Hardy-Muckenhoupt Bounds for Laplacian Eigenvalues

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Abstract

We present two graph quantities $\Psi(G, S)$ and $\Psi_2(G)$ which give constant factor estimates to the Dirichlet and Neumann eigenvalues, $\lambda(G, S)$ and $\lambda_2(G)$, respectively. Our techniques make use of a discrete Hardy-type inequality due to Muckenhoupt.

2012 ACM Subject Classification Mathematics of computing → Spectra of graphs; Mathematics of computing → Approximation algorithms; Mathematics of computing → Graph algorithms

Keywords and phrases Hardy, Muckenhoupt, Laplacian, eigenvalue, effective resistance

Digital Object Identifier 10.4230/LIPIcs.APPROX-RANDOM.2019.8

Category APPROX

Funding *Gary L. Miller*: Supported partially by NSF Grants CCF–1637523 AitF.
Noel J. Walkington: Supported by NSF Grants DMS–1418991 and DMREF–1729478.
Alex L. Wang: Supported partially by NSF Grants CCF–1637523 AitF.

Acknowledgements We would like to thank Timothy Chu for many helpful discussions.

1 Introduction

Possibly one of the most important constants of a graph is λ_2, the fundamental eigenvalue of its graph Laplacian. In computer science, this quantity is used to analyze the mixing time of random walks [14], Markov chains [16], the convergence of Laplacian solvers [11, 18, 20], the performance of spectral clustering [22] and more. The quantity λ_2 is also important in other domains: in quantum mechanics it is related to the uncertainty principles [13], and in numerical analysis arises in the analysis of partial differential equations [2]. As such, it is often necessary to give analytic estimates of this quantity.

In this paper we reexamine an inequality originating with the work of Hardy [9] and show its connection to the eigenvalues of the graph Laplacian. Using this tool, we provide an alternative to Cheeger's inequality and give a 4-approximation of λ_2 in a general setting.

Let $G = (V, E)$ be a connected graph and let $\mu \in \mathbb{R}^V_{>0}$ and $\kappa \in \mathbb{R}^E_{>0}$ be functions on the vertices and edges respectively. We will think of our graphs as spring mass systems where vertex v has mass μ_v and edge e has spring constant κ_e. The Laplacian matrix is defined as $L = D - A$ where D is the weighted diagonal degree matrix and A is the weighted

Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2019).
Editors: Dimitris Achlioptas and László A. Végh; Article No. 8; pp. 8:1–8:19
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

adjacency matrix. Let M be the diagonal mass matrix. Then, the generalized eigenvalues of L with respect to M have a nice interpretation. Specifically, solutions of the generalized eigenvalue problem

$$Lx = \lambda M x$$

correspond to modes of vibration of the associated spring mass system. When the spring mass system is connected, λ_2 is the fundamental mode of vibration[1]. In this paper we will refer to λ_2 as the Neumann eigenvalue to emphasize the implicit boundary assumptions. For an introduction to spring mass systems and the Laplacian, see chapter 5 of [21].

Another interpretation of the weighted graph comes from electrical systems. In this interpretation, we will treat κ_e as the conductance of edge e and $\frac{1}{\kappa_e}$ as its resistances. In this paper we will go back and forth between these two interpretations and will refer to κ_e as either a conductance or a spring constant.

The following result, known as Cheeger's inequality, can be traced back to [1, 4, 5]. Define the isoperimetric constant[2] of G to be

$$\Phi(G) = \min_{A} \left\{ \frac{\sum_{e \in E(A,\bar{A})} \kappa_e}{\min(\mu(A), \mu(\bar{A}))} \;\middle|\; A, \bar{A} \neq \varnothing \right\}.$$

Here, and in the rest of the paper, $\bar{A}$ denotes the complement of A

Then in the case of the normalized Laplacian (i.e, when $\mu_v = d_v$, the degree of v), we can bound λ_2 by

$$\frac{\lambda_2}{2} \leq \Phi \leq \sqrt{2\lambda_2}, \qquad \text{or equivalently,} \qquad \frac{\Phi^2}{2} \leq \lambda_2 \leq 2\Phi.$$

It is well known that both sides of the bound are tight up to constants (see [6] for simple examples). Thus we see that Φ fails to give good control over λ_2 when both quantities are small.

