A Unified Geometric Approach to Graph Separators

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Abstract

We propose a class of graphs called k-overlap graphs. Special cases of k-overlap graphs include planar graphs, k-nearest neighbor graphs, and earlier classes of graphs associated with finite element methods. We prove a separator bound of $O(k^{1/d}N^{(d-1)/d})$ for k-overlap graphs embedded in d dimensions. This result unifies several earlier separator results including Lipton and Tarjan's 1979 result for planar graphs. All our arguments are based on geometric properties of embeddings. Our separator bounds come with randomized linear-time and randomized NC algorithms. Moreover, our bounds are the best possible up to the leading term.

1 Introduction

Graph partitioning is a fundamental problem in Computer Science that has many important applications including Numerical Analysis (Lipton, Rose and Tarjan;[LRT79]), VLSI design (Ullman;[Ull84]) and even Turing machine theory (Paterson;[Pat72]).

Recently several groups of authors (Vavasis;[Vav90], Miller and Thurston;[MT90b], Miller and Vavasis;[MV91]) have proposed classes of graphs that can be embedded in d dimensions and that have $O(N^{(d-1)/d})$ *separators.* "$O(N^{d-1/d})$ separators" means that for an N-node graph in the class, there exists a subset of nodes of size $O(N^{(d-1)/d})$ whose removal disconnects the graphs into two roughly equally-sized components. See below for the formal definition. For the applications mentioned in the last paragraph, $d = 2$ and $d = 3$ are the interesting cases, in which case the bounds are $O(N^{1/2})$ and $O(N^{2/3})$ respectively.

All of these earlier classes of graphs have the disadvantage that, when specialized to two dimensions, they apparently do not contain all planar graphs. This is a serious drawback because the earliest and best-known separator result is Lipton and Tarjan's, [LT79], theorem that all planar graphs have $O(N^{1/2})$ separators. Moreover, these classes contained only graphs with bounded degree.

In this report we propose a new class of graphs, *overlap graphs*, with the following properties:

1. In two dimensions, planar graphs are special cases of overlap graphs.

2. In d dimensions for $d \geq 2$. any finite subgraph of the infinite d-dimensional grid graph is an overlap graph.

3. The overlap graphs in d-dimensions have $O(N^{(d-1)/d})$ separators.

To our knowledge, this is the first time that a class of graphs has been proposed with these three very natural properties. In addition. as we argue below, overlap graphs include Miller and Vavasis's density graphs as a special case. The proof that planar graphs are special cases of overlap graphs relies on recent deep theorems by Andreev and Thurston [And70a, And70b, Thu88] characterizing all planar graphs in a novel geometric fashion.

Our proof techniques are novel and has the potential of being applied to other problems. Our bounds for density graphs are better than Miller and Vavasis's bounds and in fact achieve matching lower bounds except for low-order terms. In order to achieve tight bounds. we use arguments that take advantage of slight differences between the various p-norms when applied to high-dimensional vectors, a technique that appears to be new and is interesting on its own.

Finally, we will argue that a generalization of overlap graphs, called *k-overlap graphs*. include k-nearest-neighbor graphs as a special case and our bound is also optimal in terms of k.

2 Definitions

The notion of a separator introduced in the last section has been in well-known since 1979. The following definition formalizes this idea and introduces the notation that will be used for the remainder of the paper.

Definition 2.1 (Separators) *A subset of vertices C of a graph G with n vertices is an $f(n)$–* **separator** *that* **δ-splits** *if the vertices of $G - C$ can be partitioned into two sets A and B such that there are no edges from A to B, $|A|$, $|B| \leq \delta n$, and $|C| \leq f(n)$, where f is a function and $0 < \delta < 1$.*

Separator results for families of graphs closed under the subgraph operation immediately lead to divide-and-conquer recursive algorithms for a variety of applications. In general, the efficiency of such algorithms depends on a δ bounded away from 1 and $f(n)$ a slowly-growing function.

Two of the most well-known families of graphs which have small separators are trees and planar graphs. It is known that a tree has a single vertex separator that 2/3–splits. Lipton and Tarjan [LT79] proved that any planar graph has a $\sqrt{8n}$–separators that 2/3–splits. They also give a linear time algorithm for finding such a separator. Many interesting extensions of this work have been made [Dji82, Mil86, Gaz86, GMS7] and separator theorem had also been obtained also for graphs with bounded genus [GHT82. HM86]. Very recently, Alon, Seymour and Thomas [AST90] proved the following interesting separator theorem: all graphs with no minor isomorphic to the h–clique have an $h^{3/2}\sqrt{n}$–separator. Many applications of separator theorems have been given for VLSI layout [Lei83a, Lei83b], finite element method and numerical analysis [LRT79, PR85].

The development of computational geometry and numerical analysis calls for deeper understanding of separator properties for graphs embedded in fixed dimensional space. especially in 2–space and 3–space. Although, the planar separator theorem is applicable to many interesting families of graphs embedded in 2-space, we shall show that there are some natural graphs in 2–space, e.g., k-nearest neighborhood graphs, which are neither planar. nor with bounded genus, nor with bounded minor. In general. none of the above separator theorems are useful for graphs in 3–space.

