On deleting vertices to make a graph of positive genus planar

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Abstract. This paper contains a proof that an n-vertex graph of genus $g > 0$ contains a set of $O(\sqrt{gn})$ vertices whose removal leaves a planar graph.

1. Introduction

Many results for graphs of known or bounded genus $g > 0$ have been derived from related results for planar graphs. Sometimes planar results have pointed the way for graphs embedded on other surfaces; examples include embedding and isomorphism testing [7,8,12], and Kuratowski's theorem and the recent finiteness result of a forbidden subgraph characterization for every surface [14]. Sometimes planar results are actually central to the extended result; for example the separator theorem for graphs of bounded genus [9] relies on the planar separator theorem [11].

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Thus one approach to problems on graphs of positive genus is to reduce the graphs to planar ones, to use planar results and techniques, and to extend these results to the original graphs.

In this paper we consider the problem of finding a small set of vertices whose removal from an n-vertex graph of genus g leaves a planar graph. The results of [1] show that $g\sqrt{2n} = O(g\sqrt{n})$ vertices can always be removed from a graph on a surface of genus g to leave a planar graph. In [9] this result was improved to $O(\sqrt{gn}\log g)$, and it was conjectured that $O(\sqrt{gn})$ vertices are sufficient. In this paper we prove the latter conjecture. Similar results have been announced by H. N. Djidjev [3,6]; our work extends some ideas of [3] where a partial proof for finding a $O(\sqrt{gn})$ "planarizing" set is given.

Theorem 1. If G is an n-vertex graph embedded on a surface of genus $g > 0$, then there is a set of at most

$$26\sqrt{gn} - 13\sqrt{n/g} = O(\sqrt{gn})$$

vertices whose removal leaves a planar graph.

Most of the steps of this proof are constructive, and in a subsequent paper we will show how to implement these ideas as an algorithm that finds this set of vertices in an embedded graph. The algorithm runs in time linear in the number of edges of the graph.

The result of Theorem 1 is best possible up to constants since it is known that embedded graphs satisfy the following separator theorems and that up to constants these results are best possible.

Theorem 2. (Lipton and Tarjan [11]; Djidjev [4]) If G is a planar graph with n vertices, then there is a set of $O(\sqrt{n})$ vertices whose removal leaves no component with more than 2n/3 vertices.

Theorem 3. (Djidjev [5]; Gilbert, Hutchinson and Tarjan [9]) If G is a graph of genus $g > 0$ with n vertices, then there is a set of $O(\sqrt{gn})$ vertices whose removal leaves no component with more than 2n/3 vertices.

If there were a set of vertices in a graph of positive genus whose removal left a planar graph and whose order was smaller than $O(\sqrt{gn})$, then by removing these vertices and using the planar separator theorem one would have a smaller order separator for graphs of positive genus. This argument also shows that Theorems 1 and 2 imply Theorem 3; the algorithmic implementations are similarly related. However the proof of Theorem 1 and related algorithm are more intricate and involve constants larger than those in [9].

In section 2 we present background for this work, the graph theory lemmas and order arithmetic needed for the proof of Theorem 1, which is presented in section 3.

2. Background in topological graph theory and order arithmetic

We use the terminology of [2] and [15]. The main definitions follow. A graph is said to embed on a surface of genus $g \geq 0$ if it can be drawn on the sphere with g handles, denoted S(g), so that no two edges cross. The genus of a graph G is the least integer g for which G embeds on S(g). A face of an embedding of G on S(g) is a connected component of $S(g) \setminus G$ and is called a 2-cell if it is contractible. An embedding is called a 2-cell embedding if every face is a 2-cell and a triangulation if every face is bounded by three edges. An example of a triangulation of the torus ($g = 1$) is shown in Figure 1a. These embedding terms can also be defined in a strictly combinatorial way. Indeed, they must be so defined for the algorithmic implementation.

