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Isomorphism of *k*-Contractible Graphs. A Generalization of Bounded Valence and Bounded Genus

GARY L. MILLER *

Department of Mathematics,
Massachusetts Institute of Technology,
Cambridge, Massachusetts 02139

A polynomial time isomorphism test for graphs called k-contractible graphs for fixed k is included. The class of k-contractible graphs includes the graphs of bounded valence and the graphs of bounded genus. The algorithm uses several new ideas including: (1) It removes portions of the graph and replaces them with groups which are used to keep track of the symmetries of these portions. (2) It maintains with each group a tower of equivalence relations which allow a decomposition of the group. These towers are called towers of Γ_k-actions.

INTRODUCTION

The author and other researchers independently (Filotti and Mayer, 1979; Lichtenstein, 1980; Miller, 1980) have presented polynomial time algorithms for isomorphism testing of graphs of bounded genus. These algorithms are based on fairly complicated analysis of embeddings of graphs on two dimensional surfaces. Since then Luks has presented a polynomial time algorithm for isomophism testing of graphs of bounded valence (Luks, 1980). The ideas used in the bounded valence algorithm are very appealing. They showed relationships between computational group theory and graph isomorphism. The existance of a common generalization between these two subcases of graph isomorphism has been an open question since Luks' work (Babai, 1981). We show that the class of graphs called the k-contractible graphs contain the graphs of bounded genus. Since they trivially contain the graphs of bounded valence, these graphs form a common generalization of the two cases. We give a polynomial time algorithm for testing isomorphism of these graphs.

The paper consists of five sections. First, the preliminaries contain basic

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definition plus the notation of a tower of Γ_k-actions which will be used throughout. Second, intersecting groups with cosets contain the basic group theoretic algorithms used in the isomorphism tests. The third section includes the notion of a graph where the symmetries at a vertex are not arbitrary but restricted to a group. This last notion will be used in the fourth section to decompose a class of graphs where at intermediate stages the graphs are those with restricted vertex symmetries. The graph for which this contraction procedure works will be called the k-contractible graphs. The fifth section shows that the k-contractible graph contain the graphs of genus $\dot{\varepsilon}k$ for some $\varepsilon > 0$.

1. PRELIMINARIES

Throughout this paper graphs will be denoted by G, H, and K; groups by A, B, and C; and sets by X, Y, and Z. Graphs may have multiple edges but no self loops. It will be important that they be allowed to have multiple edges. The edges and vertices of G will be denoted by $E(G)$ and $V(G)$, respectively. The edges common to some vertex v or set of vertices will be denoted by $E(v)$. Let ME denote the multiple edge equivalence relation on G, i.e., $e\,\mathrm{ME}\,e'$ if e and e' are common to the same points. The valence of a given vertex v will be the number of vertices adjacent to v, i.e., the number of edges ignoring multiplicity. Let G be a graph and $Y \subseteq V$. We say two edges e and e' of G are Y equivalent if there exists a path from e to e' avoiding points of Y.

DEFINITION. The graph Br induced from an equivalence class of Y-equivalent edges will be called a bridge, or a bridge of the pair (G, Y). The *vertex frontier* of Br is the vertices of Br in Y, while the *edge frontier* of Br is the set of edges of Br common to the vertex frontier. A bridge is trivial if it is a single edge.

The main graph-theoretic construction we shall use is contracting nonfrontier edges to a point.

DEFINITION. If $Y \in V$, the vertices of G, then *Contract*(G, Y) will be the graph obtained from G by identifying the nonfrontier (internal) vertices of Br for each bridge Br of G, and removing self loops. If $Y = \varnothing$ then Contract(G, Y) is a single vertex.

Let Y_k be the vertices of G with valence greater than k. Intuitively k-contractible graphs are those graphs for which the successive application of Contract(G, Y_k) yields a single point. Here we keep track of the symmetries of the bridges with groups. We attempt to render this idea.

Let $\mathrm{Sym}(X)$ denote the group of all permutations of $\boldsymbol{X}$. We let S_n denote the symmetric group on a set of size $\boldsymbol{n}$. The group A is a permutation group on X if $A \subseteq \mathrm{Sym}(X)$. The degree of A is $|X|$, while, the order is $|A|$. Let π be an equivalence relation on X. Let $A(\pi)$ denote the subgroup of A which stabilizes the equivalence classes X/π of π, i.e., $A(\pi) = \{a \in A \mid x\pi\alpha(x)$ for all $x \in X\}$. If $Y \subseteq X$ there are two natural equivalence relations defined by Y. We shall let Y denote the relation $\{xYy \mid x, y \notin Y$ or $x = y\}$. While $\hat{Y}$ will denote the relation $\{x\hat{Y}y \mid x, y \notin Y$ or $x, y \in Y\}$. Thus $A(Y)$ is the subgroup of A which fixes Y pointwise, while $A(\hat{Y})$ is the subgroup which stablizes Y. The relation $\hat{X}$ will often be denoted by id. We say A preserves π if $x\pi y$ implies $\alpha(x)\pi\alpha(y)$ for all $a \in A$. The subgroup of A preserving π we will denote by $A[\pi]$. A is primitive if it only preserves the trivial equivalence relations X and $\hat{X}$. We say the relation π contains π' if $x\pi'y$ implies $x\pi y$ for all x and y in X; denote by $\pi' \leqslant \pi$. If π is an equivalence relation then X/π denotes the equivalence classes of π. The restriction of an equivalence relation π to Y is denoted by $\pi \restriction Y$. Formally, $\pi \restriction Y$ is defined by $x\pi \restriction Yy$ if $x, y \notin$ Y or $x, y \in Y$ and $x\pi y$. If $Y \leqslant X$ then Y/π are the equivalence classes restricted to $\boldsymbol{Y}$.

The equivalence classes of the relation π defined by, $x\pi y$ if for some $a \in A$, $\alpha(x) = y$, are called the orbits of A. The induced action of A on some orbit Y is the image of A in $\mathrm{Sym}(Y)$ which we identify with the quotient group $A/A(Y)$. In general we shall let $A \restriction Y$ denote the faithful action of $A(\hat{Y})$ on Y, i.e., $A(\hat{Y})/A(Y)$.

An isomorphism is a surjective map which sends edges to edges and vertices to vertices, and preserves incidence and other possible structure. Groups will be permutation groups and they will act from the left. It is easy to see that the isomorphisms from G onto G‘, when G is isomorphic to G’, can be written as σA where a is a arbitrary isomorphism of G onto G’ and A is the group of automorphisms of G. The properties of σA are so similar to a formal coset of A we shall call σA a ***coset.*** In general σA is a ***coset of X onto Y*** if A is a subgroup of $\mathrm{Sym}(X)$ and σ is a surjective map from X to Y.