Example 2.2 *Let G be graph formed by a $2 \times \sqrt{n} \times \sqrt{n}$ grids in 3–space (see Figure 1). Clearly, G has a $2\sqrt{n}$–separator. However, G is a graph with genus $\Omega(\sqrt{n})$. It also has a minor isomorphic to $\sqrt{n}$–clique.*

Figure 2 shows an 8–nearest neighborhood graph which has an $O(\sqrt{n})$–separator. But it has genus $\Omega(\sqrt{n})$ and a minor isomorphic to the $\sqrt{n}$–clique.

Figure 1: A 3–dimensional graph with large genus and large minor but small separator

Figure 2: An example of 8–nearest neighborhood graph in 2–space

The more recent papers mentioned last section on d-dimensional separators attempt to use geometric information in the existence proofs and separators, as opposed to the combinatorial approaches mentioned in the last few paragraphs. This paper also takes the geometric point of view. However, it introduces many new techniques.

The family of graphs is defined based on a notion of a *neighborhood system*, which induces a special case of an overlap graph.

Definition 2.3 (Neighborhood System) *Let $P = \{p_1, \ldots, p_n\}$ be points in $\mathbb{R}^d$. A k-***neighborhood system** *for P is a set, $\{B_1, \ldots, B_n\}$, of closed balls such that (1) B_i is centered at p_i and (2) For each i the interior of B_i contain at most k points from P.*

In the full paper we will discuss more general neighborhood systems.

The *intersection graph* induced by this neighborhood system is the undirected graph with one node for each ball, and an edge when two balls intersect.

We now define the main class of graphs under consideration, α-overlap graphs. For this definition, we introduce the following notation. If B is a ball of radius r in $\mathbb{R}^d$, then $\alpha \cdot B$ denotes the ball with the same center as B but radius αr.

Definition 2.4 (Overlap Graph) *Let $\alpha > 0$ and let $\{B_1, \ldots, B_n\}$ be a k-neighborhood system for $P = \{p_1, \ldots, p_n\}$. The* **(α, k)-overlap graph** *for the k-neighborhood system $\{B_1, \ldots, B_n\}$ is the undirected graph $G = (V, E)$ with $V = \{1, \ldots, n\}$ and*

$$E = \{(i,j) | (B_i \cap (\alpha \cdot B_j) \neq \emptyset) \text{ and } ((\alpha \cdot B_i) \cap B_j \neq \emptyset)\}.$$

For simplicity, we call a $(1, k)$-overlap graph a *k-intersection graph.* In the case that $\alpha = 1$ and $k = 1$, and no two balls in the neighborhood system have a common point in their interior, we have the family of graphs known as *sphere-packings*; this interesting class of graphs will be discussed later.

Note that given a set P of n points in $\mathbb{R}^d$, we can uniquely define for each point $p \in P$ the largest sphere centered at p whose interior contains at most k points of P (provided $n > k$). These balls immediately lead to an instance of a k-overlap graph, which we call the *k-overlap* graph of the points.

The following is the main theorem for our report.

Theorem 2.5 (Main) *Let G be an (α, k)-overlap graph for some fixed d. Then G has an $O(\alpha \cdot k^{1/d} \cdot n^{(d-1)/d} + q(\alpha, k, d))$-separator that $\frac{d+1}{d+2}$-splits. Further, such a separator that $\frac{d+1+\epsilon}{d+2}$-splits can be computed in random constant time, using linear number of processors, for any $1/n^{1/2d} < \epsilon < 1$.*

The function $q(\alpha, k, d)$ depends exponentially on d but is independent of n. Since the interesting cases are when $d = 2$ or $d = 3$ and when n is large, this term should be considered low order.

The remainder of the paper is organized as follows. Section 3 presents applications of overlap graphs including a a geometric proof of the planar separator theorem. Matching lower bound on the size of separator for overlap graphs is also given. Section 4 presents some important geometric lemmas which are used in the proof of the main separator theorem in Section 5. Section 6 extends the main result to non-Euclidean space. Section 7 lists some open questions.

3 Applications of Overlap Graphs

In this section, we show that the class of overlap graphs includes many natural classes of graphs as a special case. In particular, it contains the set of all planar graphs, density graphs, and k-nearest neighborhood graphs.

3.1 A Geometric Proof of the Planar Separator Theorem

Let R be a region in the plane or on the 2–sphere. A *circle packing* in R is a collection of closed disks $D_1, \ldots, D_n$ contained in R as having disjoint interiors. The *nerve* of a circle packing is the embedded 1–complex whose vertices are the centers of the disks and whose edges are the geodesic segments joining the centers of the tangent disks and passing through the point of tangency.

Theorem 3.1 (Andreev and Thurston) *For each triangulated planar graph G there is a sphere packing on the plane whose nerve is isomorphic to G.*

Therefore, every planar graph is subgraph of a 1-overlap graph in 2–space.

