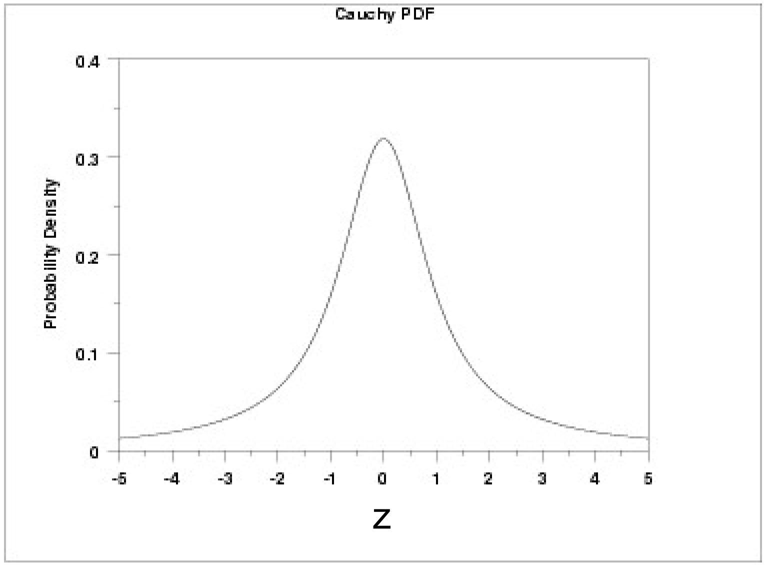
- There is a probability space over (d log d) $\times$ n matrices R such that for any n$\times$d matrix A, with probability at least 99/100 we have for all x:

$$|Ax|_1 \leq |RAx|_1 \leq d \log d \cdot |Ax|_1$$

A dense R that works:

The entries of R are i.i.d. Cauchy random variables, scaled by 1/(d log d)

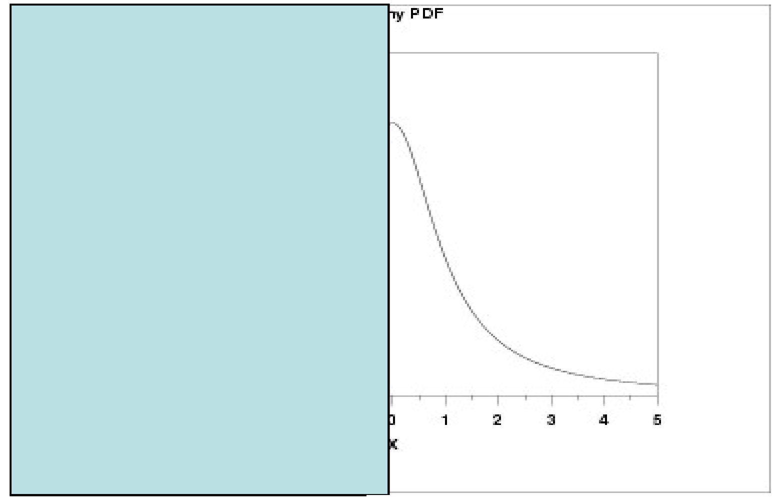
Cauchy Random Variables



- pdf(z) = $1/(\pi(1+z^2))$ for z in $(-\infty, \infty)$
- Undefined expectation and infinite variance
- 1-stable:
  - If $z_1, z_2, \ldots, z_n$ are i.i.d. Cauchy, then for $a \in R^n$,
    $a_1 \cdot z_1 + a_2 \cdot z_2 + \ldots + a_n \cdot z_n \sim |a|_1 \cdot z$, where z is Cauchy
- Can generate as the ratio of two standard normal random variables

Proof of Sketching Theorem

- By 1-stability,
  - For all rows r of R,
    - $<r, Ax> = |Ax|_1 \cdot Z / (d \log d)$,
      where Z is a Cauchy