In this paper, we introduce the Neumann content, $\Psi_2(G)$, of a graph G (see Section 6 for a formal definition).

$$\Psi_2(G) \approx \min_{A,B} \left\{ \frac{\kappa(A,B)}{\min(\mu(A), \mu(B))} \;\middle|\; A, B \neq \varnothing,\ A \cap B = \varnothing \right\}$$

where $\kappa(A, B)$ is the effective resistance between the sets A and B. When $B = \overline{A}$, it can be shown that $\kappa(A, \overline{A}) = \sum_{e \in E(A,\overline{A})} \kappa_e$, thus the Neumann content can be thought of as a relaxation of the isoperimetric constant. We will show that $\Psi_2(G)$ gives a constant factor estimate of λ_2 even in a much more general setting.

Along the way, we will consider another eigenvalue problem, which we refer to as the Dirichlet problem (see Section 2). This is a variant of the Laplacian eigenvalue problem where we hold a particular boundary set of vertices, S, to zero. In this setting, we will define the Dirichlet content, $\Psi(G, S)$, which allows us to estimate the Dirichlet eigenvalue.

Specifically, we will prove the following theorems.

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[1] The quantity λ_2 is referred to in the literature under various names: the algebraic connectivity, the Fiedler value, the fundamental eigenvalue, etc.

[2] The quantity Φ is often referred to as the conductance of the graph or the Cheeger constant. In this paper we will refer to Φ as the isoperimetric constant and reserve the term conductance for the conductance of an edge.

▶ **Theorem 1.** *Let (G, S) be a nondegenerate weighted graph with boundary. Let $\lambda(G, S)$ be the Dirichlet eigenvalue and let $\Psi(G, S)$ be the Dirichlet content of (G, S). Then*

$$\frac{\Psi}{4} \leq \lambda \leq \Psi.$$

▶ **Theorem 2.** *Let G be a nondegenerate weighted graph. Let $\lambda_2(G)$ be the Neumann eigenvalue and let $\Psi_2(G)$ be the Neumann content of G. Then,*

$$\frac{\Psi_2}{4} \leq \lambda_2 \leq \Psi_2.$$

It can be shown that the constants in both of these theorems are optimal. In particular, there exist nondegenerate weighted graphs (with and without boundary) for which $\lambda(G, S) = \Psi(G, S)$ and $\lambda_2(G) = \Psi_2(G)$. This shows that the constant 1 in the upper bound is optimal. There also exist sequences of nondegenerate weighted graphs (with and without boundary) for which $\frac{\lambda(G,S)}{\Psi(G,S)} \to \frac{1}{4}$ and $\frac{\lambda_2(G)}{\Psi_2(G)} \to \frac{1}{4}$. This shows that the constant $\frac{1}{4}$ in the lower bound is optimal. See Appendex A for these constructions.

▶ Remark 3. The proof strategy we apply is general and the theorems can be extended to the p-Laplacian[3] for $1 < p < \infty$. The proofs for the case of a general p are almost identical to the proofs for the case of $p = 2$, which we present in this paper, and thus will be omitted. More specifically, with the appropriate definitions for the Dirichlet and Neumann p-contents, both theorem statements above will hold after replacing the 4 in the denominator of the lower bound with $pq^{p/q}$, where q is the Hölder dual of p. The constants in this setting are also optimal.

1.1 Related work

A very recent independent paper [19] introduced a quantity $\rho(G)$ specifically in the case of the normalized Laplacian, i.e., when $\mu_v = d_v$. In this setting, the Neumann content $\Psi_2(G)$ is equivalent to the definition of $\rho(G)$ up to constant factors: $\frac{\Psi_2}{2} \leq \rho \leq \Psi_2$. In [19], it is proved that

$$\frac{\rho}{25600} \leq \lambda_2 \leq 2\rho.$$

This is a special subcase of our Theorem 35 with weaker constants.

The application of the Hardy-Muckenhoupt inequality to estimating the Dirichlet eigenvalue was noted in [15]. In that paper, the authors showed how to bound the Dirichlet eigenvalue on an infinite path graph by the (infinite path analogue of) Ψ. Specifically,

$$\frac{\Psi}{4} \leq \lambda \leq 2\Psi.$$

This is a special subcase of our Theorem 28 with weaker constants.

In contrast with the above related work, we can show that our constants are optimal (see Appendix A).