Corollary 3.2 *Every planar graph has an $O(\sqrt{n})$–separator.*

3.2 Separator for k-nearest Neighborhood Graphs

Let $P = \{p_1, \ldots, p_n\}$ be a set of n points in $\mathbb{R}^d$. For each $p_i \in P$ and $k \in N$, let $N_k(p_i)$ be the set of k-nearest neighbors of p_i in P (ties are broken arbitrarily).

Definition 3.3 (The k-nearest Neighborhood Graph) *A k-nearest neighborhood graph of $P = \{p_1, \ldots, p_n\}$ in $\mathbb{R}^d$, denoted by $G_{P,k}(V, E)$, is a graph with $V = \{1, ..., n\}$, and*

$$E = \{(i, j) | p_i \in N_k(p_j) \text{ or } p_j \in N_k(p_i)\}.$$

Immediately from the the definitions,

Lemma 3.4 *For any set of n points P in* $\mathbb{R}^d$*, the k-nearest neighborhood graph of P is a subgraph of the k-overlap graph defined by P.*

Corollary 3.5 *All k-nearest neighborhood graphs have an* $O\left(k^{1/d}n^{\frac{d-1}{d}}\right)$*-separator.*

We have shown that the about separator bound is tight using the following example. Let P be the set of all points of the $m \times m \times \cdots \times m$ regular grid in $\mathbb{R}^d$ and let G be the kth nearest neighborhood graph for the points P. Using the methods described in Leighton [Lei83a] we get a lower bound of $\Omega(k^{1/d}m^{d-1})$.

3.3 Separator for Density Graphs

Definition 3.6 (Density Graphs) *Let G be an undirected graph and let* π *be an embedding of its nodes in* $\mathbb{R}^d$*. Then we say that* π *is an embedding of* **density** α *if the following inequality holds for all vertices v in G. Let u be the closest node to v. Let w be the farthest node from v that is connected to v by an edge. Then*

$$\frac{||\pi(w) - \pi(v)||}{||\pi(u) - \pi(v)||} \leq \alpha.$$

In general, G is a **density graph** *if there exist a* π *and* $\alpha > 0$ *such that* π *is an embedding of density* α*.*

It has been proven by Miller and Vavasis [MV91] that all density graphs has an $O\left(\alpha^{d(d-1)}n^{\frac{d-1}{d}}\right)$-separator. Immediately from the definition,

Lemma 3.7 *Each* α*-density graphs is a subgraph of* $(\alpha, 1)$*-overlap graph.*

Corollary 3.8 *Let G be a density graph in* $\mathbb{R}^d$*. Then G has an* $O\left(\alpha \cdot n^{\frac{d-1}{d}}\right)$*-separator.*

Hence, our result greatly improves the one of Miller and Vavasis in the term of α and our bound is optimal in terms of α. This answer an open problem in [MV91] in the affirmative.

4 Some Geometric Lemmas

In this section, we state a set of basic geometric lemmas. Their proofs can be found in Appdenix A.

Define the *kissing number* τ_d be the maximum number of nonoverlapping unit balls in $\mathbb{R}^d$ that can be arranged so that they all touch a central unit ball [CS88]. For each positive real δ, let $A_d(\delta)$ be the maximum number of points that can be arranged on a unit sphere, such that the distance between each pair of points is at least δ.

Lemma 4.1 (Point Coverage Lemma) *Let* $B_1, \ldots, B_n$ *be n closed balls with centers* $p_1, \ldots, p_n$*, respectively, in* $\mathbb{R}^d$*. If for all* $1 \leq i \leq n$, $|(\text{int } B_i) \cap \{p_1, \ldots, p_n\}| \leq k$*, then for all* $p \in \mathbb{R}^d$*,*

$$|\{i : p \in B_i\}| \leq \tau_d k.$$

The following lemma can be proven similarly to that of Lemma 4.1.

Lemma 4.2 (Ball Coverage Lemma) *Let $B_1, \ldots, B_n$ be n closed balls with centers $p_1, \ldots, p_n$, respectively, in $\mathbb{R}^d$. If for all $1 \le i \le n$, $|\text{int } B_i \cap \{p_1, \ldots, p_n\}| \le k$, then for all ball $B \in \mathbb{R}^d$ (say with center p and radius r),*

$$|\{i : B_i \cap B \neq \emptyset \text{ and } p_i \in R^d - 2 \cdot B\}| \le A_d(1/2)k.$$

Proofs to the following two lemmas are omitted and will be given in the full version of the paper.