Throughout the paper we shall either restrict the groups considered or the way they may act.

DEFINITION. For $k \geqslant 2$, let Γ_k denote the class of groups A such that all the composition factors of A are subgroups of S_k.

We shall use the following fact about the primitive actions of Γ_k groups.

THEOREM 1 (Babai, Cameron, and Palfy, to appear). ***There is a function $\lambda(k)$ such that any primitive action of $A \in \Gamma_k$ of degree n has order at most n^{λ}.***

As in Miller (to appear) we could have used Luks' characterization of the p-Sylow subgroups of primitive groups in Γ_k.

We also restrict the way arbitrary groups can act.

Definition. A group A acting on a set X is a ***Γ_k-action*** if for all $x \in X$ the subgroup $A(x) \in \Gamma_k$.

We extend the notation of a Γ_k-action to a tower of such actions. The sequence $(\pi_0, \ldots, \pi_t)$ is a tower of equivalence relations on X if $X = \pi_0 \geqslant \cdots \geqslant \pi_t = \hat{X}$. We shall often write a tower as $(\pi_1, \ldots, \pi_t)$ where it is understood that $\pi_0 = X$. This gives a useful generalization of Γ_k-actions.

Definition. $(A, \pi_0, \ldots, \pi_t)$ is a ***tower of Γ_k-actions*** if:

(1) The sequence $(\pi_0, \ldots, \pi_t)$ is a tower of equivalence relations on X.

(2) $A \leqslant \mathrm{Sym}(X)$.

(3) A preserves π_i for $0 \leqslant i \leqslant t$.

(4) For each $Y \in X/\pi_i$, $0 \leqslant i < t$, $A(\hat{Y})$ acting on Y/π_{i+1} is a Γ_k-action.

We write (4) more formally.

(4') If $S_i \in X/\pi_i$ and $S_{i+1} \in S_i/\pi_{i+1}$ then

$$A(\hat{S}_{i+1})/A(\pi_{i+1} \upharpoonright S_i) \in \Gamma_k.$$

It is easy to see that (4) and (4') are equivalent since the natural homomorphism from $A(\hat{S}_{i+1})$ into $\mathrm{Sym}(S_i/\pi_{i+1})$ has kernel $A(\pi_{i+1} \upharpoonright S_i)$.

We shall prove some simple closure properties about towers of Γ_k-actions. We state them as lemmas.

Lemma 1. *If $(A, \pi_0, \ldots, \pi_t)$ is a tower of Γ_k-actions and $B \subseteq A$ then $(B, \pi_0, \ldots, \pi_t)$ is a tower of Γ_k actions.*

Proof. It is clear that $(B, \pi_0, \ldots, \pi_t)$ satisfies the first three conditions. We show it satisfies (4'). Let $S_i \in X/\pi_i$ and $S_{i+1} \in S_i/\pi_{i+1}$. We must show that $B(\hat{S}_{i+1})/B(\pi_{i+1} \upharpoonright S_i) \in \Gamma_k$. But, $B(\hat{S}_{i+1}) \subseteq A(\hat{S}_{i+1})$ and $B(\pi_{i+1} \upharpoonright S_i) = B(\hat{S}_{i+1}) \cap A(\pi_{i+1} \upharpoonright S_i)$. So by the second isomorphism theorem and the correspondence theorem $B(\hat{S}_{i+1})/A(\pi_{i+1} \upharpoonright S_i) \cap B(\hat{S}_{i+1})$ is isomorphic to a subgroup of $A(\hat{S}_{i+1})/A(\pi_{i+1} \upharpoonright S_i)$. The latter quotient group is in Γ_k. Therefore, the first quotient group is in Γ_k, since Γ_k is closed under taking subgroups.

Lemma 2. *If $(A, \pi_0, \ldots, \pi_t)$ is a tower of Γ_k-actions and $T \subseteq X = \pi_0$ then $(A \upharpoonright T, \pi_0 \upharpoonright T, \ldots, \pi_t \upharpoonright T)$ is a tower of Γ_k-actions.*

Proof. Since $A \restriction T = A(\hat{T}) \restriction T$ and by the previous lemma, towers of Γ_k-actions are closed under taking subgroup. We may assume that $A = A(\hat{T})$. Let $T_i \in T/\pi_i$, $0 \leqslant i < t$, and $T_{i+1} \in T_i/\pi_{i+1}$. Since T_i, $T_{i+1} \neq \varnothing$ and they are subsets of equivalence classes of π_i and π_{i+1}, respectively, there exists unique elements $S_i \in X/\pi_i$ and $S_{i+1} \in S_i/\pi_{i+1}$ containing T_i and T_{i+1}, respectively. We must show that $A(\hat{T}_{i+1})/A(\pi_i \restriction T_i) \in \Gamma_k$. We have the following chain of inclusions

$$A(\hat{S}_{i+1}) = A(\hat{T}_{i+1}) \supseteq A(\pi_i \restriction T_i) \supseteq A(\pi_i \restriction S_i).$$

So our quotient is a section of a Γ_k group and thus Γ_k.

2. Intersecting Groups and Cosets

In this section we give several polynomial time algorithms for intersecting groups presented in different ways with a group which is given as a tower of Γ_k-actions. We say $(\sigma A, \pi_1, ..., \pi_t)$ is a coset which is a tower of Γ_k-actions from X to Y if $(A, \pi_1, ..., \pi_t)$ is a tower of Γ_k-actions on X and σ is a surjective map from X to Y. We list the first three problems of interest.

Problem 1. ***The color symmetries in a tower of Γ_k-actions.***
Input. Sets X and Y with coloring C and a coset $(\alpha A, \pi_1, ..., \pi_t)$ which is a tower of Γ_k-actions from X to Y.
Find. Coset $\alpha B \subseteq \sigma A$ which preserves color.

Problem 2. ***Graph Isomorphisms in a tower of Γ_k-actions.***
Input. Two graphs G and G' and a coset $(\sigma A, \pi_1, ..., \pi_t)$ which is a tower of Γ_k-actions from $V(G)$ to $V(G')$.
Find. Coset $\alpha B \subseteq \sigma A$ of isomorphisms from G to G'.

We list a third problem which we will not need here but whose polynomial time solution will follow easily from the idea in this paper and the ideas in Miller *(to* appear) and may have application elsewhere.

Problem 3. ***Hypergraph Isomorphisms in a tower of Γ_k-actions.***
This problem is the same as Problem **2** with graphs replaced with hypergraphs.
We first five a polynomial time algorithm for Problem 1.

