ISOMORPHISM TESTING AND CANONICAL FORMS FOR k-CONTRACTABLE GRAPHS
(A Generalization of Bounded Valence and Bounded Genus)

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Abstract. This paper includes polynomial time isomorphism tests and canonical forms for graphs called k-contractable graphs for fixed k. The class of k-contractable graphs includes the graphs of bounded valence and the graphs of bounded genus. The algorithm uses several new ideas including: (1) it removes portions of the graph and replaces them with groups which are used to keep track of the symmetries of these portions; (2) it maintains with each group a tower of equivalence relation which allows a decomposition of the group. These towers are called a tower of Γ_k actions. It considers the canonical intersection of groups.

1. INTRODUCTION

The author, and independently other researchers [FM80, L180, M180], have presented polynomial time algorithms for isomorphism testing of graphs of bounded genus. These algorithms are based on fairly complicated analyses of embeddings of graphs on two dimensional surfaces. Since then, Luks has presented a polynomial time algorithm for isomorphism testing of graphs of bounded valence [Lu80]. The ideas used in the bounded valence algorithm are very appealing. They showed relationships between computational group theory and graph isomorphism. The existence of a polynomial time algorithm which in a natural way tests isomorphism of graphs of bounded valence and bounded genus has been an open question [Ba81]. We show that the class of graphs, called the k-contractable graphs, contains the graphs of bounded genus. They trivially contain the graphs of bounded valence. Therefore, these graphs form a common generalization of the two classes. We give a polynomial time algorithm for testing isomorphism of these graphs.

More recently, several authors [BL83, FSS83] have shown that graphs of bounded valence have p-time constructable canonical forms. We also show how these ideas can be extended to the k-contractable graphs.

We give the first of several definitions of k-contractable graphs. Later definitions of k-contractable graphs will enlarge the class of graph which will also be called k-contractable for each k. Consider the following three operations on a graph, G.

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The first operation, removing leaves, simply removes all leaves and their associated edges. The second operation, removing multiple vertices, identifies vertices of valence two which have the same neighbors. The third operation, contracting edges of valence $\leq k$, simultaneously identifies the end points of edges common to two vertices whose valence is $\leq k + 1$. It then removes selfloops and multiple edges.

Definition 1.1. The graph G is k-contractable if the application of the above three operations eventually reduces G to a point.

The definition seems to be dependent on the order in which one applies these three operations. Nevertheless, in the last section we show that any graph which is fixed by all three operations has its genus bounded below by $\Omega(k)$. Diagram 1 describes an infinite periodic planar graph. From this graph we can construct arbitrarily large graphs of genus 1 which are not 10-contractable. These graphs are 6-contractable using a slightly stronger definition of the third operation.

The paper consists of four other sections. Section 2, the preliminaries, contains basic definition plus the notation of a tower of Γ_k actions which will be used throughout. Section 3, Canonical ordering and set of orderings, contains the basic group theoretic algorithms used in the isomorphism tests. Section 4 includes the notation of a graph where the symmetries at a vertex are not arbitrary but restricted to a group. This last notion will be used in section 5 to decompose a class of graphs where at intermediate stages the graphs are those with restricted vertex symmetries. The graph for which this contraction procedure works will be called the k-contractable graphs. Section 6 shows that the k-contractable graph contains the graphs of genus εk for some $\varepsilon > 0$.

2. PRELIMINARIES

2.1 Graph Theoretic Preliminaries

Throughout this paper graphs will be denoted by G, H, and K; groups by A, B, and C; and sets by X, Y, and Z. Graphs may have multiple edges but no selfloops. It will be important that they are allowed to have multiple edges. The edges and vertices of G will be denoted by E(G) and V(G), respectively. The edges common to some vertex v or set of vertices will be denoted by E(v). Let ME denote the multiple edge equivalence relation on G, i.e. eMEe' if e and e' are common to the same points. The valence of a given vertex v will be the number of vertices adjacent to v, i.e. the number of edges ignoring multiplicity. Let G be a graph and $Y \subseteq V$. We say two edges e and e' of G are Y equivalent if there exists a path from e to e' avoiding points of Y.

Definition 2.1. The graph Br induced from an equivalence class of Y-equivalent edges will be called a bridge, or a bridge of the pair (G,Y). The vertex frontier of Br is the vertices of Br in Y, while the edge frontier of Br is the edges of Br common to the vertex frontier. A bridge is trivial if it is a single edge.

The main graph theoretic construction we shall use is contracting nonfrontier edges to a point.

Definition 2.2. If $Y \subseteq V$, the vertices of G, then Contract(G,Y) will be the graph obtained from G by identifying the nonfrontier (internal) vertices of Br for each bridge Br of G, and removing selfloops. If $Y = \emptyset$ then Contract(G,Y) is a single vertex.

Let Y_k be the vertices of G with valence greater than k. The intuition behind the k-contractable graphs is it is those graphs for which the successive application of Contract(G,Y_k) yields a single point. Here we keep track of the symmetries of the bridges with groups We attempt to render this idea.