Other methods for estimating λ_2 have been proposed. A method for lower bounding λ_2 based on path embeddings is presented in [7, 8, 10]. In this method, a graph with known eigenstructure is embedded into a host graph. Then the fundamental eigenvalue of the host graph can be estimated in terms of the eigenstructure of the embedded graph and the "distortion" of the embedding. For a review of path embedding methods, see the introduction in [8].

[3] We refer the curious reader to [3] for basic background on this topic.

1.2 Roadmap

In section 2, we set notation and discuss background related to weighted graphs, Laplacians, the eigenvalue problems, interpreting graphs as electrical networks, and minimum energy extensions. In section 3, we introduce Muckenhoupt's weighted Hardy inequality. In section 4, we introduce the Hardy quantity and the Dirichlet content and show how Muckenhoupt's result can be used to bound the Dirichlet eigenvalue on a path graph. In section 5, we extend the bounds on the Dirichlet eigenvalue from path graphs to arbitrary graphs. Finally in section 6, we introduce the two-sided Hardy quantity and the Neumann content and extend the bounds on the Dirichlet eigenvalue on a graph to the Neumann eigenvalue on a graph.

2 Preliminaries

2.1 Miscellaneous notation

For $A \subseteq V$, we denote by $\overline{A}$ the complement of A in V.

2.2 Vertex and edge weighted graphs

We collect some definitions and notation we will use related to weighted graphs.

▶ **Definition 4.** *A weighted graph is $G = (V, E, \mu, \kappa)$ where (V, E) forms an undirected graph, $\mu \in \mathbb{R}^V_{\geq 0}$ and $\kappa \in \mathbb{R}^E_{>0}$. We call μ_v the mass of vertex v and κ_e the conductance*[4] *of edge e.*

▶ **Definition 5.** *A weighted graph with boundary is a pair (G, S) where G is a weighted graph and $S \subseteq V$ is a proper nonempty subset of the vertices.*

We will make the following assumptions on our graphs. This will ensure that the appropriate eigenvalue quantities exist and are nonzero.

▶ **Definition 6.** *A nondegenerate weighted graph is a weighted graph $G = (V, E, \mu, \kappa)$ where (V, E) forms a connected graph, $|V| \geq 2$, and $\mu \in \mathbb{R}^V_{>0}$.*

▶ **Definition 7.** *A nondegenerate weighted graph with boundary is a weighted graph with boundary (G, S) where every connected component of G contains some $s \in S$, $\left|\overline{S}\right| \geq 1$, and $\mu_v > 0$ for all $v \in \overline{S}$.*

For notational simplicity, we extend μ to subsets of vertices, $\mu(A) = \sum_{v \in A} \mu_v$.

2.3 Laplacians

▶ **Definition 8.** *Let G be a weighted graph. Define $d_v = \sum_{(u,v) \in E} \kappa_{(u,v)}$ to be the degree of vertex v. Let D be the diagonal degree matrix $D_{v,v} = d_v$. Let $A \in \mathbb{R}^{V \times V}$ be the adjacency matrix of G, i.e. $A_{u,v} = \kappa_{(u,v)}$ if $(u, v) \in E$ and 0 otherwise. Then the Laplacian matrix corresponding to G is*

$$L = D - A.$$

[4] As we are dealing with spring mass systems, perhaps it would be better to refer to these quantities as spring constants and compliances. Nonetheless, we have chosen to refer to these quantities as conductances and resistances as this is the terminology most commonly found in the spectral graph theory literature.

Note that the quadratic form associated with L is

$$x^\top Lx = \sum_{(u,v)\in E} \kappa_{(u,v)}(x_u - x_v)^2.$$

▶ **Definition 9.** *Let G be a weighted graph. The mass matrix corresponding to G is the diagonal matrix $M(G)$ where $M_{v,v} = \mu_v$.*

2.4 The generalized Laplacian eigenvalue problem

▶ **Definition 10.** *Let G be a nondegenerate weighted graph. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{|V|}$ be the generalized eigenvalues of L with respect to M. We refer to λ_2 as the Neumann eigenvalue of G, denoted $\lambda_2(G)$ and we refer to an associated eigenvector as a Neumann eigenvector.*

Nondegeneracy ensures that $\lambda_2(G)$ exists as $|V| \geq 2$ and $\lambda_2(G) > 0$ by connectivity.

We state a version of the Courant-Fischer min-max theorem. This will allow us to give variational characterizations of eigenvalues.