Theorem 2. ***The color symmetries in a tower of Γ_k-actions is polynomial time constructable for fixed k.***

The algorithm to foliow has two phases. The first phase is used to reduce the group action on S/π_i to a group which in Γ_k. While the second phase just applies Luks' color isomorphism algorithm to an action in Γ_k.

Procedure $C_S(\sigma A, \pi_0,\dots, n_t, i)$

(1) *If* $S = \{x\}$ for some $x \in X$ *then*
return = σA if the color of **x** equals colors of $\sigma(x)$
$\varnothing$ otherwise.

(2) *If* A is not transitive on S *then*

(a) pick an A-stable partition of S say S_1, S_2.
(b) ***return***

$$C_{S_2}(C_{S_1}(\sigma A, \pi_0,\dots, \pi_t, i), \pi_0,\dots,\pi_t, i).$$

(3) *If* $S \subseteq S_i \in X/\pi_i$ for some S_i *then*

(a) Pick $S' \in S/\pi_{i+1}$

(b) Compute $A(\hat{S}')$ and coset representatives of $A(\hat{S}')$ in A, say, $\sigma_1,\dots,\sigma_l$.

(c) ***Return***

$$\bigoplus_{j=1}^{l} C_S(\sigma\sigma_i A(\hat{S}'), \pi_0,\dots, \pi_t, i+1).$$

(4) (a) Find a primitive block system of A on S say $\pi \geqslant \pi_i \upharpoonright S$.

(b) Compute $A(\P)$ and coset representatives $\sigma_1,\dots,\sigma_l$ of $A(\P)$ in A.

(c) ***return***

$$\bigoplus_{j=1}^{l} C_S(\sigma\sigma_i A(\P), \pi_0,\dots, \pi_t, i).$$

If we set $S = X$ then $C_S(\sigma A, \pi_0,\dots, \pi_t, 0)$ will return with the coset of elements of σA which preserve color, where $(A, \pi_0,\dots, \pi_t)$ is a tower of Γ_k-actions. By Lemmas 1 and 2 every recursive call of, say, $C_S(\sigma A, \pi_0,\dots, \pi_t, i)$ will satisfy:

(i) $(A \upharpoonright S, \pi_0 \upharpoonright S,\dots, \pi_t \upharpoonright S)$ is a tower of Γ_k-actions.

(ii) If $|S/\pi_i| > 1$ then $A \upharpoonright S/\pi_i$ is in Γ_k.

(iii) If $|S/\pi_i| = 1$ then $A \upharpoonright S/\pi_{i+1}$ is a Γ_k-action.

These three facts give us the following time analysis. Let $T(n, m)$ be the number of recursive calls of C_S where $n = |S|$ and $m = |S/\pi_i|$. We obtain the following three inequalities

(1) $T(n, m) \leqslant T(n_1, m) + T(n_2, m)$ if A is not transitive on S for some n_1 and n_2, where $n_1 + n_2 = n$.

(2) $T(n, 1) \leqslant mT(n, m)$ by step (3) of the procedure.

(3) $T(n,m) \leqslant l^{\lambda+1}T(n/l, m/l)$ by steps **(4)** and (1) where λ is the constant from Theorem 1 and l is from step **(4).**

We rewrite (3) as:

(3′j $T(n, mj \leqslant m^{\lambda+1}T(n/m, 1j$.

Combining (2) and (3') gives $T(n, 1) \leqslant m^{\lambda+2}T(n/m, 1)$. Thus, $T(n, 1) \leqslant n^{\lambda+2}$. *So C* implements at most $n^{\lambda+2}$ recursive calls. Since each call is implementable in polynomial time C is a polynomial time algorithm for fixed k.

We now apply this solution to the color problem to obtain a solution to the graph isomophism problem in a tower of Γ_k-actions. The isomorphism problem is reduced to the color problem by lifting the group or coset action to an action on unordered pairs of vertices and viewing the edges as a coloring of these pairs. Since any graph can be viewed as a directed graph we consider ordered pairs. We only need to know that a tower of Γ_k-actions can be lifted to ordered pairs.

Given A acting on X we define an equivalence relation on X^2 from one on X. If π is an equivalence relation on X we define the relation π^2 on X by $(x, y)\ \pi^2(x', y')$ if $x\pi x'$ and $y\pi y'$. This gives

LEMMA 3. *If* $(A, X, \pi_1,\ldots, n,)$ *is a tower of* Γ_k*-actions then* $(A, X^2, \pi_1^2,\ldots, \pi_l^2)$ *is a tower of* Γ_k*-actions.*

Proof: **As** in previous lemmas we need only consider condition (4'). Suppose $(A, \pi_0,\ldots, \pi_l)$ is **a** Γ_k-action on X. Let $S_i^2 \in X^2/\pi_i^2$ and $S_{i+1}^2 \in S_i^2/\pi_{i+1}^2$. Now, $S_i^2 = S_i \times S_i'$ and $S_{i+1}^2 = S_{i+1} \times S_{i+1}'$, where S_i, $S_i' \in X/\pi_i$ and $S_{i+1} \in S_i/\pi_{i+1}$ and $S_{i+1}' \in S_i'/\pi_{i+1}$. Using this notion $A(\hat{S}_{i+1}^2) = A(\hat{S}_{i+1}) \cap A(\hat{S}_{i+1}')$ and $A(\pi_i^2 \upharpoonright S_i^2) = A(\pi_i \upharpoonright S_i) \cap A(\pi_i \upharpoonright S_i')$. Thus we need only show that if B/B^* and C/C^* are in Γ_k then $B \cap C/B^* \cap C^* \in \Gamma_k$. We do this with a lemma.

LEMMA 4. *If* $B^* \lhd B$, $C^* \lhd C$, *and* C/B^*, $C/C^* \in \Gamma_k$ *then* $B \cap C/B^* \cap C^* \in \Gamma_k$

Proof. The proof follows by standard techniques. We give a proof for completeness. The group $B \cap C \cap C^* = B \cap C^* \lhd B \cap C$ and $B \cap C/B \cap C^*$ is isomorphic to a subgroup of C/C^*. Therefore, $B \cap C/B \cap C^* \in \Gamma_k$. Similarly, $B \cap C/B^* \cap C \in \Gamma_k$ and $B \cap C^*/B^* \cap C^* \in \Gamma_k$. By the correspondence theorem $B \cap C^*/C^* \cap C^*$ is isomorphic to a normal subgroup of $B \cap C/B^* \cap C^*$ and $B \cap C/B \cap C^*$ is isomorphic to its quotient. Since $B \cap C/B^* \cap C^*$ contains a normal subgroup in Γ_k and the subgroup's quotient is also in Γ_k it must be in Γ_k by the Jordan–Holder theorem.