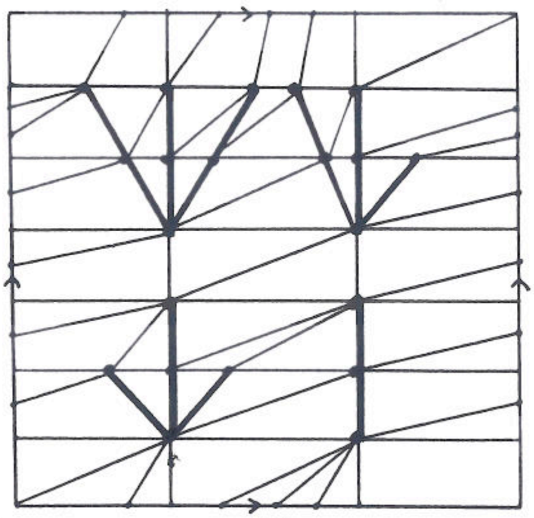
A set of vertices whose removal from a graph G leaves a planar graph is called a planarizing set for G. An important planarizing set is a set of vertices whose induced subgraph leaves all other vertices in regions that are 2-cells.

Embedded graphs on nonplanar surfaces can contain three fundamental types of simple cycles. A cycle is called contractible if it can be continuously deformed on the surface into a point; otherwise it is called noncontractible. A simple noncontractible cycle may be either a separating cycle or a nonseparating cycle according as it does or does not divide the surface into two disjoint pieces. Figure 2 shows all three types of cycles in a graph on the double torus. The Euler-Poincaré Formula will be used to distinguish among these type of cycles; it is also crucial for other parts of the proof.

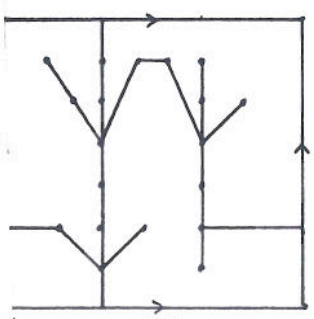
Euler-Poincaré Formula. If G has a 2-cell embedding on $S(g)$, $g \geq 0$, then $n - e + f = 2 - 2g$ where n, e and f are, respectively, the number of vertices, edges and faces of the embedded graph.

The number, $2 - 2g$, is known as the Euler characteristic of $S(g)$.

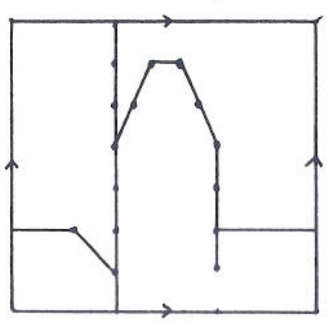
The proof of Theorem 1 will be by induction on g. First we look for a short, $O(\sqrt{n/g})$, noncontractible cycle in the embedded graph, and if such a cycle is present we can remove it and proceed by induction on graphs of smaller genus. If the graph contains no short noncontractible cycle, then we find a spanning



(a)
A triangulation of the torus with a spanning forest of radius 2 with 4 components

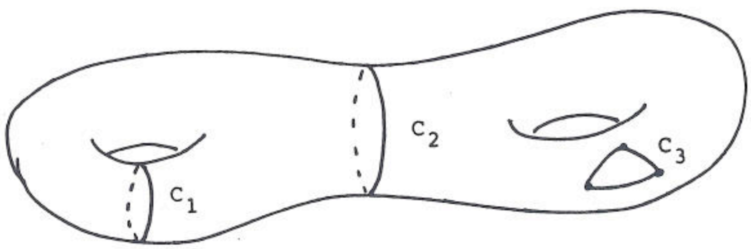


(b)
fter deleting nonforest dges until one 2-cell emains



(c)
A planarizing subgraph

Figure 1.



C_1: Noncontractible and nonseparating
C_2: Noncontractible and separating
C_3: Contractible

Figure 2.

forest of small radius and with few components. By a forest of radius r we mean that every vertex is joined to a root by a path with at most r edges. The next lemma is a generalization of a result in [9] on spanning trees of embedded graphs.

Lemma 4. Suppose the n-vertex graph G has a 2-cell embedding on $S(g)$, $g > 0$, and suppose G has a spanning forest F of radius r with $d \geq 1$ components. Then G contains a planarizing set of at most $4gr + (d-1)(2r+1) + 1$ vertices.