▶ **Theorem 11** (Courant-Fischer)**.** *Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric matrices and suppose $B \succ 0$. Let $\lambda_1 \leq \cdots \leq \lambda_n$ be the ordered generalized eigenvalues of A with respect to B. Let $k \in [n]$ and let S denote a subspace of $\mathbb{R}^n$. Then,*

$$\lambda_k = \min_S \max_x \left\{ \frac{x^\top Ax}{x^\top Bx} \,\middle|\, \dim(S) = k,\ x \in S,\ x \neq 0 \right\}.$$

Furthermore, suppose $v_1, \ldots, v_{k-1}$ are orthogonal eigenvectors corresponding to $\lambda_1, \ldots, \lambda_{k-1}$ then

$$\lambda_k = \min_x \left\{ \frac{x^\top Ax}{x^\top Bx} \,\middle|\, \begin{array}{l} x^\top Bv_i = 0,\ \forall i \in [k-1] \\ x \neq 0 \end{array} \right\}$$

and x is a generalized eigenvector with eigenvalue λ_k if and only if x is a minimizer of this second expression.

Noting that 1, the all ones vector, is a generalized eigenvector of L with respect to M with eigenvalue 0, we may apply the Courant-Fischer theorem to get a variational characterization of the Neumann eigenvalue and its eigenvectors.

▶ **Lemma 12.** *Let G be a nondegenerate weighted graph. Then*

$$\lambda_2(G) = \min_{x\in\mathbb{R}^V} \left\{ \frac{x^\top Lx}{x^\top Mx} \,\middle|\, x^\top M1 = 0,\ x \neq 0 \right\}.$$

Furthermore, x is a Neumann eigenvector of G if and only if x is a minimizer in this optimization problem.

The expression $\frac{x^\top Lx}{x^\top Mx}$ plays a large role in our analysis. This quantity is known as the Rayleigh quotient.

We will also consider the Laplacian eigenvalue problem on weighted graphs with boundaries. This corresponds to fixing the value of x at the boundary to zero.

▶ **Definition 13.** *Let (G, S) be a nondegenerate weighted graph with boundary. Let $L_{\overline{S}}$ be the submatrix of L associated with the complement of S and let $M_{\overline{S}}$ be the corresponding submatrix of M. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{|\overline{S}|}$ be the generalized eigenvalues of*

$$L_{\overline{S}}x = \lambda M_{\overline{S}}x.$$

We refer to λ_1 *as the Dirichlet eigenvalue of* (G, S)*, denoted* $\lambda(G, S)$*. Let* $x \in \mathbb{R}^{\overline{S}}$ *be an associated eigenvector and extend it to* $\mathbb{R}^V$ *by zeros. We will refer to* $x \in \mathbb{R}^V$ *as a Dirichlet eigenvector.*

Nondegeneracy ensures that $\lambda(G, S)$ exists as $\left|\overline{S}\right| \geq 1$ and $\lambda(G, S) > 0$ by connectivity.

Again by Courant-Fischer, we can give a variational characterization of the Dirichlet eigenvalue.

▶ **Lemma 14.** *Let* (G, S) *be a nondegenerate weighted graph with boundary. Then,*

$$\lambda(G, S) = \min_{x \in \mathbb{R}^V} \left\{ \frac{x^\top L x}{x^\top M x} \,\middle|\, x\restriction_S = 0,\, x \neq 0 \right\}.$$

Furthermore, x *is a Dirichlet eigenvalue if and only if* x *is a minimizer in this optimization problem.*

Note that the masses of vertices $s \in S$ play no role in either characterization. We will often neglect to assign masses to vertices in the boundary when convenient.

2.5 Graphs as electrical networks

Given a weighted graph G, we can think of its edges as electrical conductors where edge e has conductance κ_e. Thinking of $x \in \mathbb{R}^V$ as an assignment of voltages to the vertices of our electrical network, we have that

$$x^\top L x = \sum_{(u,v) \in E} \kappa_{(u,v)} (x_u - x_v)^2$$

is the power dissipated in our system. Drawing inspiration from physics, we define the effective resistance between two sets of vertices in terms of the minimum power required to maintain a unit voltage drop.