The proof is most easily seen by following the quotients in Diagram 1.

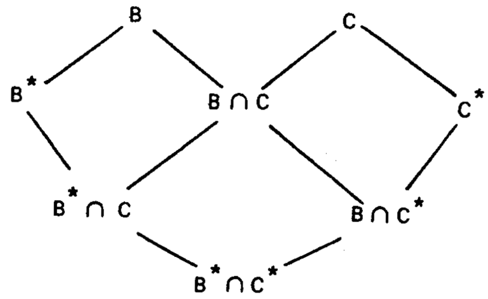

DIAGRAM 1

These arguments actually prove a strong fact which we will need. If R is a relation on X and 7 is a relation on Y then define $\pi \cdot 7$ on $X \times Y$ by (x, y) $\pi \cdot \tau(x', y')$ if $x\pi x'$ and $y\tau y'$.

LEMMA 5. *If* $(A, X, \pi_1, \ldots, \pi_t)$ *and* $(B, Y, 7, \ldots, \tau_t)$ *are towers of* Γ_k*-actions then so is* $(A \times B, X \times Y, \pi_1 \cdot \tau_1, \ldots, \pi_t \cdot \tau_t)$.

These observations give

THEOREM 3. *Graph isomorphisms in a tower of* Γ_k*-actions in polynomial-time constructable for fixed k.*

Combining Theorem 3 with the algorithm in Theorem 3 from Miller (to appear) gives

THEOREM 4. *Hypergraph isomorphisms in a tower of* Γ_k*-actions is polynomial time constructable for fixed k.*

Let A act on X and B act on Y. The ***amalgamated intersection*** of A and B or simply the intersection of A and B, $A \cap B$, is the group of elements in $\mathrm{Sym}(X \cup Y)$ which stabilizes X and Y and whose action on X is in A and on Y it is in B. In general, if σA is a coset from X to X' and γB is a coset from Y to Y' then the (amalgameted) intersection of σA and γB is the set of surjective maps a from $X \cup Y$ to $X' \cup Y'$ such that $a \restriction X \in \sigma A$ and $a \restriction Y \in \gamma B$. It is easily seen that $\sigma A \cap \gamma B = \emptyset$ or $\alpha(A \cap B)$ for any $a \in \sigma A \cap \gamma B$. Using these notions we define

PROBLEM 4. *The (amalgameted) intersection of a coset with a tower of* Γ_k*-actions*

Input. $(\sigma A, \pi_0, \ldots, \pi_t)$ a tower of Γ_k-actions and γB a coset.

Find. Generators for $\sigma A \cap \gamma B$.

In general $\sigma A \cap \gamma B$ will not be a tower of Γ_k-actions but we can still solve Problem **4** in polynomial time.

THEOREM 5. ***If $(\sigma A, \pi_1, \ldots, \pi_t)$ is a tower of Γ_k-actions and γB is an arbitrary coset then the amalgamated intersection $\sigma A \cap \gamma B$ is polynomial time constructable.***

Proof. Since the ideas of the algorithm are a natural combination of those used in computing the color symmetries in a tower of Γ_k-actions and standard techniques (Luks, 1980) we only sketch the proof.

Let $(\sigma A, \pi_0, \ldots, \pi_t)$ and γB be as in the hypothesis of the lemma. Suppose σA is a coset from X to X' and γB is a coset from Y to Y'.

Using the algorithm for computing the color isomorphism in a tower of Γ_k-actions we can compute the coset contained in σA which sends $X \cap Y$ to $X' \cap Y'$. Since the intersection of this coset with γB will be the same we may assume that σA sends X n Y to $X' \cap Y'$.

Let (a,γ)A X B be a coset from $X \times Y$ to X' X Y', and $(\lambda_1$, A,) C be a subcoset of (a,γ)$A \times B$. Let P_{r_1} and P_{r_2} be the projections of C on X and Y, respectively. Consider the following predicate on $(\lambda_1$, A,) C, where Z is $P_{r_1}(C)$-stable and contained in $X \cap Y$.

$$J_Z((\lambda_1, \lambda_2)\, C) = \{(\alpha, \beta) \in (\lambda_1, \lambda_2)\, C \mid \alpha \restriction Z = B \restriction Z\}.$$

We list some simple facts about J.

(1) $J_Z((\lambda_1, \lambda_2)\, C) = \varnothing$,
$= (\lambda_1', \lambda_2')\, J_Z(C)$ for any $(\lambda_1', \lambda_2') \in J_Z((\lambda_1, \lambda_2)\, C)$.

(2) If $P_{r_1}(C)$ is not transitive on Z with stable partition Z_1, Z_2 then $J_Z((\lambda_1, \lambda_2)\, C) = J_{Z_1}(J_{Z_2}((\lambda_1, \lambda_2)\, C))$.

(3) If $A^* \subseteq P_{r_1}(C)$ with coset representatives $\sigma_1, \ldots, \sigma_l$ in $P_{r_1}(C)$ then

$$C_Z((\lambda_1, \lambda_2)\, C) = \bigoplus_{i=1}^{l} C_Z((\lambda_1 \sigma_i, \lambda_2 \beta_i)\, C^*),$$

where $C^* = \{(\alpha, \beta) \in C \mid \alpha \in A^*\}$ and $(\sigma_i, \beta_i) \in C$, $1 \leqslant i \leqslant l$.

(4) If $Z = \{z\}$ then

$$C_Z((\lambda_1, \lambda_2)\, C) = O,$$
$$= (A_r, \lambda_2 \beta)\, C(z, z), \qquad \text{otherwise},$$

where $\beta \in P_{r_2}(C)$ such that $\beta(z) = \lambda_2^{-1} \lambda_1(z)$.

Using these four facts about J and the recursive structure used in the color isomorphism algorithm we can compute the intersection in $n^{\lambda+2}$ recursive calls of the form of fact (4). Since we can compute the stabilizer of (z, z) in C and find β in (4) using Sim's algorithm analyzed in Furst, Hopcroft, and Luks (1980).