Proof: We call the edges of F and $G \setminus F$ forest and nonforest edges, respectively. We begin by deleting nonforest edges from G one by one until the remaining graph is embedded with exactly one face; as shown in [9] this can be accomplished so that the final face is a 2-cell. (An example is shown in Figure 1 with $d = 4$, $r = 2$ and $g = 1$.) Next we successively delete (nonroot) vertices of degree one and their incident edge (necessarily a forest edge). If G had originally e edges and f faces, we are left with a subgraph G' of G with n' vertices, e' edges and f' faces where

$$n' \le n, \quad e' \le e, \quad f' = 1, \quad \text{and} \quad n' - e' + 1 = 2 - 2g.$$

Let $F' = F \cap G'$ be the remaining spanning forest of radius $r' \le r$ with $d' \le d$ components. Thus

$$e' = n' - 1 + 2g = (n' - d') + (2g + d' - 1),$$

and the e' edges of G' consist of $(n' - d')$ forest edges and $(2g + d' - 1)$ nonforest edges.

Now the spanning forest F' has d' roots, and each nonforest edge of G' joins two vertices of F' at distance at most r' from a root. Furthermore, by construction every vertex of G' lies on some path from a nonforest edge to a root of F'. We estimate the number of vertices of G'. First there are d' roots of F'. Then to every nonforest edge $w = \{u_1, u_2\}$ we associate the at most $2r'$ (nonroot) vertices that lie on the path from u_i to a root, $i = 1, 2$. Thus

$$\begin{aligned} n' &\le d' + (2g + d' - 1)(2r') \\ &\le d + (2g + d - 1)(2r) \\ &= 4gr + (d - 1)(2r + 1) + 1. \end{aligned}$$

If these n' vertices of G' are removed, the remaining graph lies in the one 2-cell face of G' and so is planar. ∎

When the graph contains no $O(\sqrt{n/g})$ noncontractible cycle, we proceed by finding a breadth first spanning tree of presumably too large a radius and then break it into a spanning forest of small radius, $r = O(\sqrt{n/g})$, with few components, $d = O(g)$.

<u>Lemma 5.</u> If G is a connected graph with n vertices, then G has a spanning forest of radius r with at most $\lceil n/(r+1) \rceil$ components.

Proof: Let T be a spanning tree of G of radius s with root t; we are done if $s \le r$. Pick a leaf z of T at distance s from t, and let x be the vertex at distance r from z along the path from z to t. Remove from G the vertex x and all its ancestors; this discarded part of G can be covered by one tree of radius r. The remaining graph is connected with at most $n-r-1$ vertices and by induction its vertices can be covered by at most $\lceil (n-r-1)/(r+1) \rceil = \lceil n/(r+1) \rceil - 1$ trees of radius at most r. Thus G can be covered by at most $\lceil n/(r+1) \rceil$ trees of radius r. ■

<u>Corollary 6.</u> [3] A graph G with n vertices and each connected component having at least m vertices has a spanning forest of radius r with at most $\lfloor n/(r+1) + n/m \rfloor$ components.

Proof: Suppose G has k connected components with $n_1, n_2, \ldots, n_k$ vertices each. Then $n_1 + \ldots + n_k = n$ and $n \geq km$. By Lemma 5 each component can be covered by at most $\lceil n_i/(r+1) \rceil$ trees and so G can be covered by at most

$$\sum_{i=1}^{k} \lceil n_i/(r+1) \rceil \leq \sum_{i=1}^{k} (n_i/(r+1)) + 1$$
$$= \frac{n}{r+1} + k$$
$$\leq \frac{n}{r+1} + \frac{n}{m}. \blacksquare$$

The next two lemmas give detailed information on the growth rate of the function $f(g,n) = 2\sqrt{gn} - \sqrt{n/g}$. This will be necessary for our induction steps.

<u>Lemma 7.</u> For all $g > 1$ and $n > 0$

$$2\sqrt{(g-1)n} - \sqrt{n/(g-1)} + \sqrt{n/g} \leq 2\sqrt{gn} - \sqrt{n/g}.$$

Proof: Since

$$1/\sqrt{g} - 1/(2\sqrt{g-1}) < 1/(2\sqrt{g}) < 1/(\sqrt{g} + \sqrt{g-1}) = \sqrt{g} - \sqrt{g-1},$$

it follows that

$$2\sqrt{n/g} - \sqrt{n/(g-1)} < 2\sqrt{gn} - 2\sqrt{(g-1)n},$$

and the lemma follows. ■

Lemma 8. Let g, n, x, y and d be positive integers satisfying $0 < g < n$, $0 < d \leq \sqrt{n/g}$, $0 < x < g$, and $0 < y < n - d$. Then

$$2\sqrt{xy} - \sqrt{y/x} + 2\sqrt{(g-x)(n-y-d)} - \sqrt{(n-y-d)/(g-x)} + d \leq 2\sqrt{gn} - \sqrt{n/g}.$$