Z

- $RAx = (|Ax|_1 \cdot Z_1, \ldots, |Ax|_1 \cdot Z_{d \log d}) / (d \log d)$,
  where $Z_1, \ldots, Z_{d \log d}$ are i.i.d. Cauchy

- $|RAx|_1 = |Ax|_1 \sum_j |Z_j| / (d \log d)$
  - The $|Z_j|$ are half-Cauchy

- $\sum_j |Z_j| = \Omega(d \log d)$ with probability 1-exp(-d log d) by Chernoff

- But the $|Z_j|$ are heavy-tailed…

Proof of Sketching Theorem

- $\sum_j |Z_j|$ is heavy-tailed, so $|RAx|_1 = |Ax|_1 \sum_j |Z_j| / (d \log d)$ may be large
- Each $|Z_j|$ has c.d.f. asymptotic to $1-\Theta(1/z)$ for z in $[0, \infty)$
- There *exists* a well-conditioned basis of A
  - Suppose w.l.o.g. the basis vectors are $A_{*1}, \ldots, A_{*d}$
- $|RA_{*i}|_1 = |A_{*i}|_1 \cdot \sum_j |Z_{i,j}| / (d \log d)$
- Let $E_{i,j}$ be the event that $|Z_{i,j}| \leq d^3$
  - Define $Z'_{i,j} = |Z_{i,j}|$ if $|Z_{i,j}| \leq d^3$, and $Z'_{i,j} = d^3$ otherwise
  - $E[Z_{i,j} \mid E_{i,j}] = E[Z'_{i,j} \mid E_{i,j}] = O(\log d)$
- Let E be the event that for all i,j, $E_{i,j}$ occurs
  - $\Pr[E] \geq 1 - \frac{\log d}{d}$
- What is $E[Z'_{i,j} \mid E]$?

Proof of Sketching Theorem

- What is $\mathrm{E}\left[\mathrm{Z}'_{\mathrm{i,j}} \mid \mathrm{E}\right]$?

- $$\begin{aligned}\mathrm{E}\left[\mathrm{Z}'_{\mathrm{i,j}}\middle|\mathrm{E}_{\mathrm{i,j}}\right] &= \mathrm{E}\left[\mathrm{Z}'_{\mathrm{i,j}}\middle|\mathrm{E}_{\mathrm{i,j}},\mathrm{E}\right]\Pr[\mathrm{E}\mid\mathrm{E}_{\mathrm{i,j}}] + \mathrm{E}\left[\mathrm{Z}'_{\mathrm{i,j}}\middle|\mathrm{E}_{\mathrm{i,j}},\neg\mathrm{E}\right]\Pr[\neg\mathrm{E}\mid\mathrm{E}_{\mathrm{i,j}}] \\ &\geq \mathrm{E}\left[\mathrm{Z}'_{\mathrm{i,j}}\middle|\mathrm{E}_{\mathrm{i,j}},\mathrm{E}\right]\Pr[\mathrm{E}\mid\mathrm{E}_{\mathrm{i,j}}] \\ &= \mathrm{E}\left[\mathrm{Z}'_{\mathrm{i,j}}\middle|\mathrm{E}\right]\cdot\left(\frac{\Pr\left[\mathrm{E}_{\mathrm{i,j}}\middle|\mathrm{E}\right]\Pr[\mathrm{E}]}{\Pr\left[\mathrm{E}_{\mathrm{i,j}}\right]}\right) \\ &\geq \mathrm{E}\left[\mathrm{Z}'_{\mathrm{i,j}}\middle|\mathrm{E}\right]\cdot\left(1-\frac{\log \mathrm{d}}{\mathrm{d}}\right)\end{aligned}$$

- So, $\mathrm{E}\left[\mathrm{Z}'_{\mathrm{i,j}}\middle|\mathrm{E}\right] = \mathrm{O}(\log \mathrm{d})$
- $|RA_{*i}|_1 = |A_{*i}|_1 \cdot \sum_j |Z_{i,j}| / (d \log d)$
- With constant probability, $\sum_i |RA_{*i}|_1 = O(\log d) \sum_i |A_{*i}|_1$