In the case where B is also a tower of Γ_k-actions then we can return not only the intersection but a tower of Γ_k-actions. The following lemma will **suffice.**

LEMMA **6.** *If* $(A, X, \pi_1, \ldots, \pi_t)$ *and* $(B, Y, \tau_1, \ldots, \tau_t)$ *are tower of* Γ_k-*actions* then $(A \cap B, X \cap Y, \pi_1 \cap \tau_1, \ldots, \pi_k \cap \tau_k)$ *is a tower of* Γ_k-*actions.*

Proof: Let A and B be as in hypothesis of lemma. The lemma follows by Lemmas 1 and 2 for the actions of $A \cap B$ on $X - Y$ and $Y - X$. *So* without loss of generality assume that $X = Y$ and $A = B = A \cap B$. Let $S_i \in X/\pi_i \cap \tau_i$ and $S_{i+1} \in S_i/\pi_{i+1} \cap \tau_{i+1}$. As in other cases, $S_i = S_i^\pi \cap S_i^\tau$ and $S_{i+1} = S_{i+1}^\pi \cap S_{i+1}^\tau$, where $S_i^\pi \in X/\pi_i$, $S_i^\tau \in Y/\tau_i$, $S_{i+1}^\pi \in S_i^\pi/\pi_{i+1}$ and $S_{i+1}^\tau \in S_i^\tau/\tau_{i+1}$. The group $A(\hat{S}_{i+1}) = A(\hat{S}_{i+1}^\pi) \cap A(\hat{S}_{i+1}^\tau)$ and $A(\pi_{i+1} \cap \tau_{i+1} \restriction S_i) = A(\pi_{i+1} \restriction S_i^\pi) \cap A(\tau_{i+1} \restriction S_i^\tau)$. By Lemma 4, $A(\hat{S}_{i+1})/A(\pi_{i+1} \cap \tau_{i+1} \restriction S_i) \in \Gamma_k$. This proves the lemma.

3. SYMMETRIES OF A VERTEX GIVEN BY A GROUP

In Miller (1979) we discussed gadgets, graphs which were used to denote symmetries or as data structures for syrnmetrics. Here we reverse those ideas and replace bridges or gadgets of a graph by a group or coset which will represent the symmetries of the frontier of a bridge. We shall apply these ideas to testing isomorphism of graphs which we will call k-contractible. Here, we present an algorithm which under certain conditions tests isomorphism of graphs where the vertices have specified symmetries. We make these notions precise in what follows. It seems crucial that the graphs considered have multiple edges. Throughout this section the graphs are assumed to have multiple edges.

DEFINITION. A graph with ***specified symmetry*** is a graph G, a set of permutation groups $A_v \leqslant \mathrm{Sym}(E(v))$, where v is a vertex of G plus a list of maps σ_{ij}, where $\sigma_{ij} = \varnothing$ or $\sigma_{ij}: E(v_i) \to E(v_j)$ which is 1–1 and onto.

The σ_{ij}'s are also required to be ***consistent.*** That is,

(1) The relation π defined by $v_i \pi v_j$ if $\sigma_{ij} \neq \varnothing$ is an equivalence relation.

(2) For all v_i and v_j in $V(G)$ if $\sigma_{ij} \neq \varnothing$ then $\sigma_{ij} A_{v_i} \sigma_{ij}^{-1} = A_{v_j}$.

We could have included symmetries of the edges of G but these symmetries can easily be handled by direction and color. We will in general consider symmetries between two graphs, say (G, G'). The pair (G, G') have specified symmetry if the graph $G \cup G'$ has specified symmetry.

DEFINITION. A graph G has its ***vertex symmetries given by towers of Γ_k-actions*** if G has specified symmetry where A, is given in the form $(A_v, \pi_1, \ldots, \pi_t)$, a tower of Γ_k-actions, and the map σ_{ij} for $\sigma_{ij} \neq \varnothing$ preserve this structure. We shall say the ***symmetries are Γ_k on the multiple edges*** if for every vertex $v \in V(G)$ the induced action of A, on $E(v)/ME$ is a Γ_k-action. We shall often refer to these graphs as simply Γ_k-graphs.

The map $f: G \to G'$ is an isomorphism between two graphs with specified symmetry if (1) f is a simple graph isomorphism between G and G' and (2) if $f(v_i) = v_j$ then $f \upharpoonright E(v_i) \in \sigma_{ij} A_{v_i}$. Since the symmetries of (G, G') are consistent it is easily seen that the automorphisms of a specified graph form a group and the isomorphism between two graphs form a so-called coset of the automorphism of G. We state the main theorem of this section.

THEOREM 6. ***The coset of isomorphisms for Γ_k-graphs is polynomial time constructable for fixed*** k.

The assumption that the symmetries are Γ_k-actions on the multiple edge relation will ensure that the induced action on vertices will be tractable. By identifying multiple edges the symmetries become Γ_k-actions. In this case, the edge stabilizer of a connected graph is a group in Γ_k. We state this well-known fact as a lemma.

LEMMA 7. ***If G is connected with specified symmetries which are Γ_k-actions then the automorphisms of G which stabilize an edge form a group in Γ_k.***

The algorithm will use the leveling idea. By standard techniques we may assume the two graphs G and G' are connected. We pick an e in $E(G)$ and compute the coset of isomorphisms from e to e' for each edge e' in $E(G')$. We shall level the edges and vertices by their distance from e and e', respectively. Using this leveling we shall inductively construct the coset of partial isomorphisms from edges to edges and from vertices to vertices. We begin the formal construction.

Label each edge of G with the integer which is the distance (the number of vertices in a shortest path) that the edge is from e, e.g., e is labeled ***0.*** Label each vertice of G with the integer which is the number of edges it is from e, e.g., the end points of e are labelled 1. Similarly, label the edges and vertices of G'. We say an edge is ***even*** if both end points are labeled the same, otherwise we say the edge is ***odd.***

Let G_i be the induced graph on edges labeled $\leqslant i$. That is, G_i is the graph on vertices labeled $\leqslant i+1$ and edges labeled $\leqslant i$. The graph $\bar{G}_i$ consists of (1) G_i; (2) all odd edges labeled $i+1$ where the end point labeled $i+2$ has been replaced with a new distinct vertex for each edge; (3) two copies of each even edge labeled $i+1$ where one copy is attached to one end point labeled $i+1$ and the other copy is attached to the other end point labeled $i+1$. Again, the other end point of these even edges is a new vertex. The vertex symmetries for vertices labeled $\leqslant i$ of G_i will be those of G. While, the vertices labeled $i+1$ will have no restriction on symmetries. The vertex symmetries of $\bar{G}_i$ for vetices labeled $\leqslant i+1$ will be those of G. Again, the symmetrices of vertices labeled $i+2$ will not be constrained. Analogously construct G_i' and $\bar{G}_i'$.

Let $\mathrm{Iso}(G_i, G_i')$ be the isomorphisms from G_i to G_i' which send e to e'. Similarly $\mathrm{Iso}(\bar{G}_i, \bar{G}_i')$ is the set of isomorphisms from $\bar{G}_i$ to $\bar{G}_i'$ which send e to e'. The $\mathrm{Iso}(G_0, G_0')$ is a trivial coset of a group of order 2.