Proof: Multiplying the inequality by $\sqrt{x}$, $\sqrt{g}$ and $\sqrt{g-x}$, we must show that

$$(2x-1)\sqrt{y}\sqrt{g-x}\sqrt{g} + (2g-2x-1)\sqrt{n-y-d}\sqrt{x}\sqrt{g} + d\sqrt{x}\sqrt{g-x}\sqrt{g}$$
$$\leq (2g-1)\sqrt{n}\sqrt{x}\sqrt{g-x}. \qquad (1)$$

First we find the maximum value of the left hand side of (1) as a function of d: let $f(d) = (2g-2x-1)\sqrt{n-y-d}\sqrt{x}\sqrt{g} + d\sqrt{x}\sqrt{g-x}\sqrt{g}$. Then the maximum value of $f(d)$ occurs when $d = (n-y) - (g-x) + 1 - \frac{1}{4(g-x)}$. At this value of d, $n - y - d = (g-x) - 1 + \frac{1}{4(g-x)} < (g-x)$ since $(g-x)$ is an integer. Thus the left hand side of (1) is bounded by

$$(2x-1)\sqrt{y}\sqrt{g-x}\sqrt{g} + (2g-2x-1)\sqrt{n-y-d}\sqrt{x}\sqrt{g} + d\sqrt{x}\sqrt{g-x}\sqrt{g}$$
$$< \sqrt{g-x}\,[(2x-1)\sqrt{y}\sqrt{g} + (2g-2x-1)\sqrt{x}\sqrt{g} + d\sqrt{x}\sqrt{g}] \qquad (2)$$
$$\leq \sqrt{g-x}\,[(2x-1)\sqrt{y}\sqrt{g} + (2g-2x-1)\sqrt{x}\sqrt{g}$$
$$+ ((n-y) - (g-x) + 1 - 1/(4(g-x))\sqrt{x}\sqrt{g}] \qquad (3)$$

Next we find the maximum value of (3) as a function of y: let $f(y) = (2x-1)\sqrt{y}\sqrt{g} + (n-y)\sqrt{x}\sqrt{g}$. Then the maximum value of $f(y)$ occurs at $y = x - 1 + \frac{1}{4x} < x$ since x is an integer. Thus (2) is bounded by

$$\sqrt{g-x}\,[(2x-1)\sqrt{x}\sqrt{g} + (2g-2x-1)\sqrt{x}\sqrt{g} + d\sqrt{x}\sqrt{g}]$$
$$\leq \sqrt{g-x}\,[(2g-2)\sqrt{x}\sqrt{g} + \sqrt{n}\sqrt{x}] \quad (\text{since } d \leq \sqrt{n/g})$$
$$< \sqrt{x}\sqrt{g-x}\,[(2g-1)\sqrt{n}] \quad (\text{since } g < n).$$

This last line is the desired right hand side of line (1). ■

3. The main result

We begin by looking for a $O(\sqrt{n/g})$ noncontractible cycle. Given any simple cycle C we perform the following operation and analysis to determine whether C is contractible or not, separating or not. We can imagine "cutting" the surface along C, then "sewing" in two discs, keeping a copy of C on the boundary of each disc. Call the resulting graph G(C); it may no longer be a triangulation.

Suppose one component $G_1(C)$ of G(C) has n' vertices, e' edges and f' faces. Set $g' = \frac{1}{2}(2 - n' + e' - f')$, the genus of the surface on which $G_1(C)$ is embedded. If $g' = 0$ or g, the cycle C was contractible. If $g' = g - 1$ and G(C) is connected, then C was noncontractible and nonseparating. G(C) is embedded on a surface of genus $g - 1$, and a planarizing set for G(C) together with the vertices of C forms a planarizing set for G. Finally if $0 < g' < g$ and G(C) is not connected, then C was noncontractible and separating. The component $G_1(C)$ is embedded on a surface of genus g' and $G(C) \backslash G_1(C)$ is embedded on a surface of genus $g - g'$. A planarizing set for G will consist of a planarizing set for each component of G(C) together with the vertices of C; see Figure 2.