Lemma 4.3 *Let $a_1, \ldots, a_n$ be nonnegative numbers, and suppose $p \ge 1$. Then*

$$\left(\sum_{i=1}^{n} a_i\right)^p \le p\left[\sum_{i=1}^{n} a_i \left(\sum_{j=i}^{n} a_j\right)^{p-1}\right]$$

Lemma 4.4 *Let $\ldots, m_{-1}, m_0, m_1, m_2, \ldots$ be a doubly infinite sequence of nonnegative numbers such that each m_i is bounded above by θ and such that at most a finite number of m_i's are nonzero. Let $d \ge 2$ be an integer. Then*

$$\left(\sum_{k=-\infty}^{\infty} m_k 2^{-k(d-1)}\right)^{d/(d-1)} \le c_d \theta^{1/(d-1)} \sum_{k=-\infty}^{\infty} m_k 2^{-kd}$$

where c_d is a positive number depending on d.

5 Geometric constructions of the Main Theorem

By applying Theorem 2.5 with the choice that $\alpha = 1$, we obtain,

Theorem 5.1 *Let $P = \{p_1, \ldots, p_n\}$ be a set of n points in $\mathbb{R}^d$ and G be their k-overlap graph P. Then G has an $O(k^{1/d} n^{\frac{d-1}{d}})$-separator.*

For simplicity we shall focus on the proof of Theorem 5.1. Observe that these graphs may still have unbounded degree and are interesting on their own. Also, we shall show that these results are best possible in term of n, α, and k.

The basic idea to prove the Main Theorem is to first construct a real-valued function f based on the structure of the given graph G; then to show that there is a $(d-1)$-sphere S in $\mathbb{R}^d$ splits the vertices of G not on S into two sets, the interior and exterior of S, each of size at most $(\frac{d+1}{d+2})n$, such that the cost of S, denoted as $Cost_f(S)$, is bounded from above by $O\left(\alpha k^{1/d} n^{\frac{d-1}{d}}\right)$, where

$$Cost_f(S) = \int_{v \in S} (f(v))^{d-1} (dv)^{d-1};$$

Such a sphere S is called a *continuous separator* of G based on f; then to deduce a vertex separator of the underlining graph from the continuous one, such that the size of the vertex separator is linearly bounded by the cost of the continuous separator.

Notice, however, in order to deduce a vertex separator from the continuous counterpart, the continuous function f must be *faithful* in the sense that the cost of a continuous separator models faithfully the size of a vertex separator of the underlining graph. In other words, the continuous function f encodes faithfully some combinatorical properties related to separators of of the underlining graph.

The above basic idea is taken from Miller and Thurston [MT90b], however, our specific construction is quite different and more sophisticated, and it contains many noval ideas. Our construction of the real-valued function is derived from the one used by Miller and Vavasis [MV91] for density graphs. but we shall show that our construction is more 'faithful' to the structure of the underlining graphs. Because of this, our results can be applied to much larger class of graphs, as well as when applied to the density graphs, it gives the best possible dependence on the density, which improving the result of Miller and Vavasis.

5.1 Computing a Continuous Separator

Let $f(x)$ be a real valued nongative function defined on $\mathbb{R}^d$ such that f^k is integrable for all $k = 1, 2, 3, \ldots$. Such an f is called a *cost function.* The total cost of the system is

$$\text{Total–Cost}(f) = \int_{v \in \mathbb{R}^d} (f(v))^d (dv)^d \leq \infty$$

Similarly, for all $(d-1)$–sphere S, the cost of S is

$$Cost_f(S) = \int_{v \in S} (f(v))^{d-1} (dv)^{d-1}$$

A $(d-1)$-sphere S is a *δ-splitting sphere* of a set P of n distinct points in $\mathbb{R}^d$ if S splits the points of P not on S into two sets, the interior and exterior of S, each of size at most δn.

Theorem 5.2 (Miller and Thurston) *If f is a cost function on $\mathbb{R}^d$ and P a set of n distinct points in $\mathbb{R}^d$ then there is a $(\frac{d+1}{d+2})$-splitting sphere S of P such that*

$$Cost_f(S) = O\left((\text{Total-Cost}(f))^{\frac{d-1}{d}} \right)$$

Further, using a result of Miller and Teng [MT90a] for approximating center points in fixed dimension, it can be shown that

Lemma 5.3 *If f is a cost function on $\mathbb{R}^d$ and P a set of n distinct points in $\mathbb{R}^d$ then a $(\frac{d+1+\epsilon}{d+2})$-splitting sphere S of P of cost*

$$Cost_f(S) = O\left((\text{Total-Cost}(f))^{\frac{d-1}{d}} \right)$$

can be computed in random constant time, using $O(n)$ processors, where $\frac{1}{n^{1/2d}} \leq \epsilon \leq 1$.

5.2 A Cost Function for k-overlap Graphs

We now construct a cost function f for k-overlap graphs such that $\text{Total-Cost}(f) = O(k^{1/(d-1)} n)$. We shall show that our construction can be generalized for (α, k)–overlap graphs (see Appendix C).

Let $P = \{p_1, \ldots, p_n\}$ be a set of n points in $\mathbb{R}^d$ and G be the k-overlap graph defined by P. The cost we shall define is closely related to the structure and the construction of a k-overlap graph.