2.2 Group Theoretic Preliminaries

Let Sym(X) denote the group of all permutations of X. We let S_n denote the symmetric group on a set of size n. The group A is a permutation group on X if $A \subseteq$ Sym(X). The degree of A is $|X|$, while the order is $|A|$. Let π be an equivalence relation on X. Let $A(\pi)$ denote the subgroup of A which stabilizes the equivalence classes X/π of π, i.e. $A(\pi) = \{\alpha \in A | x\pi\alpha(x) \text{ for all } x \in X\}$. If $Y \subseteq X$, we let Y denote the relation $\{xYy | x,y \notin Y \text{ or } x = Y\}$. Thus, $A(Y)$ is the subgroup of A which fixes Y pointwise, while $A[Y]$ is the subgroup which stabilizes Y We let id denote the empty relation. We say A preserves π if $x\pi y$ implies $\alpha(x)\pi\alpha(y)$ for all $\alpha \in A$. The subgroup of A preserving π we will denote by $A[\pi]$. A is primitive if it only preserves the trivial equivalence relations X and id. We say the relation π contains π' if $x\pi'y$ implies $x\pi y$ for all x and y in X, denoted by $\pi' \leq \pi$. If π is an equivalence relation then X/π denotes the equivalence classes of π. The restriction of an equivalence relation π to Y is denoted by $\pi \upharpoonright Y$. Formally, $\pi \upharpoonright Y$ is defined by $x \; \pi\upharpoonright Y y$ if $x,y \notin Y$ or $x,y \in Y$ and $x\pi y$. If $Y \subseteq X$ then Y/π are the equivalence classes of π restricted to Y.

The equivalence classes of the relation π defined by, $x\pi y$ if for some $\alpha \in A\, \alpha(x) = y$, are called the orbits of A. The induced action of A on some orbit Y is the image of A in Sym(Y) which we identify with the quotient group A/A(Y). In general we shall let $A \upharpoonright Y$ denote the faithful action of A[Y] on Y, i.e. A[Y]/A(Y).

An isomorphism is a surjective map which sends edges to edges and vertices to vertices, and preserves incidence and other possible structure. Groups will be permutation groups and they will act from the left.

2.3 Towers of Γ_k-Actions

Throughout this paper we shall either restrict the groups considered or the way they may act.

Definition 2.3. For $k \geq 2$, let Γ_k denote the class of groups A such that all the composition factors of A are subgroups of S_k.

We shall use the following fact about the primitive actions of Γ_k groups.

Theorem 2.4 [BCP ta]. There is a function $\lambda(k)$ such that any primitive action of $A \in \Gamma_k$ of degree n has order at most n^{λ}.

The main theorem seems to require that we allow all groups but restrict the way in which they can act. Even in the case of cubic graphs the edge stabilizer is a 2-group but the full group may be arbitrary.

Definition 2.5. A group A acting on a set X is a Γ_k-action if for all $x \in X$ the subgroup $A(x) \in \Gamma_k$.

We extend the notation of a Γ_k-action to a tower of such actions. The sequence $(\pi_0,\ldots,\pi_t)$ is a tower of equivalence relations on X if $X = \pi_0 \geq \ldots \geq \pi_t = \mathrm{id}$. We shall often write a tower as $(\pi_1\ldots,\pi_t)$ where it is understood that $\pi_0 = X$. This gives a useful generalization of Γ_k-actions.

Definition 2.6. $(A,\pi_0\ldots,\pi_t)$ is a tower of Γ_k-actions if:

(1) the sequence $(\pi_0,\ldots,\pi_t)$ is a tower of equivalence relations on X;

(2) $A \subseteq \mathrm{Sym}(X)$;

(3) A preserves π_i for $0 \leq i \leq t$;

(4) for each $Y \in X/\pi_i$, $0 \leq i \leq t$, the action of A[Y] on Y/π_{i+1} is a Γ_k-action.

We write (4) more formally.

(4') If $S \in X/\pi_i$ and $T \in S/\pi_{i+1}$ then $A[T]/A(\pi_{i+1} \restriction S) \in \Gamma_k$.

It is easy to see that (4) and (4') are equivalent since the natural homomorphism from $A[T]$ into $\mathrm{Sym}(S/\pi_{i+1})$ has kernal $A(\pi_{i+1} \restriction S)$.

We shall prove some simple closure properties about towers of Γ_k actions. We state them as lemmas.

Lemma 2.7. If $(A,\pi_0,\ldots,\pi_t)$ is a tower of Γ_k actions and $B \leq A$ then $(B,\pi_0,\ldots,\pi_t)$ is a tower of Γ_k actions.

Proof. It is clear that $(B,\pi_0,\ldots,\pi_t)$ satisfies the first three conditions. We show it satisfies (4'). Let $S \in X/\pi_i$ and $T \in S/\pi_{i+1}$. We must show that $B[T]/B(\pi_{i+1} \restriction S) \in \Gamma_k$. But, $B[T] \leq A[T]$ and $B(\pi_{i+1} \restriction S) = B[T] \cap A(\pi_{i+1} \restriction S)$. So by the Second Isomorphism Theorem and the Correspondence Theorem $B[T]/A(\pi_{i+1} \restriction S) \cap B[T]$ is isomorphic to a subgroup of $A[T]/A(\pi_{i+1} \restriction S)$. The latter quotient group is in Γ_k. Therefore, the first quotient group is in Γ_k, since Γ_k is closed under taking subgroups.