<u>Theorem 1.</u> If G is an n-vertex graph embedded on a surface of genus $g > 0$, then G has a planarizing set of size at most $26\sqrt{gn} - 13\sqrt{n/g}$.

Proof: We may assume that G is a triangulation since adding edges to triangulate each face can only increase the size of the planarizing set. The proof is by induction on g. In [9] it was shown that a graph has a planarizing set of at most

$6\sqrt{gn}\log g + 6\sqrt{gn}$ vertices. Thus we may assume that $g > 2$, for $g \leq 2$ implies $6\sqrt{gn}\log g + 6\sqrt{gn} < 13\sqrt{gn} \leq 26\sqrt{gn} - 13\sqrt{n/g}$ for all positive g and n.

We may also assume that $\sqrt{n/g} > (26 - \frac{13}{3}) = 21.667$, for otherwise $n \leq (26 - \frac{13}{3})\sqrt{gn} \leq 26\sqrt{gn} - 13\sqrt{n/g}$ for $g \geq 3$, and all n vertices would form a planarizing set. Thus for future reference we assume

$$1 < .046\sqrt{n/g} \tag{4}$$

$$= .046\frac{1}{g}\sqrt{gn} \leq .015\sqrt{gn} \text{ for } g \geq 3. \tag{5}$$

We begin by finding a breadth first spanning tree T with levels $L_0, L_1, \ldots, L_r$ where L_i consists of all vertices at distance i from the root t and where r is the radius of T. Let $|L_i|$ denote the number of vertices in L_i, and set $F_i \subseteq L_i$ equal to those vertices of L_i adjacent to a vertex of L_{i+1}; we call F_i the <u>frontier</u> of L_i. We also define the level of an edge $\{u,v\}$ (or of a triangle $\{a,b,c\}$) to be the maximum level of a vertex in the edge (or triangle).

<u>Lemma 9.</u> For $0 \leq i < r$, F_i induces a subgraph that consists of edge-disjoint cycles.

Sketch of proof: If F_i induces a subgraph of edge-disjoint cycles, then the modulo two sum of all edges of triangles at level $i+1$ with the edges of the cycles of F_i is clearly an edge-disjoint union of cycles and can be shown to equal F_{i+1}. (A similar result can be found in [13].) ■

We note however that this decomposition into cycles may not be unique.

Suppose the graph G contains a noncontractible cycle C of length at most $13\sqrt{n/g}$; because this parameter arises so often we define $K = 13\sqrt{n/g}$. We perform the surface cutting construction described at the beginning of this section, but in addition we delete the two copies of C and all incident edges and we triangulate the resulting, nontriangular faces. Suppose C is nonseparating and noncontractible. By induction the remaining graph has a planarizing set P of size at most $26\sqrt{(g-1)n} - 13\sqrt{n/(g-1)}$. Then $P \cup C$ forms a planarizing set for G and by Lemma 7 has size at most $26\sqrt{gn} - 13\sqrt{n/g}$. Suppose C is separating and noncontractible. Then the remaining graph consists of two graphs, say $G_1(C)$ and $G_2(C)$ with y and $n - y - |C|$ vertices, respectively and of genus x and $g - x$, respectively where $0 < x < g$. By induction $G_1(C)$ has a planarizing set P_1 of size at most $26\sqrt{xy} - 13\sqrt{y/x}$, and $G_2(C)$ has a planarizing set P_2 of size at most $26\sqrt{(g-x)(n-y-d)} - 13\sqrt{(n-y-d)/(g-x)}$. Then $P_1 \cup P_2 \cup C$ forms a planarizing set for G and by Lemma 8 (with $|C| = d$) is of size at most $26\sqrt{gn} - 13\sqrt{n/g}$.

Otherwise every noncontractible cycle in G is larger than K. For $i = 1,2,\ldots,r$ let S_i be the region of the surface formed from all triangles and their boundaries with labels at most i; cycles of F_i form the boundary between S_i and $S(g)\backslash S_i$. We set $S_0 = F_0 = \{t\}$. Suppose we cut the surface S(g) along the cycles of F_i, leaving a graph embedded on S_i with v_i vertices, e_i edges and f_i faces. Then the Euler characteristic of S_i is given by

$$E_i = v_i - e_i + f_i.$$

S_i is a subset of the sphere if and only if $E_i = 2$. See Figure 3.