▶ **Definition 15.** *Let* G *be a weighted graph and let* $A, B \subseteq V$ *be disjoint nonempty sets such that there exists a path between* a *and* b *for some* $a \in A$ *and* $b \in B$*. The effective resistance between* A *and* B*, denoted* $R(A, B)$*, and the effective conductance between* A *and* B*, denoted* $\kappa(A, B)$*, are the quantities such that*

$$\frac{1}{R(A, B)} = \kappa(A, B) = \min_{x \in \mathbb{R}^V} \left\{ x^\top L x \,\middle|\, x\restriction_A = 0,\, x\restriction_B = 1 \right\}.$$

The quantity on the right is well-defined as $x^\top L x$ *is continuous and without loss of generality, we may optimize over* $x \in [0, 1]^V$*, a compact set. Then, by the connectivity assumption,* $\kappa(A, B) \in (0, \infty)$*. Thus,* $R(A, B)$ *is also well-defined.*

If $A = \{a\}$ is a single element, we will opt to write $R(a, B)$ instead of the more cumbersome $R(\{a\}, B)$. Similarly we will write $R(A, b)$ or $R(a, b)$ where appropriate.

▶ Remark 16. When $A = \{a\}$ and $B = \{b\}$ are singleton sets, this definition agrees with the standard definition $R(a, b) = \chi_{a,b} L^+ \chi_{a,b}$. In general, we can define $R(A, B)$ in a different way. Consider contracting all vertices in A to a single vertex v_A and all vertices to a single vertex v_B. Then $R(A, B)$ is the effective resistance between v_A and v_B in the new graph. This is the definition given in [19].

2.6 Splitting edges and minimum energy extensions

Let G be a weighted graph. At times, we will split edges using vertices with zero mass. This can be done without affecting the variational quantities[5].

▶ **Lemma 17.** *Let $\alpha_i > 0$ such that $\sum_{i=1}^{k} \alpha_i = 1$. Let $\kappa > 0$ and let $\kappa_i = \kappa/\alpha_i$. Let $y_0, y_k \in \mathbb{R}$ be fixed. Then*

$$\min_{y_1,\dots,y_{k-1}} \sum_{i=1}^{k} \kappa_i (y_i - y_{i-1})^2 = \kappa (y_k - y_0)^2.$$

Furthermore, the unique optimum is achieved by $y_i^* = y_0 + \left(\sum_{j=1}^{i} \alpha_j\right)(y_k - y_0)$.

Proof. Note that $\sum_{i=1}^{k} \kappa_i (y_i - y_{i-1})^2$ is a strictly convex function as y_0 and y_k are fixed. Thus if suffices to show that y^* is a local optimum. Differentiating with respect to y_j and evaluating at y^*,

$$\begin{aligned} \frac{\partial}{\partial y_j}\left(\sum_{i=1}^{k} \kappa_i (y_i - y_{i-1})^2\right)_{y=y^*} &= 2\left(\kappa_j (y_j^* - y_{j-1}^*) - \kappa_{j+1}(y_{j+1}^* - y_j^*)\right) \\ &= 2\left(\frac{\kappa}{\alpha_j}\alpha_j - \frac{\kappa}{\alpha_{j+1}}\alpha_{j+1}\right)(y_k - y_0) = 0. \end{aligned}$$

Then y^* is the unique minimizer achieving objective value

$$\sum_{i=1}^{k} \kappa_i (y_i^* - y_{i-1}^*)^2 = \kappa (y_k - y_0)^2 \sum_{i=1}^{k} \alpha_i = \kappa (y_k - y_0)^2. \qquad \blacktriangleleft$$

Let G be a weighted graph and consider an edge (a, b) of conductance κ in G. Given $\alpha_i > 0$ summing to 1, we can split the edge (a, b) into k edges by inserting $k-1$ new vertices, removing the edge (a, b), and inserting edges $(a, c_1), (c_1, c_2), \dots, (c_{k-1}, b)$ with conductances according to the lemma. We will assign $\mu'(v) = 0$ for all newly added vertices. Let this new weighted graph be $G' = (V', E', \mu', \kappa')$.