Recall that a k-overlap graph is defined by the geometrical relations of a neighborhood system $\mathcal{B}_1, \ldots, \mathcal{B}_n$, where $\mathcal{B}_i$ is a ball centered at p_i and contains no more than k members of P in its interior. To define the cost function for G, we first define n functions, $f_1, \ldots, f_n$, where the function f_i is defined based on $\mathcal{B}_i$, as follows: let r_i be the radius of $\mathcal{B}_i$ and let $\gamma_i = 2r_i$,

$$f_i(x) = \begin{cases} 1/\gamma_i & \text{if } ||x - p_i|| \leq \gamma_i \\ 0 & \text{otherwise} \end{cases}$$

Intuitively, f_i sets up a cost on each $(d-1)$-sphere S such that the closer S is to p_i, the larger p_i contributes to the cost of S. We say a sphere S *cuts* an edge (i, j) of G if S and $\text{seg}(p_i, p_j)$, the linear segment between p_i and p_j have a common point, denoted as $I_S(p_i, p_j)$.

Notice that if S cuts an edge (i, j) of G, then by the definition of k-overlap graph, either $I_S(p_i, p_j) \in \mathcal{B}_i$ or $I_S(p_i, p_j) \in \mathcal{B}_j$. This implies, as we shall see in Subsection 5.3, that there is a constant c such that either $\int_{v \in S} (f_i(v))^{d-1} (dv)^{d-1} \geq c$ or $\int_{v \in S} (f_j(v))^{d-1} (dv)^{d-1} \geq c$.

We say a cost function f is *faithful* if for all $(d-1)$-sphere S, for all $Q \subseteq \{1, ..., n\}$,

$$\int_{v \in S} (f(v))^{d-1} (dv)^{d-1} \geq \sum_{i \in Q} \left(\int_{v \in S} (f_i(v))^{d-1} (dv)^{d-1} \right)$$

For each $(d-1)$-sphere, let

$$M(S) = \{i | \text{there exists } p_j, \text{ such that } S \text{ cuts } (p_i, p_j) \in G \text{ and } I_S(p_i, p_j) \in \mathcal{B}_i\},$$

We can conclude from the above discussion that if a cost function f is faithful, then $|M(S)| \leq Cost_f(S)$. We shall see in Subsection 5.3 that $M(S)$ is a separator for G if S is a continuous one. Thus, the cost function should defined to be the minimum function that is faithful. Let us start with some notations.

Let $a_1, \ldots, a_n$ be n nonnegative numbers. Define the L_p^{th} *norm* of $a_1, \ldots, a_n$, denoted as $L_p(a_1, \ldots, a_n)$, to be

$$L_p(a_1, \ldots, a_n) = \left(\sum_{i=1}^{n} |a|^p \right)^{1/p}, \quad \text{where} p > 0$$

The following lemma states the relationship between different norms.

Lemma 5.4 $L_p \leq L_{p-1}$.

The cost function[1] of the k-overlap graph G is then defined to be the L_{d-1}^{st} norm of $f_1, \ldots, f_n$, i.e.,

$$f(x) = L_{d-1}(f_1, \ldots, f_n) = \left(\sum_{i=1}^{n} (f_i(x))^{d-1} \right)^{1/(d-1)}$$

Now, to prove Theorem 5.1, it is sufficient to prove

Lemma 5.5 *For any set P of n points in $\mathbb{R}^d$, if $f_1, \ldots, f_n$ and f are defined as above, then*

$$Total - Cost(f) = O(k^{1/(d-1)} n).$$

[1]Notice that in the construction of Miller and Vavasis for density graph, the cost function is defined to be the L_1 norm of the functions defined over each vertices.

Proof: Let V_d be the volume of a unit ball in $\mathbb{R}^d$. Clearly, $\int_{x\in\mathbb{R}^d}(f_i(x))^d(dx)^d = V_d$.

Consequently, letting

$$g(x) = L_d(f_1,\ldots,f_n) = \left(\sum_{i=1}^{n}(f_i(x))^d\right)^{1/d},$$

we have

$$\int_{x\in\mathbb{R}^d}(g(x))^d(dx)^d = V_d n$$

Therefore, Lemma 5.5 follows immediately from the followig lemma whose proof is given in Appendix B.

Lemma 5.6 *For all* $x \in \mathbb{R}^d$, $(g(x))^d \leq (f(x))^d \leq c_d 2^d(6^d\tau_d k)^{1/(d-1)} \cdot (g(x))^d$.

Consequently,

Lemma 5.7 *There exists a* $(\frac{d+1}{d+2})$*-splitting sphere* S *of* P *with* $Cost_f(S) = O\left(6\xi k^{1/d} n^{\frac{d-1}{d}}\right)$*, where* $\xi = 2^{d-1}(\tau_d)^{1/d}V_d$.

5.3 A Vertex Separator From A Continuous One

To complete the proof the Theorem 2.5 and 5.1, we shall construct a vertex separator C of G from a continuous separator S obtained from Lemma C.3. We then show that $|C| = O(Cost_f(S))$.