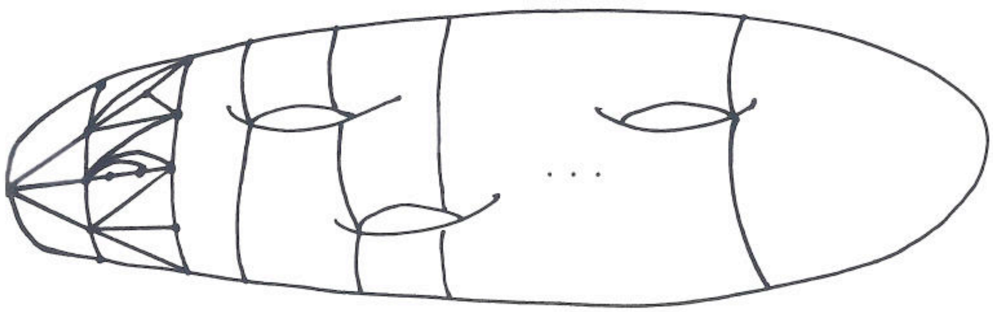


Figure 3.

Let q be the least integer such that either F_{q+1} contains a noncontractible cycle or $E_{q+1} < 2$. Figure 3 contains an example in which F_{q+1} contains noncontractible cycles. Let $p \le q$ be the largest integer such that $|L_p| \le K$; thus F_p contains only contractible cycles. Note that one cycle of F_p, call it c_p, separates the surface into a contractible region containing the root t and the noncontractible region. Finally let s be the greatest integer such that $E_{s-1} < g$, but $E_s = g$. Thus the region $S(g)\backslash S_g$ is a subset of the sphere and contains all vertices on levels s+1 and higher.

If $s > p+1$, then $|L_i| > K$ for $p < i < s$ by the definition of p and since $L_{q+1}, \ldots, L_{s-1}$ all contain noncontractible cycles of length greater than K. Let $G_{p,s}$ be the graph obtained from G by contracting all vertices on levels $L_0, L_1, \ldots, L_{p-1}$ to a new root t^* and by deleting all vertices on levels $L_{s+1}, \ldots, L_r$. If $G_{p,s}$ has radius at most $5\sqrt{n/g}$, then by Lemma 4 and line (5), $G_{p,s}$ has a planarizing set P of size at most

$$4 \cdot 5g\sqrt{n/g} + 1 \le 21.667\sqrt{gn} \le 26\sqrt{gn} - 13\sqrt{n/g}$$

for $g > 2$. Note that since $L_0 \cup \ldots \cup L_{p-1}$ is embedded in a contractible region as is $L_{s+1} \cup \ldots \cup L_r$, then P forms a planarizing set for G as well as for $G_{p,s}$.

If the radius is larger than $5\sqrt{n/g}$, we divide $G_{p,s}$ up into b "bands" of radius $r' = \lceil \sqrt{n/g} \rceil$ where $b = \lceil (s-p)/r' \rceil$. For $i = 1,\ldots,b-1$ we let

$$B_i = L_{p+(i-1)r'+1} \cup \ldots \cup L_{p+ir'}, \text{ and}$$

$$B_b = L_{p+(b-1)r'+1} \cup \ldots \cup L_s.$$

Let $|B_i| = n_i$ and for $i = 1,\ldots,b$ let L_i^* be the smallest level in B_i. Then $|L_i^*| \le n_i/r'$. For future reference we set $t = \lceil 2\sqrt{n/g} \rceil$ and note that

$$\begin{aligned} n_i &\ge Kr' && \text{since all levels have size} \ge K \\ &\ge 6tr' && \text{since } 6t \le 12\sqrt{n/g} + 6 < K \text{ by (4).} \end{aligned} \qquad (6)$$

Consider a frontier $F_i^* \subseteq L_i^*$; by Lemma 9 it consists of edge-disjoint cycles. Each component of F_i^* that contains fewer than K vertices contains only contractible cycles; for each such contractible cycle C_i we delete all vertices in its (contractible) interior. We redefine F_i^* to be $F_i^* \setminus C_i$. (In other words the vertices of C_i are no longer considered to be in the frontier.) We have thrown away only a part of the graph that lies in a contractible region. Every component of (the remaining) F_i^* has at least K vertices, and by Corollary 6 these components can each be covered by at most $\lfloor n_i/(t+1) + n_i/K \rfloor$ trees of radius at most t. For $i = 2,\ldots,b-1$, let these components be covered by trees $T_1, T_2,\ldots,T_u$.