Lemma 2.8. If $(A,\pi_0,\ldots\pi_t)$ is a tower of Γ_k actions and $Y \subseteq X = \pi_0$ then $(A \restriction Y,\ \pi_0 \restriction Y,\ \ldots,\ \pi_t \restriction Y)$ is a tower of Γ_k actions.

Proof. Since $A \restriction Y = A[Y] \restriction Y$ and by the previous lemma, towers of Γ_k actions are closed under taking subgroup. We may assume that $A = A[Y]$. Let $S' \in Y/\pi_i$, $0 \leq i \leq t$, and $T' \in S'/\pi_{i+1}$. Since $S',T' \neq \phi$ and they are subsets of equivalence classes of π_i and π_{i+1} respectively, there exists unique elements $S \in X/\pi_i$ and $T \in S/\pi_{i+1}$ containing S' and T', respectively. We must show that $A[T']/A(\pi_i \restriction S') \in \Gamma_k$. We have the following chain of inclusions

$$A[T] = A[T'] \geq A(\pi_i \restriction S') \geq A(\pi_i \restriction S) .$$

So our quotient is a section of a Γ_k group and thus Γ_k.

3. CANONICAL ORDERING OF SETS AND SET OF ORDERINGS

3.1 Canonical Forms

As in [BL83, FSS83] we say a function $CF:K \to K$, where K is a class of graphs, is a canonical form for K if:

(i) for G in K, $CF(G) \cong G$

(ii) for G,H in K, $G \cong H$ if and only if $CF(G) = CF(H)$.

In the previous papers, they considered linearly ordering the vertices as a canonical form or presentation. We shall order edges. For each edge e we shall consider it as oriented, i.e. e^{-1}, the reverse of e, is distinct from e. Formally, a canonical form of a graph G will be a linear ordering of the oriented edges of G which satisfies conditions (i) and (ii). We shall often let I,J be maps from $\{1,\dots,|E|\}$ to E. In general, I will be a map from $\{1,\dots,|X|\}$ to X, in which case we will call I an ordering of X. If $A \subseteq \mathrm{Sym}(X)$ then the pair (I,A) determines the set of ordering $\{aI \mid a \in A\}$. The algorithms considered will return all equivalent ordering, i.e. pairs (I,A). We consider several canonization problems.

Problem 3.1. String canonical forms for towers of Γ_k-actions.

Input: A colored set X with orderings (I,A) where A is given as a tower of Γ_k-actions, $(A,\pi_0,\dots,\pi_t)$.

Find:

(1) $\alpha \in A$ such that αI is canonical;

(2) group $B \subseteq A$ of elements which preserve coloring.

Formally, an algorithm CF satisfies condition (1) if whenever CF on $(I,A,\pi_0,\dots,\pi_t)$ returns αI and CF on $(\beta I,A,\pi_0,\dots,\pi_t)$ returns γI for $\beta \in A$ then $\alpha\gamma^{-1} \in B$. Thus the canonical form may be a function of $(\pi_0,\dots,\pi_t)$. The ordering I is said to be $\leq$ the order J if (X with ordering I) is lexigraphically before (X with ordering J).

Problem 3.2. Graph canonical forms restricted to a tower of Γ_k-actions.

Input: A graph G with ordering I of V(G) and a tower of Γ_k-actions $(A,\pi_0,\dots,\pi_t)$ on V(G).

Find: (1) $\alpha \in A$ such that αI is a canonical ordering of V(G);

(2) group $B \subseteq A$ of automorphisms of G in A.

We list a third problem which we will not use in this paper but whose polynomial time solution will follow easily from the ideas in this paper and the ideas in [Mlta] and may have application elsewhere.

Problem 3.3. Hypergraph canonical forms restricted to tower of Γ_k-actions.

In problem 3.3 the graph in problem 3.2 is replaced by a hypergraph.

Before presenting the fourth problem of this section we give a polynomial time solution to problem 3.1. This algorithm gives:

Theorem 3.4. String canonical forms for towers of Γ_k-actions is polynomial time constructable for fixed k.

Let $(A,\pi_0,\dots,\pi_t)$ be a tower of Γ_k-actions on a colored set X. Further, let I be an ordering of X.

We consider two procedures, CF which returns the canonical ordering and C which returns the group of symmetries. If $S \subseteq X$ we define C and CF as follows:

(i) $C_S(A) = \{\alpha \in A | \alpha$ preserves color on $S\}$;

(ii) $CF_S(I,A)$ = canonical αI, $\alpha \in A$, for coloring on S only.

Space does not permit giving the procedure for C but it follows easily from that of CF.

The algorithm will have a recursive form similar to [Lu80, BL83]. The procedure has two phases. The first phase is used to reduce the group's action on S/π_i to a Γ_k group. While the second phase just applies an algorithm similar to the color canonialization algorithm of [BL83] to an action which is Γ_k. In the procedure X, I, $\pi_0, \ldots, \pi_t$, and t will be global variables and S will be an A-stable subset of X.