Proof of Sketching Theorem

- With constant probability, $\sum_i |RA_{*i}|_1 = O(\log d) \sum_i |A_{*i}|_1$
- Recall $A_{*1}, \ldots, A_{*d}$ is a well-conditioned basis, and we showed the existence of such a basis earlier
- We will use the Auerbach basis which always exists:
  - For all x, $|x|_\infty \leq |Ax|_1$
  - $\sum_i |A_{*i}|_1 = d$
- $\sum_i |RA_{*i}|_1 = O(d \log d)$
- For all x, $|RAx|_1 \leq \sum_i |RA_{*i} x_i| \leq |x|_\infty \sum_i |RA_{*i}|_1$

  $= |x|_\infty O(d \log d)$

  $= O(d \log d) |Ax|_1$

Where are we?

- Suffices to show for all x with $|x|_1 = 1$, that $|Ax|_1 \leq |RAx|_1 \leq d \log d \cdot |Ax|_1$
- We know
  - (1) there is a γ-net M, with $|M| \leq \left(\frac{d}{\gamma}\right)^{O(d)}$, of the set $\{Ax \text{ such that } |x|_1 = 1\}$
  - (2) for any fixed x, $|RAx|_1 \geq |Ax|_1$ with probability $1 - \exp(-d \log d)$
  - (3) for all x, $|RAx|_1 = O(d \log d)|Ax|_1$

- Set $\gamma = 1/(d^3 \log d)$ so $|M| \leq d^{O(d)}$
  - By a union bound, for all y in M, $|Ry|_1 \geq |y|_1$

- Let x with $|x|_1 = 1$ be arbitrary. Let y in M satisfy $|Ax - y|_1 \leq \gamma = 1/(d^3 \log d)$

- $|RAx|_1 \geq |Ry|_1 - |R(Ax - y)|_1$

  $\geq |y|_1 - O(d \log d)|Ax - y|_1$

  $\geq |y|_1 - O(d \log d)\gamma$

  $\geq |y|_1 - O\left(\frac{1}{d^2}\right)$

  $\geq |y|_1/2$ (why?)

Outline

- Quick recap of ℓ_1-regression, and how to speed it up
- Introduction to the Streaming Model and Estimating Norms

L_1 Regression Algorithm Recap

Compute poly(d)-approximation

Compute well-conditioned basis

Sample rows from the well-conditioned basis and the residual of the poly(d)-approximation

Solve l_1-regression on the sample, obtaining vector x, and output x

We saw how to solve the above problems by sketching by a matrix of i.i.d. Cauchy random variables

Sketching to solve l_1-regression [CW, MM]

- Most expensive operation is computing R*A where R is the matrix of i.i.d. Cauchy random variables

- All other operations are in the “smaller space”

- Can speed this up by choosing R as follows:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} C_1 & & & & \\ & C_2 & & & \\ & & C_3 & & \\ & & & \cdots & \\ & & & & C_n \end{bmatrix}$$

- For all x, $\left(\frac{1}{d^2 \log^2 d}\right) |Ax|_1 \leq |RAx|_1 \leq O(d \log d)\, |Ax|_1$
- Overall time for ℓ_1-regression is nnz(A) + poly(d/ϵ)

Further sketching improvements [WZ]

- Can show you need a fewer number of sampled rows in later steps if instead choose R as follows

- Instead of diagonal of Cauchy random variables, choose diagonal of reciprocals of exponential random variables

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/E_1 & & & & \\ & 1/E_2 & & & \\ & & 1/E_3 & & \\ & & & \cdots & \\ & & & & 1/E_n \end{bmatrix}$$

- For all x, $\left(\frac{1}{d^{.5}\text{poly}(\log(nd))}\right)|Ax|_1 \le |RAx|_1 \le O(d \log d)\,|Ax|_1$