Instead of using F_1^*, we use $F_p \subseteq L_p$ and treat it in a slightly different way. Recall that $|F_p| \le |L_p| \le K$, and that F_p

contains a distinguished contractible cycle, c_p. We delete all other cycles of F_p and their contractible interiors. We cover c_p with at most $\lceil c_p/(2t+1) \rceil \leq \lceil 13\sqrt{n/g}/(4\sqrt{n/g}) \rceil = 4$ trees of radius t (i.e., by paths of 2t edges). Call these trees $P_1, \ldots, P_w$.

From these pieces we construct the desired spanning forest F of (the remainder of) $G_{p,s}$. First we cover c_p, $F_2^*, F_3^*, \ldots,$ and F_{b-1}^* with the trees $P_1, \ldots, P_w, T_1, \ldots, T_u$. Then we use the portion of the original tree T that extends from c_p up to and including vertices in $L_2^* \setminus F_2^*$ (but not including F_2^*), for i = 2 to b-2 from F_i^* up to and including $L_{i+1}^* \setminus F_{i+1}^*$, and from F_{b-1}^* up through L_s. F is a spanning forest of the remaining graph since a vertex in the level above L_p or L_i^* is either contained in a short contractible cycle and so is deleted or is adjacent only to vertices in c_p or in (the remaining) F_i^*. Each portion from the original tree T involves at most 2r' levels and so the resulting trees in F have radius at most $t + 2r' \leq 4\sqrt{n/g} + 3 \leq 4.138\sqrt{n/g}$ by (4).

Next we count the number of components of F. On levels L_2^* and up we have at most

$$\sum_{i=2}^{b-1} (|L_i^*|/t + |L_i^*|/K) \leq \sum_{i=2}^{b-1} (n_i/tr' + n_i/r'K)$$

$$\leq n/tr' + n/r'K - n_1/tr' - n_1/r'K$$

$$\leq n/tr' + n/r'K - 7 \qquad \text{from (6).}$$

The cycle c_p is covered by at most 4 trees of radius t and so in total F contains at most $d = n/tr' + n/r'K - 3$ components and $(d-1) < n/tr' + n/r'K$. By Lemma 4, $G_{p,s}$ has a planarizing set of size at most

$$4gr + (d-1)(2r+1) + 1 < 4g(t+2r') + (n/tr' + n/r'K)(2t + 4r'+1) + 1$$
$$\leq 4g(4.138\sqrt{n/g}) + (n/(2(n/g)) + n/(13(n/g)))(8\sqrt{n/g} + 7) + 1$$
$$\leq 16.55\sqrt{gn} + ((1/2)g + (1/13)g)(8.32\sqrt{n/g}) + 1 \qquad \text{by (4)}$$
$$\leq 16.55\sqrt{gn} + 4.8\sqrt{gn} + .015\sqrt{gn} \qquad \text{by (5)}$$
$$= 21.365\sqrt{gn} < 21.667\sqrt{gn}$$
$$\leq 26\sqrt{gn} - 13\sqrt{n/g} \quad \text{for } g \geq 3.$$

Thus F forms the desired planarizing set for $G_{p,s}$ and for G. ■

4. Conclusion.

In [1] a stronger result was obtained, namely that in every triangulation of a surface of genus g with n vertices there is a nonseparating noncontractible cycle of length at most $\sqrt{2n}$. We conjecture that if $g \leq n$ there is always a $O(\sqrt{n/g})$ noncontractible cycle. This would imply Theorem 1: removing such a cycle and applying the conjecture repeatedly to graphs of smaller genus would produce a $O(\sqrt{gn})$ planarizing set. In [10] the following is established.

__Theorem.__ If G is a triangulation of a surface of genus g with n vertices, then

a) if $g \leq n$, there is a $O(\sqrt{n/g}\log g)$ noncontractible cycle, and

b) if $g > n$, there is a $O(\log g) = O(\log n)$ noncontractible cycle.

In a subsequent paper we shall provide O(e)-time algorithms to find the planarizing set of Theorem 1 and the noncontractible cycle of the latter theorem.

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