The vertices in C are those points $q \in P$ such that its ball, B_q intersects S. If the edge $(p,q) \in E$ is such that p is interior to S and q is exterior then either B_q or B_p must intersect S and thus either p or q is in C. Therefore, C $(\frac{d+1}{d+2})$-splits the points interior to S from those exterior to S, excluding those points in C. The following lemma bounds the size of $|C|$ (the proof is given in Appendix D).

Lemma 5.8

$$|C| \leq 2A(1/2)k + \tau_d k + \left(\frac{4\sqrt{7}}{7}\right)^{d-1}\frac{1}{V_{d-1}}Cost_f(S).$$

Notice that $Cost_f(S) = O\left(k^{1/d}n^{\frac{d-1}{d}}\right)$ and $k \leq n$ implies $k \leq k^{1/d}n^{\frac{d-1}{d}}$. It follows that $|C_1| = O\left(k^{1/d}n^{\frac{d-1}{d}}\right)$. This complete the proof of Theorem 5.1.

6 Non-Euclidean Neighborhood Systems

In this section, we shall show that our separator theorems for the overlap graphs of Euclidean neighborhood system can be generalized to neighborhoods determined by any norm. Thus, a **neighborhood system for a norm** $||v||$ is a collection of ball $B_i = \{p \mid ||p - p_i|| \leq r_i\}$.

Observe, that any norm determines a unique symmetric closed convex body, i.e., the unit ball centered at the origin. A body B about the origin is symmetric if $p \in B$ implies that $-p \in B$. It is also true that any symmetric convex body B determines a unique norm, namely, that norm with unit ball B.

A more general neighborhood system can be obtained from translation and dilations of any closed convex body. Due to space constraints we only discuss this more general case. In a body that is not necessarily symmetric, the first issue becomes that of defining the center of the body.

Let Γ be a bounded convex body in $\mathbb{R}^d$. Let E' be the largest ellipsoid that is contained in Γ and E the smallest one that contains Γ. It is proven by Lowner and John [Lov86] that E' and E are concentric. Moreover, E arises from E' by enlargering by a factor at most d. (E', E) is called a *Lowner-John pair* and the center O_Γ of both E' and E is *Lowner-John* center.

Theorem 6.1 *For each convex body Γ in $\mathbb{R}^d$, if G is an (α, k)-overlap graph of in $S(\Gamma)$, then G has an $O(\alpha \cdot d \cdot k^{1/d} \cdot n^{(d-1)/d} + q(\alpha, k, d))$-separator that $\frac{d+1}{d+2}$-splits.*

7 Open Questions

1. What is the computational complexity for deciding whether a graph G is k–embeddable or (α, k)–embeddable?

2. Is there a polynomial time algorithm for computing the disk packing of a planar graph.

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A Proof of the Ball Coverage Lemma

Proof: (**Lemma 4.1**): We first prove the following lemma:

Lemma A.1 *The maximum number of points that can be arranged on a unit sphere whose pairwise distance is at least* 1, *is bounded above by* τ_d.

Proof: Suppose the lemma is false, i.e., there are $t > \tau_d$ points $p_1, \ldots, p_t$ on a unit sphere with center o, such that

$$\min\{||p_i - p_j|| : 1 \leq i < j \leq t\} \geq 1.$$

Let p'_i be points on the ray op_i such that $||o - p'_i|| = 2$.

Clearly, $||p'_i - p'_j|| = 2||p_i - p_j|| \geq 2$.

Let S_i be the unit sphere centered at p'_i. Because $||p'_i - p'_j|| \geq 2$, int $S_i \cap$ int $S_j = \emptyset$ for all $1 \leq i < j \leq t$, and for all $S_1, \ldots, S_t$ touch the central unit sphere, a contradiction. □

We now prove Lemma 4.1.

Proof: We first prove the lemma in the case when $k = 1$, i.e., no ball contains the center of other balls.

Suppose Lemma 4.1 is false in this case, i.e., there is a $p \in R^d$ such that there are $t > \tau_d$ balls, without loss of generality, $B_1, \ldots, B_t$ contain p.

Without loss of generality, assume p is on the boundary of all balls $B_1, \ldots, B_t$. For otherwise, we can replace B_i by a ball C_i centered at p_i with radius $||p_i - p||$. The assumption of the Lemma is still satisfied because $C_i \subset B_i$ which implies

$$|\text{int } C_i \cap \{p_1, \ldots . p_n\}| \leq |\text{int } B_i \cap \{p_1, \ldots, p_n\}| \leq k.$$

Let $d = \min\{||p - p_i|| : 0 \leq i \leq t\}$. By proper linear transformation, we can assume $d = 1$. Let S_p be the sphere centered at p with radius $d = 1$.

Let p'_i be the intersection of the ray pp_i with the sphere S_p. Let B'_i be the ball centered at p'_i with radius d.

We observe, for all i, int $B'_i \cap \{p'_1, \ldots . p'_n\} = \{p'_i\}$.