<u>Procedure</u>: $CF_S(I,A,i)$.

<u>Begin</u>: (1) <u>If</u> $S = \{X\}$ for some $x \in X$ <u>then return</u> I;

(2) <u>If</u> A is not transitive on S <u>then</u>

(a) "Canonically" pick an A-stable partition of S w.r.t. I, say, S_1, S_2;

(b) <u>Return</u> $CF_{S_2}(CF_{S_1}(I,A,i), C_{S_1}(A),i)$;

(3) If $S \subseteq T \in X/\pi_i$ for some T <u>then</u>

(a) "Canonically" pick $S' \in S/\pi_{i+1}$ w.r.t. I;

(b) Compute $A' = A[S']$ and right coset representatives of A' in A, say, $\{\sigma_1, \ldots, \sigma_\ell\}$;

(c) <u>Return</u> $\operatorname{Min}_j\{CF_S(\sigma_j I, A', i+1)\}$;

(4) (a) Find a canonical primitive block system w.r.t. I on S, say, $\pi \geq \pi_i \mid S$;

(b) Compute $A' = A(\pi)$ and right coset representatives $\{\sigma_1, \ldots, \sigma_\ell\}$ of A' in A;

(c) <u>Return</u> $\operatorname{Min}_j\{CF_S(\sigma_j I, A', i)\}$.

<u>End</u>

Since the algorithm parallels earlier algorithms it follows that $CF_X(I,A,0)$ will compute a string canonical form.

To see that CF runs in polynomial time we must bound the size of ℓ whenever step (4) is implemented. We state three facts which follow for any recursive call of CF, say, $CF(\sigma,A,i)$ by lemmas 2.7 and 2.8.

(i) $(A \upharpoonright S, \pi_0 \upharpoonright S, \ldots, \pi_t \upharpoonright S)$ is a tower of Γ_k-actions;
(ii) If $|S/\pi_i| > 1$ then $A \upharpoonright S/\pi_i$ is in Γ_k;
(iii) If $|S/\pi_i| = 1$ then $A \upharpoonright S/\pi_{i+1}$ is a Γ_k-action.

By a straightforward calculation one can show that the number of recursive calls is bounded by $n^{\lambda+2}$, where λ is the constant from theorem 2.4. This completes the proof of theorem 3.4.

To apply the solution to the string canonical forms to the graph canonical forms, we simply consider the group as acting on pairs of vertices. Since edges are pairs of vertices we color these pairs according to whether they are edges or nonedges. But this is the string canonical form problem. We need only show that a tower of Γ_k-actions on X can be "lifted" to a tower of Γ_k-actions on X^2.

We state this lifting in a slightly stronger form which we will need later. If π is a relation on X and τ is a relation of Y then define $\pi \cdot \tau$ on $X \times Y$ by $(x,y)\pi\cdot\tau(x',y')$ if $x\pi x'$ and $y\tau y'$. Thus, a polynomial time algorithm follows for problem 3.2 from the following lemma.

Lemma 3.5. If $(A,X,\pi_1,\ldots,\pi_t)$ and $(B,Y,\pi_1,\ldots,\pi_t)$ are towers of Γ_k-actions then so is $(A\times B, X\times Y, \pi_1\cdot\tau_1,\ldots,\pi_t\cdot\tau_t)$.

3.2 Canonical Group Intersection

In this section we consider the problem of canonically intersecting two permutation groups. The solution again will parallel the work of [Lu80, BL83]. In practice the two groups may not be acting on the same set. Thus, we consider a slight generalization of the problem.

Let A act on X and B act on Y. The amalgamated intersection of A and B or simply the intersection of A and B, $A \cap B$, is the group of elements in $\mathrm{Sym}(X \cup Y)$ which stabilizes X and Y and whose action on X is in A and on Y is in B.

We shall say a function CIN is canonical form for $A \cap B$ if it returns ordering satisfying condition (1).

(1) $\mathrm{CIN}(I,J,A,B) = (\alpha I,\beta J)$ and $\mathrm{CIN}(aI,bJ,A,B) = (\alpha' I,\beta' J)$ implies $\alpha'\alpha^{-1} \in A$ and $\beta'\beta^{-1} \in B$ for $a \in A$ and $b \in B$.

This gives the fourth problem.

Problem 3.6. Canonical intersection of an arbitrary group with a tower of Γ_k-actions.

Input: Two orderings (I,A) and (J,B) where A is given as a tower of Γ_k-actions.

<u>Find:</u> $\alpha \in A$ and $\beta \in B$ such that $(\alpha I, \beta J)$ is canonical.

We do not require solutions to canonical intersections to return a linear ordering of $X \cup Y$. But, there are many natural orderings of $X \cup Y$ determined by αI and βJ. For the applications here we find the two orderings more natural.

The algorithm for problem 3.6 will be a combination of the group intersection algorithm in [Lu80] and the canonization algorithms of [BL83, FSS83].