Fun Fact about Cauchy Random Variables

- Suppose you have i.i.d. copies $R_1, \ldots, R_n$ of a random variable with mean 0 and variance σ^2
- What is the distribution of $\frac{\sum_i R_i}{n}$?
- By Central Limit Theorem, this approaches a normal random variable N(0, $\sigma^2/n)$
- Intuitively, the variance is decreasing and the average is approaching its expectation
- Now suppose you have i.i.d. copies $R_1, \ldots, R_n$ of a standard Cauchy random variable
- What is the distribution of $\frac{\sum_i R_i}{n}$?
- It's still a standard Cauchy random variable!

Outline

- Introduction to the Streaming Model
- Estimating Norms in the Streaming Model

Turnstile Streaming Model

- Underlying n-dimensional vector x initialized to 0^n
- Long stream of updates $x_i \leftarrow x_i + \Delta_j$ for Δ_j in {-M, -M+1, …, M-1, M}
  - $M \leq \text{poly}(n)$
- Throughout the stream, x is promised to be in $\{-M, -M+1, \ldots, M-1, M\}^n$
- Output an approximation to f(x) with high probability over our coin tosses
- Goal: use as little space (in bits) as possible
  - Massive data: stock transactions, weather data, genomes

Testing if x = 0^n

- How can we test, with probability at least 9/10, over our random coin tosses, if the underlying vector $x = 0^n$?

- Can we use O(log n) bits of space?

- We saw that for any fixed vector x, if S is a CountSketch matrix with $O(\frac{1}{\epsilon^2})$ rows, then $|Sx|_2^2 = (1 \pm \epsilon)|x|_2^2$ with probability at least 9/10

- If we set $\epsilon = \frac{1}{2}$, we use O(log n) bits of space to store the O(1) entries of Sx

- We can store the hash function and sign function defining S using O(log n) bits

Testing if x = 0^n

- Is there a deterministic, i.e., zero-error, streaming algorithm to test if the underlying vector $x = 0^n$ with o(n log n) bits of space?

- Theorem: any deterministic algorithm requires $\Omega(n \log n)$ bits of space

- Suppose the first half of the stream corresponds to updates to a vector a in $\{0, 1, 2, \ldots, \text{poly}(n)\}^n$

- Let S(a) be the state of the algorithm after reading the first half of the stream
  - If |S(a)| = o(n log n), there exist a$\neq$ a$'$ for which S(a) = S(a')

- Suppose the second half of the stream corresponds to updates to a vector b in $\{0, -1, -2, \ldots, -\text{poly}(n)\}^n$

- The algorithm must output the same answer on a+b and a'+b, so it errs in one case

Example: Recovering a k-Sparse Vector

- Suppose we are promised that x has at most k non-zero entries at the end of the stream
- k is often small – maybe we see all coordinates of a vector a followed by all coordinates of a *similar* vector b, and a-b only has k non-zero entries
- Can we recover the indices and values of the k non-zero entries with high probability?
- Can we use k poly(log n) bits of space?
- Can we do it deterministically?

Example: Recovering a k-Sparse Vector

- Suppose A is an s x n matrix such that any 2k columns are linearly independent
- Maintain $A \cdot x$ in the stream
- Claim: from $A \cdot x$ you can recover the subset S of k non-zero entries and their values
- Proof: suppose there were vectors x and y each with at most k non-zero entries and $A \cdot x = A \cdot y$
- Then A(x-y) = 0. But x-y has at most 2k non-zero entries, and any 2k columns of A are linearly independent. So x-y = 0, i.e., x = y.
- Algorithm is deterministic given A. But do such matrices A exist with a small number s of rows?