Suppose this is not true, i.e., there exists i and $j \in \{1, ..., t\}$ such that $p'_i \in$ int B'_j. It follows that $||p'_i - p'_j|| < d$. This implies $p'_j \in$ int B'_i as well.

If $||p - p_i|| = ||p - p_j||$, then

$$||p_i - p_j|| = \frac{||p'_i - p'_j||}{||p - p'_i}||p - p_i|| < ||p - p_i||.$$

Thus $p_i \in$ int B_j and $p_j \in$ int B_i which contradicts with the assumption that $p_i \notin$ int B_j.

Without loss of generality, we assume $||p - p_i|| > ||p - p_j||$.

Let q be a point on the ray pp_i such that $||p - q|| = ||p - p_j||$. We have

$$||q - p_j|| = \frac{||p'_i, p'_j||}{||p - p'_i||}||p - p_i|| < ||p - p_j||.$$

By the triangle inequality,

$$||p_i - q|| + ||q - p_j|| \geq ||p_i - p_j||.$$

Thus

$$||p_i - p|| = ||p_i - q|| + ||q - p|| > ||p_i - q|| + ||q - p_j|| \geq ||p_i - p_j||.$$

This implies that $p_j \in \text{int } B_i$, a contradiction. Therefore for all i, int $B'_i \cap \{p'_1, \dots, p'_n\} = \{p'_i\}$. This implies that if Lemma 4.1 is false for the case $k = 1$, then there are $t > \tau_d$ points on the unit sphere, whose pairwise distance is no less than 1. This contradicts with the Lemma A.1. Hence, Lemma 4.1 holds when $k = 1$.

Suppose that Lemma 4.1 is false for some $k > 1$, i.e., there are $t > \tau_d k$ closed balls, without loss of generality, $B_1, \dots, B_t$ contain a point $p \in \mathbb{R}^d$.

Define a subset Q of $\{p_1, \dots, p_n\}$ by the following procedure.

In the following procedure, initially, $P = \{p_1, \dots, p_t\}$ and $Q = \emptyset$.

1. **while** $P \neq \emptyset$
   (a) Let $q \in P$ with the largest $||q - p||$;
   (b) $Q = Q \cup \{q\}$;
   (c) $P = P - \text{int } B_q$, (where B_q stands for the closed ball centered at q);

Because that no ball contains more than k points from $\{p_1, \dots, p_t\}$ in its interior, we have

$$|Q| \geq \lceil \frac{t}{k} \rceil \geq \tau_d + 1.$$

We now observe that for all $q \in Q$, int $B_q \cap Q = \{q\}$.

Let $Q = \{q_1, \dots, q_m\}$ such that for all $i < j$, q_i is put Q in the above procedure before q_j, we have, for all $j > i$, $q_j \notin \text{int } B_{q_i}$. Further, for all $i < j$, $q_i \notin \text{int } B_{q_j}$. This is so, because $||q_i - q_j|| \geq ||q_i - p|| \geq ||q_j - p||$.

Consequently, if there are more than $\tau_d k$ balls covered p, then there are more than τ_d balls that cover p and no ball cover the center of other ball in its interior.

This contradicts with Lemma 4.1 when $k = 1$. Therefore. Lemma 4.1. □

B Proof of Lemma 5.6

Proof: The first inequality follows immediately from the definitions of f and g and Lemma 5.4.

For the second inequality, we focus on a particular point $p \in \mathbb{R}^d$. Notices that if $g(p) = 0$, then, $f(p) = 0$ as well. The inequality follows.

Now, assume $g(p) > 0$.

Define

$$M_l = \left\{ i \in \{1, \dots, n\} : 2^{-l} \leq f_i(p) \leq 2^{-l+1} \right\}$$

for all $l : -\infty \leq l \leq \infty$.

Because that $\cup_{-\infty \leq l \leq \infty} M_l = \{i : f_i(p) \neq 0\}$ and M_l's are pairwise disjoint, each indices $i : f_i(p) \neq 0$ occurs in exactly one of M_l's.

Let $m_l = |M_l|$. We claim $m_l \leq 6^d \tau_d k$.

We now proof the claim.

For each $i \in M_l$, by the definition of M_l and f_i, $2^{l-1} \leq \gamma_i \leq 2^l$, recall γ_i is twice of the radius of B_i.

Let $\mathcal{B}$ be a ball centered at p with radius $2^l + 2^{l-1}$. Since $||p - p_i|| \leq \gamma_i$, it follows $\mathcal{B}_i \subset \mathcal{B}$.