We say $(I,J) \leq (I_1,J_1)$ if I and I_1 first enumerate $X \wedge Y$ and $J^{-1}I$ is lexographically before $J_1^{-1}I_1$.

The problem of simply computing the intersection of A and B when A is a tower of Γ_k-actions is also polynomial computable. The algorithm proposed is a natural combination of the algorithm for color symmetries in a tower of Γ_k-actions and the group intersection algorithm in [Lu80] for Γ_k groups. Due to space constraints we will not present it here. The canonization procedure will call this algorithm.

Let C be a subgroup of $A \times B$ acting on $X \times Y$. Let $Pr_1(C)$ $Pr_2(C)$ be the projections of C on X and Y, respectively. Consider the following function on C for subsets Z of $X \cap Y$.

$$IN_Z(C) = \{(\alpha,\beta) \in C | \alpha \upharpoonright Z = \beta \upharpoonright Z\} .$$

Similar to IN_Z we extend CIN. We have $CIN_Z(I,J,C)$ return $(\alpha I, \beta J) \in C$ which is canonical w.r.t. $C_Z = \{(a,b) \in C | a \upharpoonright Z = b \upharpoonright Z\}$. We give a list of recursive properties of CIN_Z which easily produces the desired polynomial time algorithm.

<u>(3.7).</u> If $Pr_1(C)$ is not transitive on Z with stable canonical partition Z_1, Z_2, then:

$$CIN_Z(I,J,C) = CIN_{Z_2}(CIN_{Z_1}(I,J,C),\ IN_{Z_1}(C)) .$$

<u>(3.8).</u> If $A^* \subseteq Pr_1(C)$ with right coset representatives $\{\alpha_1,\ldots,\alpha_\ell\}$ in $Pr_1(C)$ then:

$$CIN_Z(I,J,C) = \mathrm{Min}\{CIN_Z(\alpha_i I, \beta_i J, C^*)\}$$

where $(\alpha_i,\beta_i) \in C$ and $C^* = \{(\alpha,\beta) \in C | \alpha \in A^*\}$.

<u>(3.9).</u> If $Z = \{z\}$ then:

$$CIN_Z(I,J,C) = \mathrm{Min}\{(\alpha_i I, \beta_i J)\}$$

where (α_i, β_j) are right coset representatives of $C((z,z))$ in C.

Using these three identities about CIN and the recursive control structure of the color canonicalization algorithm we can compute the intersection and the canonical intersection in $n^{\lambda+2}$ recursive calls of identity (3.9). The stabilizer of (z,z) in C can be computed using Sim's algorithm as analyzed in [FHL].

In the case that B is also a tower of Γ_k-actions then we can return not only the intersection but a tower of Γ_k-actions. The following lemma will suffice.

Lemma 3.10. If $(A,X,\pi_1,\ldots,\pi_t)$ and $(B,Y,\tau_1,\ldots,\tau_t)$ are towers of Γ_k-actions then $(A \cap B,\ X \cap Y,\ \pi_1 \cap \tau_1,\ldots,\pi_k \cap \tau_k)$ is a tower of Γ_k-actions.

4. SYMMETRIES OF A VERTEX GIVEN BY A GROUP

In [MI79] we discussed gadgets — graphs which were used to denote symmetries or as a data structure for symmetries. Here we reverse those ideas and replace bridges or gadgets of a graph by a group or coset which will represent the symmetries of the frontier of a bridge. We shall apply these ideas to isomorphism testing and canonical forms for k-contractable graphs. Here, we present an algorithm which under certain conditions tests isomorphism of graphs where the vertices have specified symmetries. We make these notations precise in what follows. It seems crucial that the graphs considered have multiple edges. Throughout this section the graphs are assumed to have multiple edges.

Definition 4.1. A graph with specified symmetry is a graph $G = (V,E)$, a set of ordering (I_v, A_v) for each set $E(v)$, $v \in V$, and a consistent partition P of the vertices G. The partition P is consistent if vPw implies that $I_w^{-1}A_wI_w = I_v^{-1}A_vI_v$.

We next consider restriction on the way the group at each vertex can act.

Definition 4.2. A graph G has its vertex symmetries given by towers of Γ_k-actions if G has specified symmetry where A_v is given in the form $(A_v,\pi_1,\ldots,\pi_t)$, a tower of Γ_k-actions, and the partition P is consistent with this structure. We shall say the symmetries are Γ_k on the multiple edges if for every vertex $v \in V(G)$ the induced action of A_v on $E(v)/ME$ is a Γ_k-action. We shall often refer to these graphs as simply Γ_k-graphs. An ismorphism of G must preserve this structure.

Theorem 4.3. Isomorphism and canonical form for Γ_k graphs are polynomial time constructable for fixed k.

The assumption that the symmetries are Γ_k-actions on the multiple edge relation will insure that the induced action on vertices will be tractable. It would be interesting to know if this constraint can be dropped. By identifying multiple edges the symmetries become Γ_k-actions. In this case, the edge stabilizer of a connected graph is a group in Γ_k. We state this well-known fact as a lemma.

Lemma 4.4. If G is connected with specified symmetries which are Γ_k-actions then the automorphisms of G which stabilize an edge form a group in Γ_k.