We shall need inductively two conditions or facts concerning the isomorphism. First, that the automorphisms of G_i and $\bar{G}_i$ which fix e acting on the edges are written as a tower of Γ_k-actions. Second, the automorphisrns of G, leaving e fixed acting on the vertices is in Γ_k. We consider the second condition first. If we identify multiple edges of $\bar{G}_{i-1}$ but not multiple copies of the vertices and the symmetries are those induced from the vertices of $\bar{G}_{i-1}$ then the new graph satisfies the hypothesis of Lemma 7. Therefore, the group of automorphisms fixing the edge of e is in Γ_k. Now, any automorphism of $\bar{G}_{i-1}$ induces an automorphism on this graph. Since the action on the vertex is unchanged by identifying multiple edge we have the second condition for $\bar{G}_{i-1}$. If we now also identify the multiple vertices of $\bar{G}_{i-1}$ the automorphism fixing e will still be in Γ_k. But this is the same graph we obtain by identifying the multiple edges of G_i. This proves the second condition. We shall maintain the first condition throughout the construct.

We need only give a polynomial time alogorithm for constructing $\mathrm{Iso}(\bar{G}_i, \bar{G}_i')$ from $\mathrm{Iso}(G_i,$ GI) and constructing $\mathrm{Iso}(G_{i+1}, G_{i+1}')$ from $\mathrm{Iso}(\bar{G}_i, \bar{G}_i')$, where the cosets are given as towers of Γ_k-actions. We consider the latter case first.

The elements of $\mathrm{Iso}(G_{i+1}, G_{i+1}')$ are simply those elements of $\mathrm{Iso}(\bar{G}_i, \bar{G}_i')$ which preserve multiple copies of vertices and edges. That is, they preserve the relation $a \equiv b$ if a and b are copies of the same edge or vertex. We obtain the coset preserving this relation by applying the isomorphism in a tower of Γ_k-actions algorithm from Section 2. Here the graphs have vertices consisting of the vertices and edges of $\bar{G}_i$ and $\bar{G}_i'$, respectively, and the edges are the multiple copies relation. Since towers of Γ_k-actions are closed under taking subgroups the first condition is inductively satisfied for $\mathrm{Iso}(G_{i+1}, G_{i+1}')$.

We have left the construction of $\mathrm{Iso}(\bar{G}_i, \bar{G}_i')$ from $\mathrm{Iso}(G_i,$ Gf) which we must show is a tower of Γ_k-actions. Let $(A, \pi_1, ..., \pi_t) = \mathrm{Iso}(G_i, G_i')$ a tower of

Γ_k-actions. Let E_i (E_i') be the edges of $\bar{G}_i$ ($\bar{G}_i'$) which are common to a vertex labeled i The coset of maps from E_i to E_i' which preserve the symmetry of vertices labeled i can be written as a direct product of wreath products since the symmetries are consistent. That is, it will be of form $\alpha \Pi A_{ij} \mathrm{Wr}\, S$. Let γB be that coset. The coset $\mathrm{Iso}(\bar{G}_i, \bar{G}_i')$ will be the amalgamated intersection of σA and γB. By section **2** this intersection is constructable in polynomial time since A is given as a tower of Γ_k-actions. To show that the intersection can be written as a tower of Γ_k-actions we note the following. Let π_{ij} be the ith equivalence relation of vertex **j**. We claim that $\tau_i = \bigcap_j \pi_{ij}$, where V_j has label $\leqslant i+1$, form a tower for $\sigma A \cap \gamma B$. By induction it is true for A where the intersection is taken over vertices labeled $\leqslant i$. In the construction of B we used the symmetric group in the wreath product. But we know that the induced action on the vertices labeled $i+1$ given by A is in Γ_k. So we may restrict the wreath product to this Γ_k group. Let B' be this smaller group. Now, $(\gamma' B', \tau_1', \ldots, \tau_l')$ is a tower of Γ_k-actions where $\tau_i' = \bigcap_j \pi_{ij}'$, V_0 has label $i+1$. This proves the theorem.

We shall need that the coset of isomorphisms from G to G' can be written as a tower of Γ_k-actions.

Let G be a graph whose vertex symmetries are given by towers of Γ_k-actions and these symmetrices are Γ_k on the multiple edges. Suppose the vertex symmetries of G are $(A_i, \pi_{1i}, \ldots, \pi_{ti})$ for vertex V_i. Let ME be the multiple edge relation on G. In this case the automorphisms of G are a tower of Γ_k-actions.

LEMMA 8. ***If* G *and* $(A_i, \pi_{1i}, \ldots, \pi_{ti})$ *are as above and* A *is the group of automorphisms of G acting on the edges then* $(A, ME, \tau_1, \ldots, \tau_t)$, *where* $\tau_j = (\bigcap_i \pi_{ij})$ ME *for* $v_i \in V(G)$, *is a tower of* Γ_k*-actions.***

Proof. Let E be the edges of G. By Lemma 7 A acting on E/ME is a Γ_k-action. We need only show that if $E_1 \in E/\mathrm{ME}$ then $(A \restriction E_1, \tau_1 \restriction E, \ldots, \tau_k \restriction E_1)$ is a tower of Γ_k-actions. Let **x,** y be the end points of the edges E_1. It will suffice to prove the condition for the normal subgroup of $A \restriction E_1$ of index at most 2 which fixes x and y. Let $A' \subseteq A \restriction E_1$ which fixes x and y. Now $A \subseteq (A_x \cap A_y) \restriction E_1$. But $((A_x \cap A_y) \restriction E_1, \tau_1 \restriction E_1, \ldots, \tau_t \restriction E_1)$ is a tower of Γ_k-actions by Lemmas 2 and 6.

4. k-CONTRACTIBLE GRAPHS

In this section we simultaneously define the valence k-contractable graphs and present a polynomial time algorithm for testing isomorphism of these graphs. We define the valence k-contractible graphs via a decomposition algorithm. This definition is unsatisfactory since small perturbations may result in a different class of graphs. We leave it as an open problem to find a satisfactory definition. However, the definition is sufficient enough so that any perturbation will still contain the graphs of bounded genus and bounded valence. Let G be a graph with prescribed symmetries either given by towers of Γ_k-actions or unconstrained. We shall assume that two unconstrained vertices have at most one edge between them. We shall call these graphs ***Γ_k-contractible graphs.***