▶ **Definition 18.** *Let G be a weighted graph and let G' be a weighted graph constructed from G by splitting edges using the procedure described above. Let $x \in \mathbb{R}^V$. The minimum energy extension of x to V' is the vector y given by*

$$y = \arg\min_{y \in \mathbb{R}^{V'}} \left\{ y^\top L' y \,\middle|\, y\restriction_V = x \right\}.$$

Then by the above lemma it is immediate that $\min_{y \in \mathbb{R}^{V'}} \left\{ y^\top L' y \,\middle|\, y\restriction_V = x \right\} = x^\top L x$. Thus, as $\mu'(v) = 0$ for all $v \in V' \setminus V$, we have,

$$\begin{aligned} &\min_{y \in \mathbb{R}^{V'}} \left\{ \frac{y^\top L' y}{y^\top M' y} \,\middle|\, y^\top M 1 = 0,\, y \neq 0 \right\} \\ &\quad = \min_{x \in \mathbb{R}^V} \left\{ \frac{\min_{y \in \mathbb{R}^{V'}} \left\{ y^\top L' y \,\middle|\, y\restriction_V = x \right\}}{x^\top M x} \,\middle|\, x^\top M 1 = 0,\, x \neq 0 \right\} \\ &\quad = \min_{x \in \mathbb{R}^V} \left\{ \frac{x^\top L x}{x^\top M x} \,\middle|\, x^\top M 1 = 0,\, x \neq 0 \right\}. \end{aligned}$$

[5] In fact, this can be done without affecting the eigenvalue quantities provided they exist. However, proving this requires more set up than is given in this paper.

Similarly, if S is a proper nonempty subset of V, then

$$\begin{aligned}
&\min_{y \in \mathbb{R}^{V'}} \left\{ \frac{y^\top L' y}{y^\top M' y} \,\middle|\, y\restriction_S = 0,\, y \neq 0 \right\} \\
&\quad = \min_{x \in \mathbb{R}^{V}} \left\{ \frac{\min_{y \in \mathbb{R}^{V'}} \left\{ y^\top L' y \,\middle|\, y\restriction_V = x \right\}}{x^\top M x} \,\middle|\, x\restriction_S = 0,\, x \neq 0 \right\} \\
&\quad = \min_{x \in \mathbb{R}^{V}} \left\{ \frac{x^\top L x}{x^\top M x} \,\middle|\, x\restriction_S = 0,\, x \neq 0 \right\}.
\end{aligned}$$

▶ **Definition 19.** *Let G be a nondegenerate weighted graph and let G' be a weighted graph constructed from G using the procedure described above. The Neumann eigenvalue of G' is*

$$\lambda_2(G') = \min_{y \in \mathbb{R}^{V'}} \left\{ \frac{y^\top L' y}{y^\top M' y} \,\middle|\, y^\top M 1 = 0,\, y \neq 0 \right\}.$$

A vector y is a Neumann eigenvector of G' if y is a minimizer of this optimization problem.

▶ **Definition 20.** *Let (G, S) be a nondegenerate weighted graph with boundary and let G' be a weighted graph constructed from G using the procedure described above. The Dirichlet eigenvalue of (G', S) is*

$$\lambda(G', S) = \min_{y \in \mathbb{R}^{V'}} \left\{ \frac{y^\top L' y}{y^\top M' y} \,\middle|\, y\restriction_S = 0,\, y \neq 0 \right\}$$

A vector y is a Dirichlet eigenvector of (G', S) if y is a minimizer of this optimization problem.

3 Muckenhoupt's weighted Hardy inequality

The following theorem, due[6] to Muckenhoupt [17], relates the $L^2(\mathbb{R}_{\geq 0}, \kappa)$ norm of a function and the $L^2(\mathbb{R}_{\geq 0}, \mu)$ norm of the "running integral" of the function.[7] In other words, this theorem characterizes the boundedness of the Hardy operator. In this paper we will refer to this inequality as Muckenhoupt's weighted Hardy inequality (see [12] for a more thorough account of the development and history of the Hardy inequality).

▶ **Theorem 21** (Muckenhoupt 1972)**.** *Let μ, κ be measurable functions from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}_{>0}$. Let C be the smallest (possibly infinite) constant such that for all $f \in L^1_{loc}(\mathbb{R}_{\geq 0})$,*

$$\int_0^\infty \mu(x) \left(\int_0^x f(t)\, dt \right)^2 dx \leq C \int_0^\infty \kappa(x) f(x)^2\, dx.$$

Let

$$B = \sup_{r>0} \left(\int_r^\infty \mu(x)\, dx \right) \left(\int_0^r \frac{1}{\kappa(x)}\, dx \right).$$

Then $B \leq C \leq 4B$. In particular, C is finite if and only if B is finite.