We explain the canonical form algorithm here. It will use the leveling idea. By standard techniques we may assume the graph G is connected. For each edge e in E(G) we compute the canonical form from e. We shall level the edges and vertices by their distance from e. Using this leveling we shall inductively construct the partial automorphisms from edges to edges and from vertices to vertices. We begin the formal construction.

Label the edge e' of G with the integer which is the distance (the number of vertices in a shortest path) from e to the edge e', e.g. e is labeled 0. Vertices of G are labeled with the integer which is the number of edges they are from e, the end points of e are labeled 1. An edge is even if both end points are labeled the same, otherwise it is odd.

Let G_i be the induced graph on edges labeled $\le i$. That is, G_i is the graph on vertices labeled $\le i + 1$ and edges labeled $\le i$. The graph $\bar{G}_i$ consists of: (1) G_i; (2) all odd edges labeled $i + 1$ where the end-point labeled $i + 2$ has been replaced with a new distinct vertex for each edge; (3) two copies of each even edge labeled $i + 1$ where one copy is attached to one end point labeled $i + 1$ and the other copy is attached to the other end point labeled $i + 1$. Again, the other end-point of these even edges is a new vertex. The vertex symmetries for vertices labeled $\le i$ of G_i will be those of G while the vertices labelled $i + 1$ will have no restriction on symmetries. The vertex symmetries of $\bar{G}_i$ for vertices labeled $\le i + 1$ will be those of G. Again, the symmetri of vertices labeled $i + 2$ will not be constrained.

Let $\mathrm{Auto}(G_i)$ be the automorphisms of G_i which fix e. Similarly, $\mathrm{Auto}(\bar{G}_i)$ are the automorphisms of $\bar{G}_i$ which fix e.

We shall need inductively two conditions or facts concerning the isomorphism. First, that the automorphisms of G_i and $\bar{G}_i$ which fix e acting on the edges are written as a tower of Γ_k-actions. Second, the automorphism of G_i leaving e fixed acting on the vertices is in Γ_k. We consider the second condition first. If we identify multiple edges of $\bar{G}_{i-1}$ but not multiple copies of the vertices and the symmetries are those induced from the vertices of $\bar{G}_{i-1}$, then the new graph satisfies the hypothesis of lemma 7. Therefore, the automorphism of the graph fixing the edge e are in Γ_k. Now, any automorphism of $\bar{G}_{i-1}$ induces an automorphism on this graph. Since the action on the vertex is unchanged by identifying multiple edges we have the second condition for $\bar{G}_{i-1}$. If we now also identify the multiple vertices of $\bar{G}_{i-1}$, the automorphism fixing e will still be in Γ_k. But this is the same graph we obtain by identifying the multiple edges of G_i. This proves the second condition. We shall maintain the first condition throughout the construct.

We need only give a polynomial time algorithm for constructing Auto($\bar{G}_i$) from Auto(G_i) and constructing Auto(G_{i+1}) from Auto($\bar{G}_i$) where the groups are given as towers of Γ_k-actions. We consider the latter case first.

The elements of Auto(G_{i+1}) are simply those elements of Auto($\bar{G}_i$) which preserve multiple copies of vertices and edges. That is, they preserve the relation $a \equiv b$, a and b are copies of the same edge or vertex. We obtain the canonical form for this relation by applying the graph canonical form restricted to a tower of Γ_k-actions algorithm (3.2). Here the graph has vertices consisting of the vertices and edges of $\bar{G}_i$ and the edges are the multiple copies relation. Since towers of Γ_k-actions are closed under taking subgroups the first condition is inductively satisfied for Auto(G_{i+1}).

We have left the construction of Auto($\bar{G}_i$) from Auto(G_i) which we must show is a tower of Γ_k-actions. Let $(A,\pi_1,\ldots,\pi_t)$ = Auto(G_i), a tower of Γ_k-actions. Let E_i be the edges of $\bar{G}_i$ which are common to a vertex labeled i. The maps on E_i which preserve the symmetry of vertices labeled i can be written as a direct product of wreath products since the symmetries are consistent. That is, it will be of form $\Pi A_{ij}\mathrm{Wr}S$. Let B be this group. The group Auto($\bar{G}_i$) will be the amalgamated intersection of A and B. By (3.6) this intersection and its canonical form are constructable in polynomial time. To show that the intersection can be written as a tower of Γ_k-actions we note the following. Let π_{ij} be the i^{th} equivalence relation of vertex j. We claim that $\tau_i = \bigcap_j \pi_{ij}$, where V_j has label $\leq i + 1$, form a tower for $A \cap B$. By induction, assume it is true for A where the intersection is taken over vertices labeled $\leq$ i. In the construction of B we used the symmetric group in the wreath product. But we know that the induced action on the vertices labeled i + 1 given by A is in Γ_k. So we may restrict the wreath product to this Γ_k group. Let B' be this smaller wreath product. Now, $(B', \tau_1', \ldots, \tau_t')$ is a tower of Γ_k-actions where $\tau_i' = \bigcap_j \tau_{ij}$, V_j has label i + 1. This proves the theorem.