We shall say a vertex v is Γ_k if the symmetries of v induce a Γ_k-action on the multiple edge relation ME otherwise it is not Γ_k. Br is a ***Γ_k-bridge*** if it is a bridge of (G,Y), where Y is the set of vertices which are not Γ_k in G. A Γ_k-bridge is not formally a Γ_k-graph for two reasons. First, the unconstrained internal vertices have no associated tower of relations. This problem is remedied by allowing these unconstrained vertices to "inherit" the symmetries from their neighbors. Note that, each neighbor v' of an unconstrained vertex v either shares only a single edge with v' or else its symmetries are a tower of Γ_k-actions. Thus, the symmetries of an unconstrained vertex can be restricted to a product of wreath products where the wreath products are of the form AW_rS_k, where A is the inherited symmetry from a neighboring vertex. Now, $k' \leqslant$ k since the number of neighbors of an unconstrained internal vertex for a Γ_k-bridge must k' $\leqslant$ k. Second, we have not specified the symmetry of frontier vertices. We transform Br into a modified bridge so that it has the form of a Γ_k-graph. For each vertex v of Br common to a frontier vertex v' of Br we introduce a new copy of the frontier vertex $\bar{v}'$ and have edges between v and v' go between v and $\bar{v}'$. We now view these new frontier vertices as unconstrained vertices and allow them to inherit the symmetrices of their neighbors. Let Br be the bridge obtained from Br by the above construction.

Given two Γ_k-bridges Br and Br' we can compute the isomorphism between them by first constructing Br and $\bar{\text{B}}$r$'$ then applying the isomorphism test for Γ_k-graphs. This procedure will return with a coset in the form of a tower of Γ_k-actions. Applying the isomorphism test for towers of Γ_k-actions we can compute the subcoset of isomorphism from Br to Br', which sends multiple copies of vertices to corresponding multiple copies of vertices. But these are the isomorphisms from Br to Br'.

LEMMA 9. *The isomorphisms between two Γ_k-bridges is polynomial time constructible and the coset can be written as a tower of Γ_k-actions.*

This gives a natural decomposition of a Γ_k-contractible graph say G. For each Γ_k-bridge Br of G we identify the internal vertices of Br and remove self-loops. We denote this graph by ***Contract$_k$(G)***. For each pair of Γ_k-bridges Br and Br' the coset of vertex symmetrices between Contract(Br) and Contract$_k$(Br$'$) will be the tower of Γ_k-actions induced by the isomorphisms between Br and Br'. If Br = G then the procedure returns a single point. It follows by standard arguments that any isomorphism between two contractible graphs G and G' induces an isomorphism from $\mathrm{Contract}_k(G)$ to $\mathrm{Contract}_k(G')$ where it is understood that the two Contract constructions are done simultaneously. And conversely, any isomorphism from $\mathrm{Contract}_k(G)$ to $\mathrm{Contract}_k(G')$ can be extended to an isomorphism of G onto G'. Both directions are polynomial time implementable for fixed **k.**

If we apply Contract, to a tree of valence $>k+1$ then the procedure will simply return the original graph. Graphs that are sent to themselves under the procedure will be called fixed points. We introduce a decomposition procedure analogous to the tree isomorphism algorithm. For each pair of vertices (x, y) which are not leaves, valence 1 ignoring multiplicity, we restrict the symmetries between x and y to those that send leaves to leaves and preserve symmetry of a leaf. For these symmetries construct the induced action on the edges which are not common to a leaf. We then remove all vertices of valence 1 and their edges from G. Call this procedure ***Remove Leaves.*** Again, Remove Leaves preserves isomorphism, i.e., G and G' are isomorphic if and only if Remove Leaves (G) and Remove Leaves (G') are isomorphic. The two reductions Contract, and Remove Leaves will not be sufficient to reduce graphs of bounded genus *to* a point. We introduce a third reduction which will correspond to a generalized 3-connected decomposition.

Two vertices of valence **2,** ignoring multiple edges, are multiple if their neighboring vertices are the same. This gives a natural equivalence relation on valence **2** vertices. An equivalence class will simply be called a set of multiple vertices. Let $v_1,..., v_l$ be a set of multiple vertices of G. We consider two cases. First, suppose that the common neighbors, x and y, have unconstrained symmetries. This case will not occur for graphs that are 3-connected originally. Here we replace $v_1,..., v_l$ with a vertex z of valence **2** counting multiples. The symmetries of z will either be trival **or** the cyclic group of order **2.** We must also determine which pairs of multiple vertices are isomorphic. Since the symmetries at x and y are unconstrained the isomorphic pairs are easily determined from the $\tau_{ij's}$. We decompose the second case into two subcases: (1) x and y both have constrained symmetries or (2) only x or only y have constrained symmetries. We discuss in more detail the first subcase. The second subcase is simpler and follows easily from ideas from the first subcase. We may assume that the tower of Γ_k-actions of x and y, say $(A, \pi_1,..., \pi_l)$ and $(B, \tau_1,..., \tau_l)$, preserve the multiple edge relation. We must construct a coset of symmetries for each pair of sets of edges $E_1 = E(v_1,..., v_l)$, $E_2 = E(v'_1,..., v'_l)$, where $v'_1,..., v'_l$ is another set of multiple vertices. Let x' and y' be the neighbors of $v'_1,..., v'_l$. We can restrict our attention to the subcoset of index at most 2 which sends x to x' and y to y'. Let σ and γ be the given symmetries from x to x' and y to y', respectively. An isomorphism which sends x to x' and y to y' when restricted to

$E(x, y)$, can be viewed as contained in $(\sigma, \gamma) A \times B$ acting on $E(x) \times E(y)$. By Lemma 5 $(A \times B, \pi_1\tau_1, ..., \pi_t\tau_t)$ is a tower of Γ_k-actions. Using the color symmetry algorithm we can compute subcoset which preserves the coloring $\{(e, e') \in E(x) \times E(y) | e$ and e' share a vertex or they are the same edge$\}$. Let λC be this coset. The coset restricted to $E(v_1, ..., v_t)$ will send vertices to vertices and be a tower of Γ_k-actions. We now intersect this restriction of λC with the natural picture of wreath products of towers of Γ_k-actions given by the symmetries of $v_1, ..., v_t$ onto $v_1', ..., v_t'$.

The third reduction called ***Remove Multiple Vertices*** will consist of identifying multiple vertices and introducing the symmetries on these new vertices as described above.

We include in the last reduction one other case. If v is of valence 2 with neighbors x and y and these vertices also contain common edges we shall introduce v as a midpoint for these edges and modify the symmetries as in the other cases.

The graph G is a fixed point of these three reductions if it is a fixed point of each reduction. Note that we may arrive at a different fixed point depending on the order which we apply these reductions. We consider fixed points of the following procedure.

Procedure Reduction, (G)

If $G \neq$ Remove Leaves(G) ***then***
$G \leftarrow$ Remove Leaves(G)
else If $G \neq$ Remove Multiple Vertices(G) ***then***
$G \leftarrow$ Remove Multiple Vertices(G)
else $G \leftarrow \text{Contract}_k(G)$.