Example: Recovering a k-Sparse Vector

- Vandermonde matrix A with s = 2k rows and n columns. $A_{i,j} = j^{i-1}$

$$\begin{bmatrix} 1 & 1 & 1 & \ldots \\ 1 & 2 & 3 & \ldots \\ 1 & 4 & 9 & \ldots \\ 1 & 8 & 27 & \ldots \end{bmatrix}$$

- Determinant of 2k x 2k submatrix of A with set of columns equal to $\{i_1, \ldots, i_{2k}\}$ is: $\prod_j i_j \prod_{j<j'} (i_j - i_{j'}) \neq 0$, so any 2k columns of A are linearly independent

- But entries of A are exponentially increasing – how to store A and $A \cdot x$?

- Just store $A \cdot x$ mod p for a large enough prime p = poly(n)

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Example Problem: Norms

- Suppose you want $|x|_p^p = \Sigma_{i=1}^n |x_i|^p$
- Want Z for which $(1-\varepsilon) |x|_p^p \leq Z \leq (1+\varepsilon) |x|_p^p$ with probability > 9/10
- p = 1 corresponds to total variation distance between distributions
- p = 2 useful for geometric and linear algebraic problems
- $p = \infty$ is the value of the maximum entry, useful for anomaly detection, etc.

Example Problem: Euclidean Norm

- Want Z for which $(1-\varepsilon)\ |x|_2^2 \le Z \le (1+\varepsilon)\ |x|_2^2$
- Sample a random CountSketch matrix S with $1/\epsilon^2$ rows
- Can store S efficiently using limited independence
- If $x_i \leftarrow x_i + \Delta_j$ in the stream, then $Sx \leftarrow Sx + \Delta_j S_{*,i}$
- At end of stream, output $|Sx|_2^2$
- With probability at least 9/10, $|Sx|_2^2 = (1 \pm \epsilon)|x|_2^2$
- Space complexity is $1/\epsilon^2$ words, each word is O(log n) bits

Example Problem: 1-Norm

- Want Z for which (1-Ɛ) $|x|_1 \leq Z \leq$ (1+Ɛ) $|x|_1$
- Sample a random Cauchy matrix S?
- Can store S with $\frac{1}{\epsilon}$ words of space [Kane, Nelson, W]
- If $x_i \leftarrow x_i + \Delta_j$ in the stream, then $Sx \leftarrow Sx + \Delta_j S_{*,i}$
- Space complexity is $1/\epsilon^2$ words, each word is O(log n) bits
- At end of stream, output $|Sx|_1$?
- *Cauchy random variables have no concentration…*

1-Norm Estimator

- Probability density function f(x) of |C| for a Cauchy random variable C is $f(x) = \frac{2}{\pi(1+x^2)}$
- Cumulative distribution function F(z):

$$F(z) = \int_0^z f(x)dx = \frac{2}{\pi}\arctan(z)$$

- Since $\tan(\pi/4) = 1$, F(1) = ½, so median(|C|) = 1
- If you take $r = \frac{\log\left(\frac{1}{\delta}\right)}{\epsilon^2}$ independent samples $X_1, \ldots, X_r$ from F, and $X = \text{median}_i X_i$, then F(X) in [1/2-ϵ, 1/2+ ϵ] with probability 1-δ
- $F^{-1}(X) = \tan\left(\frac{X\pi}{2}\right) \in [1 - 4\epsilon, 1 + 4\epsilon]$

p-Norm Estimator

- Can achieve $1/\epsilon^2$ words of space for p-norm estimation for any 0 < p < 2
- Proof is similar to 1-norm estimation, and uses p-stable distributions, which exist only for 0 < p < 2
- No closed form expression for their probability density function but they are efficiently sampleable:
  - If $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $r \in [0,1]$ are uniformly random, then

$$\frac{\sin(p\,\theta)}{\cos^{\frac{1}{p}}\theta}\left(\frac{\cos(\theta(1-p))}{\ln\left(\frac{1}{r}\right)}\right)^{\frac{1-p}{p}}$$ is a sample from a p-stable distribution!

- Can discretize them and store a sketching matrix of samples from the p-stable distribution using limited independence