We shall need that the full group of automorphisms of G can be written as a tower of Γ_k-actions.

Let G be a graph whose vertex symmetries are given by towers of Γ_k-actions and these symmetries are Γ_k on the multiple edges. Suppose the vertex symmetries of G are $(A_i, \pi_{1i}, \ldots, \pi_{ti})$ for vertex V_i. Let ME be the multiple edge relation on G. In this case the automorphisms of G are a tower of Γ_k-actions.

<u>Lemma 4.5</u>. If G and $(A_i, \pi_{1i}, \ldots, \pi_{ti})$, are as above and A is the group of automorphisms of G acting on the edges, then $(A, ME, \tau_1, \ldots, \tau_t)$, where $\tau_i = (\bigcap_j \pi_{ij}) \cap ME$ for $v_i \in V(G)$, is a tower of Γ_k-actions.

5. k-CONTRACTABLE GRAPHS

In this section we simultaneously define the valence k-contractable graphs and present a polynomial time algorithm for testing isomorphisms of these graphs. We define the valence k-contractable graphs via a decomposition algorithm. The definition is unsatisfactory since small perturbations may result in a different class of graphs. We leave it as an open problem to find a satisfactory definition. The definition is sufficient enough so that any perturbation will still contain the graphs of bounded genus and bounded valence. Let G be a graph with prescribed symmetries either given by towers of Γ_k-actions or unconstrained. We shall assume that two unconstrained vertices have at most one edge between them. We shall call these graphs <u>Γ_k-contractable graphs</u>.

We shall say a vertex v is Γ_k if the symmetries of v induce a Γ_k-action on the multiple edge relation ME; otherwise it is not Γ_k. Br is a <u>Γ_k-bridge</u> if it is a bridge of (G,Y) where Y is the set of vertices which are not Γ_k in G. A Γ_k-bridge is not formally a Γ_k-graph for two reasons. First, the unconstrained vertices have no associated tower of relations. This problem is remedied by allowing these unconstrained vertices to "inherit" the symmetries from their neighbors. Note that each neighbor v' of an unconstrained vertex v either shares only a single edge with v' or else its symmetries are a tower of Γ_k-

actions. Thus, the symmetries of an unconstrained vertex can be restricted to a product of wreath products where the wreath products are of the form AW_rS_k, where A is the inherited symmetry from a neighboring vertex and $k' \le k$. Second, we have not specified the symmetry of frontier vertices. We transform Br into a modified bridge so that it has the form of a Γ_k-graph. For each vertex v of Br common to a frontier vertex v' of Br, we introduce a new copy of the frontier vertex $\bar{v}'$ and have edges between v and v' go between v and $\bar{v}'$. We now view these new frontier vertices as unconstrained vertices and allow them to inherit the symmetries of their neighbor. Let $\bar{B}r$ be the bridge obtained from Br by the above construction.

Given any Γ_k-bridge Br, we can compute the automorphisms and canonical form by first constructing $\bar{B}r$ and applying the isomorphism test for Γ_k-graphs. This procedure will return with a group in the form of a tower of Γ_k-actions. Applying (3.2) we can compute the subgroup and canonical form of Auto($\bar{B}r$) which sends multiple copies of vertices to corresponding multiple copies of vertices. This will be Auto(Br) and its canonical form.

Lemma 5.1. The automorphism and canonical form for a Γ_k-bridge is polynomial time constructable and the automorphisms can be written as a tower of Γ_k-actions.

This gives a natural decomposition of Γ_k-contractable graphs, say G. For each Γ_k-bridge Br of G we identify the internal vertices of Br and remove selfloops. We denote this graph by Contract(G).

The vertex symmetries of this new vertex will be the induced action of Auto(Br) on the frontier edges. This will be a tower of Γ_k-actions. We call this graph Contract$_k$(G) which is a Γ_k-contractable graph. By standard argument Contract$_k$(G) is a canonical operation on G.

If we apply Contract$_k$ to a tree of valence > k + 1 then the procedure will simply return the original graph. Graphs that are sent to themselves under a procedure will be called fixed points. We introduce a decomposition procedure analogous to the tree isomorphism algorithm. The procedure Remove Leaves will remove leaves and let their neighbors "inherit" their symmetries. The details will appear in the final paper.

We introduce a third reduction which corresponds to a generalized 3-connected decomposition.

Two vertices of valence 2, ignoring multiple edges, are multilpe if their neighboring vertices are the same. This gives a natural equivalence relation on valence 2 vertices. An equivalence class will simply be called a set of multiple vertices. Let $v_1,\ldots,v_\ell$ be a set of multiple vertices of G. Multiple vertices can be identified with the new vertex "inheriting" the symmetries of the multiple vertices and their common neighbors. The third reduction is called Remove Multiple Vertices. Again, the details will appear in the final paper.

The graph G is a fixed point of these three reductions if it is a fixed point of each reduction. Note that we may arrive at a different fixed point depending on the order which we apply these reductions. For specificity's sake, suppose we consider fixed points of the following procedure.

Procedure: $\text{Reduction}_k(G)$.