Definition. ***G is a k-contractible graph*** if successive applications of Reduction applied to G yield a singleton.

From the discussion above we get

Theorem 7. ***Isomorphism for k-contractible graphs is polynomial time testable.***

5. The Bounded Genus Case

Here we shall show that graphs of bounded genus are k-contractible graphs.

This will demonstrate that the k-contractible graphs form a class of graphs which is a common generalization of the bounded valence and the bounded genus graphs. The containment will follow by showing that the fixed graphs under the reduction operation have the property that their genus grows linearly in k. Throughout the rest of the discussion let G be a fixed point. Since neither the genus nor the fact that G is a fixed point is affected by multiple edges we may assume without loss of generality that G has no multiple edges. We may assume that G has no vertices of valence **2.** The procedure Remove Multiple vertices will ensure that no pair of vertices will share both an edge and a vertex of valence 2. Not all vertices of G will have valence $\geqslant k+1$. Let S be the set of vertices of G with valence $\leqslant k$. Then, S will be an independent set. Let v' denote the number of vertices of G in $V-S$. We first state a relationship between genus and k-contractible graphs as defined in the previous sections. We shall actually prove a slightly stronger result for a stronger notion of k-contractible.

THEOREM 8. *If G is a fixed point of Reduction,-, then the genus of G, g, satisfies*

$$2g \geqslant ((k/2-6)/6)\, v' + 2.$$

To **see** that such a large value of k is necessary consider the infinite tiling of the plane in Diagram 2. Since this tiling is periodic we can construct arbitrarily large graphs of genus 1 which are fixed points of Reduction,, **.**

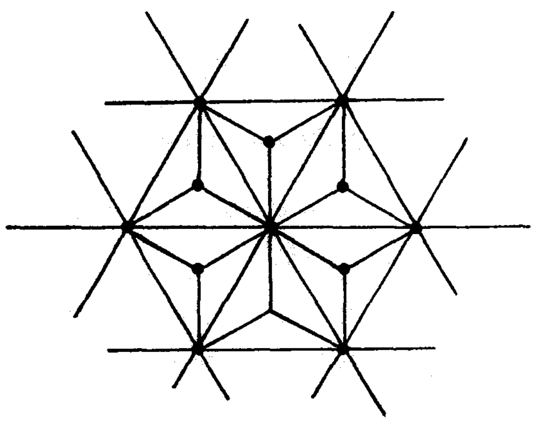

DIAGRAM *2*

Let V' denote the vertices $V-S$. We distinguish two types of edges of G. The edges between points of V' will be called type A edges and those between V‘ and S will be called type B. Since we can easily distinguish these two types we can include in Reduction a procedure which restricts the symmetries at each vertex to the coset which preserves type. Call this new reduction procedure Reduction’. For Reduction: the example from Diagram **2** will contract **to** a point. For Reduction$'$ we **get** the following result.

Theorem 9. *If G is a fixed point of Reduction; then the genus of G, g, satisfies*

$$2g \geqslant ((k-6)/6)\, v' + 2.$$

Note that we are using Reduction; since fixing one edge at a vertex will only in general effect one of the two types of edges at that point. This gives the following corollary for k-contractible graphs with respect to Reduction'.

Corollary. *If $k > 4g + 2$ and $g \geqslant 1$ then the k-contractible graphs include the graphs of genus g. For $g = 0$ (the planar case) $k \geqslant 5$ will suffice.*

To see the corollary we simple note that $v' \geqslant 3$ since a fixed point can have no multiple valence 2 vertices.

***Proof of Theorem* 9.** The proof uses standard counting argument based on Euler's formula $2g = e - f - v + 2$, where g is the genus of some embedding and e, f, and v are the numbers of edges, faces, and vertices, respectively.

We divide the vertices of V' into two sets. Let a_v and b_v be the number of edges of type A and type B common to v, respectively. Define $V_1 = \{v \in V' \mid a_v > b_v\}$ and $V_2 = \{v \in V' \mid b_v \geqslant a_v\}$. Throughout the rest of the proof we fix some embedding of G of genus g with f faces. Let a_1 and a_2 be the number of occurrences of type A edges common to a vertex in V_1 and V_2, respectively. Similarly, define b_1 and b_2 to be the number of edges of type B common to vertices of V_1 and V_2, respectively. Thus, $e = (a_1 + a_2)/2 + b_1 + b_2$. Using these notions we prove an upper bound on the number of faces.

Lemma. $f \leqslant \frac{1}{3}a_1 + \frac{2}{3}b_1 + \frac{1}{2}a_2 + \frac{1}{2}b_2$.

Proof: For each corner in $V - S$ (i.e., a triple (e, x, e') where e, e' are edges common to $x \in V - S$ and e' follows e in the cyclic ordering at x) we assign it weight $\frac{1}{2}$ if either e or e' is of type B and otherwise assign it weight $\frac{1}{3}$. Let γ be the sum over all the corners in $V - S$.

We claim that $f \leqslant \gamma$. The claim follows by showing that the sum of the weights of the corners of each face is greater or equal to one. Consider two cases. First, suppose the face contains an edge of type B. The face must then contain two type B edges and therefore weight $\geqslant 1$. Second, suppose all edges on the face are of type A. The face must then contain three type A edges with three corners contributing $\frac{1}{3}$ each to the sum.

We next show the RHS $\geqslant \gamma$. We do this by considering the arrangement of the edges at a vertex which maximizes the sum. It should be clear that the sum is maximum when they are alternately type A and type B. **Thus,** for $v \in V_2$ the weight at v is $\leqslant (a_v + b_v)/2$ and the sum over all vertices of V_2

has weight $(a_2 + b_2)/2$. For $v \in V_1$ we must have $a_v - b_v$ type AA corners. So the weight at v is $(a_v - b_v)/3 + (2b_v)/2$ and the sum over all vertices in V_1 has weight $\leqslant a_1/3 + 2b_1/3$. This proves the lemma.

Now $|S| \leqslant (b_1 + b_3)/3$ since the valence at each point of S is $\geqslant 3$. Substituting the inequalities into Euler's formula gives

$$2g \geqslant a_1/6 + b_2/6 - |V_1| - |V_2| + 2.$$

Now, $a_1 \geqslant k|V_1|$ since every vertex in V_1 has valence in type A at least k times and a_1 is the occurrence of type A edges in V_1. Recall that if an edge has both end points in V_1 then it contributes 2 to a_1. Similary $b_2 \geqslant k\,|V_2|$ this gives $2g \geqslant ((k-6)/6)\,|V'| + 2$, Theorem 9.

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