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[6] A similar theorem may have been known previous to Muckenhoupt. Indeed [17] cites the work of Tomaselli and Artola, however we were unable to obtain copies of these papers.
[7] The original theorem deals more generally with L_p norms and Borel measures – see [17].

We will state and prove a finite, discrete version of the above inequality in the following section. Our proof will be stated in the language of graph Laplacians but closely follows the structure of [15, 17] and is only included for completeness.

We first sketch, non-rigorously, why this theorem may be useful. Suppose we have a differentiable function g satisfying $g(0) = 0$. Then taking $f = \frac{d}{dx} g$, and rearranging the above theorem, we have that

$$\frac{1}{C} \leq \frac{\int_0^\infty \kappa(x) g'(x)^2 \, dx}{\int_0^\infty \mu(x) g(x)^2 \, dx} \approx \frac{\sum_{i=1}^\infty \kappa_i (g_i - g_{i-1})^2}{\sum_{i=1}^\infty \mu_i g_i^2}.$$

Note that the right hand side is the Rayleigh quotient of the Laplacian of a weighted infinite path graph (cf. Lemma 14 above). Then minimizing over g, we have that $1/C$ corresponds to a Dirichlet eigenvalue and the bound $B \leq C \leq 4B$ allows us to estimate this eigenvalue.

4 The Dirichlet problem on path graphs

Throughout this section, let $P = (V, E, \mu, \kappa)$ be a weighted path graph. Let the vertices be numbered $\{v_0, v_1, \ldots, v_N\}$ for some $N \geq 1$ and let the boundary set be $S = \{v_0\}$. Let the edges be $E = \{(v_i, v_{i-1}) \,|\, i \in [N]\}$ and let edge (v_i, v_{i-1}) have conductance $\kappa_i > 0$. Let vertex v_i have mass $\mu_i > 0$.

It is immediate that (G, S) is a nondegenerate weighted graph with boundary.

4.1 The Hardy quantity and the Dirichlet content

Let $A \subseteq V \setminus S$ be a set of vertices disjoint from the boundary. Consider the graph consisting of two vertices v_S, v_A. Let v_A have mass $\mu(A)$ and let the edge (v_S, v_A) has conductance $\kappa(S, A)$. Then the Dirichlet eigenvalue of this two node system with boundary set $\{v_S\}$ is given by $\frac{\kappa(S,A)}{\mu(A)}$. We will define the Dirichlet content of G, $\Psi(G)$, to be the minimum such quantity over choices of A and, for historical reasons, we will define the Hardy quantity to be $H = \Psi^{-1}$.

▶ **Definition 22.** *Let (G, S) be a nondegenerate weighted graph with boundary. The Dirichlet content of (G, S), denoted $\Psi(G, S)$, is*

$$\Psi(G, S) = \min_{A \subseteq V} \left\{ \frac{\kappa(S, A)}{\mu(A)} \,\middle|\, A \neq \varnothing,\, A \cap S = \varnothing \right\}.$$

▶ **Definition 23.** *Let (G, S) be a nondegenerate weighted graph with boundary. The Hardy quantity of (G, S), denoted $H(G, S)$, is $H(G, S) = \Psi(G, S)^{-1}$, i.e.*

$$H(G, S) = \max_{A \subseteq V} \{ R(S, A)\mu(A) \,|\, A \neq \varnothing,\, A \cap S = \varnothing \}.$$

In a path graph, it suffices to optimize over tail sets. This gives us a second characterization of H (and thus Ψ) on path graphs.

▶ **Lemma 24.** *Let (P, v_0) be a nondegenerate weighted path graph with boundary. Let $A_k = \{v_i \,|\, i \geq k\}$ be the tail set beginning at v_k. Then*

$$H(P, v_0) = \max_{1 \leq k \leq N} R(v_0, A_k)\mu(A_k).$$

Proof. Let $A \subseteq V \setminus S$. Let $k = \min\{i \,|\, v_i \in A\}$ be the minimum element in A. Then $R(S, A) = R(S, A_k)$ and $\mu(A_k) \geq \mu(A)$ thus $R(S, A_k)\mu(A_k) \geq R(S, A)\mu(A)$. ◀

For a path graph, we have the following closed form expression for the resistance between v_0 and A_k. This is a consequence of Lemma 17.

▶ **Lemma 25.** *In a weighted path graph,* $R(v_0, A_k) = \sum_{i=1}^{k} \frac{1}{\kappa_i}$.