Notice that for all $i \in M_l$, int $\mathcal{B}_i$ contains no more than k points from $\{p_j : j \in M_l\}$. By Lemma 4.1, no point from $\mathcal{B}$ is covered by more than $\tau_d k$ balls from $\{\mathcal{B}_j : j \in M_l\}$. Therefore,

$$\tau_d k \cdot \text{vol}(\mathcal{B}) \geq \sum_{j \in M_l} \text{vol}(\mathcal{B}_j)$$

Let $V_d(r)$ be the volume of a ball in $\mathbb{R}_d$ of radius r. We have for all $j \in M_l$, $\text{vol}(\mathcal{B}_j) \geq V_d(2^{l-2})$. Consequently,

$$\tau_d k \cdot V_d(2^l + 2^{l-1}) \geq |M_l| V_d(2^{l-2}),$$

which implies $|M_l| \leq 6^d \tau_d k$, thus the **claim**.

Now, we have

$$\begin{aligned} (f(p))^d &= \left(\sum_{l=-\infty}^{\infty} \sum_{i \in M_l} f_i(p)^{(d-1)} \right)^{d/(d-1)} \\ &\leq \left(\sum_{l=-\infty}^{\infty} m_l (2^{-l+1})^{d-1} \right)^{d/(d-1)} \\ &\leq 2^d \left(\sum_{l=-\infty}^{\infty} m_l (2^{-l})^{d-1} \right)^{d/(d-1)} \end{aligned}$$

where $m_l \leq 6^d \tau_d k$.

Setting $\theta = 6^d \tau_d k$ and applying Lemma 4.4. we obtain

$$f(p)^d \leq c_d 2^d (6^d \tau_d k)^{1/(d-1)} \sum_{l=-\infty}^{\infty} m_l 2^{-ld}$$

This summation is a lower bound on $g(p)^d$ because for each $i \in M_l$. $f_i(p)^d \geq 2^{-ld}$. This concludes the proof of the lemma. □

C A Cost Function for (α, k)–overlap Graphs

Let $P = \{p_1, \ldots, p_n\}$ be a set of n points in $\mathbb{R}^d$ and G be the (α, k)–overlap graph defined by P. Let r_i be the radius of $\mathcal{B}_i$. Let $\gamma_i = 2\alpha r_i$. We define a function f_i as

$$f_i(x) = \begin{cases} 1/\gamma_i & \text{if } x \in (2\alpha) \cdot \mathcal{B}_i, \text{ i.e., } ||x - p_i|| \leq \gamma_i \\ 0 & \text{otherwise} \end{cases}$$

The cost function of the (α, k)–overlap graph G is then defined to be

$$f(x) = L_{(p-1)}(f_1, \ldots, f_n)$$

The following are two lemmas which can be proved similarly to Lemma 5.6 and 5.7. respectively.

Lemma C.1 $(g(x))^d \leq (f(x))^d \leq c_d 2^d (6^d \alpha^d k)^{1/(d-1)} \cdot (g(x))^d$.

Lemma C.2

$$\int_{x\in \mathbb{R}^d} (f(x))^d (dx)^d \leq \xi_d \alpha k^{1/d} n$$

Consequently,

Lemma C.3 *There exists a* $(\frac{d+1}{d+2})$*-splitting sphere* S *of* P *with* $Cost_f(S) = O\left(\xi \alpha k^{1/d} n^{\frac{d-1}{d}}\right)$.

D Proof of Lemma 5.11

Proof: We write C as the union of three subsets $C = C_1 \cup C_2 \cup C_3$, where

$$\begin{aligned} C_2 &= \{i \in C : p_i \in R^d - (2\cdot S)\} \\ C_3 &= \{i \in C : p_i \in (2\cdot S)\ \gamma_i \geq \text{radius}(S)\} \\ C_1 &= C - C_2 - C_3 \end{aligned}$$

It simply follows from Lemma 4.2. that $|C_2| \leq A_d(1/2)k$. Similarly, $|C_3| \leq A_d(1/2)k + \tau_d k$. We now bound the size of C_1. First notice that

$$Cost_f(S) = \int_{v\in S} \sum_{i=1}^{n} f_i(v)^{d-1}(dv)^{d-1} \geq \sum_{i\in C_1} \int_{v\in S} f_i(v))^{d-1}(dv)^{d-1}.$$

By the definition of C_1, for each $i \in C_1$, S has a common point with B_i. Further, the $radius(S) \geq \gamma_i$. This implies that the area of $S \cap (2\cdot B_i)$ is at least $\left(\frac{\sqrt{7}}{4}\gamma_i\right)^{d-1} V_{d-1}$.

Therefore,

$$\int_{v\in S} (f_i(v))^{d-1}(dv)^{d-1} \geq Area(A \cap (2\cdot B_i))\left(\frac{1}{\gamma_i}\right)^{d-1} \geq \left(\frac{\sqrt{7}}{4}\right)^{d-1} V_{d-1}.$$

Consequently,

$$Cost_f(S) \geq \sum_{i\in C_1} \int_{v\in S} f_i(v))^{d-1}(dv)^{d-1} \geq |C_1|\left(\frac{\sqrt{7}}{4}\right)^{d-1} V_{d-1}$$

Thus, $|C_1| \leq \left(\frac{4\sqrt{7}}{7}\right)^{d-1} \frac{1}{V_{d-1}} Cost_f(S)$. Detailed proof will be given in the full paper. □