If G = Remove Leaves(G) *then*
G ← Remove Leaves(G)
else if G ≠ Remove Multiple Vertices(G) *then*
G ← Remove Multiple Vertices(G)
else G ← $\text{Contract}_k(G)$.

Definition 5.2. G is a k-contractable graph if successive application of reduction applied to G yields a singleton.

From the discussion above we get

Theorem 5.3. Isomorphism for k-contractable graphs is polynomial time testable for fixed k.

6. THE BOUNDED GENUS CASE

Here we shall show that graphs of bounded genus are k-contractable graphs. Thus, demonstrating that the k-contractable graphs form a class of graphs which is a common generalization of the bounded valence and the bounded genus graphs. The containment will follow by showing that the fixed graphs under the reduction operation have the property that their genus grows linearly in k. Throughout the rest of the discussion let G be a fixed point. Since neither the genus nor the fact that G is a fixed point is effected by multiple edges, we may assume without loss of generality that G has no multiple edges. Similarly, we may assume that G has no vertices of valence 2. Not all vertices of G will have valence $\geq k + 1$. Let S be the set of vertices of G with valence $\leq k$. Then, S will be an independent set. Let v' denote the number of vertices of G in V-S. We first state a relationship between genus and k-contractable graphs as defined in the previous sections. We shall actually prove a slightly stronger result for a stronger notion of k-contractable.

Theorem 6.1. If G is a fixed point of Reduction_{k-1} then the genus of G,g satisfies

$$2g \geq ((k/2 - 6)/6)v' + 2 .$$

To see that such a large value of k is necessary consider the infinite tiling of the plane in diagram 1. Since this tiling is periodic we can construct arbitrarily large graphs of genus 1 which are fixed points of Reduction_{11}.

Let V' denote the vertices V-S. We distinguish two types of edges of G. The edges between points of V' will be called type A edges and those between V' and S will be called type B. Since we can easily distinguish these two types we can include in Reduction a procedure which restricts the symmetries at each vertex to the coset which preserves type. Call this new reduction procedure Reduction'. For $\text{Reduction}'_6$ the example from diagram 1 will contract to a point. For Reduction' we get the following result.

Theorem 6.2. If G is a fixed point of $\text{Reduction}'_k$ then the genus of G,g satisfies

$$2g \geq ((k - 6)/6)V' + 2 .$$

Note that we are using $\text{Reduction}'_k$ since fixing one edge at a vertex will only in general effect one of the two types of edges at that point. This gives the following corollary for k-contractable graphs with respect to Reduction'.

Corollary 6.3. If $k > 4g + 2$ and $g \geq 1$ then the k-contractable graphs include the graphs of genus g. For $g = 0$ (the planar case) $k = 5$ will suffice.

To see the corollary we simply note that $v' \geq 3$ since a fixed point can have no multiple valence 2 vertices.

Proof of Theorem 6.2. The proof uses standard counting argument based on Euler's formula $2g = 3 - f - v + 2$ where g is the genus of some embedding and e, f, and v are the numbers of edges, faces, and vertices, respectively.

Let G be a graph with some fixed embedding. Further assume G is fixed by Reduction$'_k$. Without loss of generality we may add new vertices of type S and new edges of type A as long as the genus is unchanged. If any vertex in V' has two consecutive edges of type A we may add to this face a valence 3 vertex of type S. On the other hand, if a vertex $v \in V'$ has two consecutive type B edges, then we may add a type A edge across this face common to v. Here we use the fact that S is an independent set. Without loss of generality we may assume that the edges at each vertex of V' are alternately of type A and then type B. So, if a equals the number of type A edges and b equals the number of type B edges then $2a = b$. Using this fact one can show:

<u>Lemma 6.4</u>. $f \le a + b/2 = 3a$.

Using the following facts: (1) $|V'| + |S| = V$, (2) $|S| \le b/3$, (3) $e = b + a$, (4) $b \ge k|V'|$, we get theorem 6.2.

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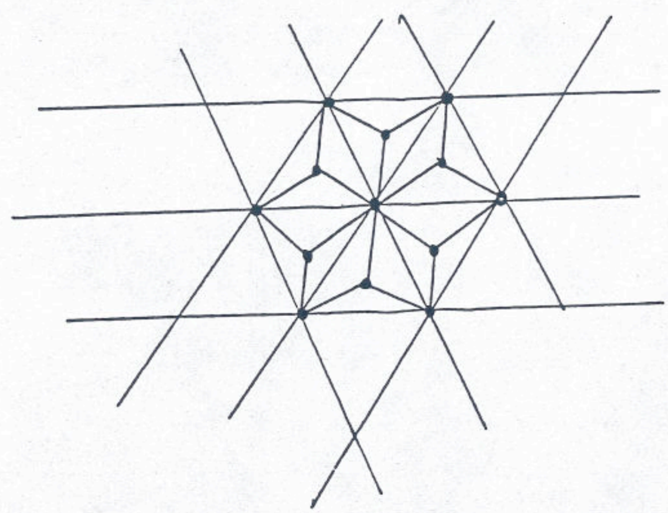

Diagram 1.

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