4.2 Bounding the Dirichlet eigenvalue

▶ **Theorem 26.** *Let* (P, v_0) *be a nondegenerate weighted path graph with boundary. Let* $\lambda(P, v_0)$ *be the Dirichlet eigenvalue and let* $H(P, v_0)$ *be the Hardy quantity of* (P, v_0). *Then,*

$$\frac{1}{4H} \leq \lambda \leq \frac{1}{H}.$$

We reiterate that the below proof has been known since [17] and is included only for completeness.

Proof. We begin by proving the upper bound. Note that if $x\restriction_A = 1$, then $x^\top M x \geq \mu(A)$ for any $A \subseteq V \setminus S$. Applying this bound to λ, we note that the numerator of the Rayleigh quotient becomes an effective conductance term.

$$\begin{aligned}
\lambda &= \min_x \left\{ \frac{x^\top L x}{x^\top M x} \,\middle|\, x_0 = 0,\, x \neq 0 \right\} \\
&\leq \min_{1 \leq k \leq N} \min_x \left\{ \frac{x^\top L x}{x^\top M x} \,\middle|\, x_0 = 0,\, x\restriction_{A_k} = 1 \right\} \\
&\leq \min_{1 \leq k \leq N} \frac{1}{\mu(A_k)} \min_x \left\{ x^\top L x \,\middle|\, x_0 = 0,\, x\restriction_{A_k} = 1 \right\} \\
&= \min_{1 \leq k \leq N} \frac{\kappa(S, A_k)}{\mu(A_k)} \\
&= H^{-1}.
\end{aligned}$$

We turn to the lower bound. Begin by rewriting x_i as a sum of differences in the denominator. Let $\alpha_j > 0$ to be fixed later. We apply Cauchy-Schwarz,

$$\begin{aligned}
\sum_{i=1}^{n} \mu_i x_i^2 &= \sum_{i=1}^{n} \mu_i \left(\sum_{j=1}^{i} (x_j - x_{j-1}) \left(\frac{\kappa_j}{\alpha_j}\right)^{1/2} \left(\frac{\alpha_j}{\kappa_j}\right)^{1/2} \right)^2 \\
&\leq \sum_{i=1}^{n} \mu_i \left(\sum_{j=1}^{i} (x_j - x_{j-1})^2 \frac{\kappa_j}{\alpha_j} \right) \left(\sum_{j=1}^{i} \frac{\alpha_j}{\kappa_j} \right).
\end{aligned}$$

Let $y_j = \left(\sum_{i=1}^{j} \frac{1}{\kappa_i}\right)^{1/2}$ and $y_0 = 0$. We will pick[8] $\alpha_j = \kappa_j(y_j - y_{j-1})$. Thus, plugging in this choice of α_j, noticing the telescoping sum and reversing the order of summation,

[8] This choice ensures that Cauchy-Schwarz is tight when $x = y$ and corresponds to the intuition that the true eigenvector is "close to" y.

$$
\begin{aligned}
\dots &\le \sum_{i=1}^{n} \mu_i \left(\sum_{j=1}^{i} (x_j - x_{j-1})^2 \frac{\kappa_j}{\alpha_j} \right) y_i \\
&= \sum_{j=1}^{n} \kappa_j (x_j - x_{j-1})^2 \frac{1}{\alpha_j} \sum_{i=j}^{n} \mu_i y_i \\
&\le \left(\sum_{j=1}^{n} \kappa_j (x_j - x_{j-1})^2 \right) \left(\max_{1\le j\le n} \frac{1}{\alpha_j} \sum_{i=j}^{n} \mu_i y_i \right).
\end{aligned}
$$

It remains to bound the final term. Note that $y_j = R(v_0, A_j)^{1/2} \le H^{1/2}\mu(A_k)^{-1/2}$. Then,

$$
\sum_{i=j}^{n} \mu_i y_i \le H^{1/2} \sum_{i=j}^{n} \mu_i \left(\sum_{k=i}^{n} \mu_k \right)^{-1/2}.
$$

Note that if $A, a \ge 0$, then $a(A+a)^{-1/2} \le 2\left((A+a)^{1/2} - A^{1/2}\right)$. Indeed, this holds by noting that $(A+ta)^{1/2}$ is concave: $a(A+a)^{-1/2} = \frac{d}{dt}\left(2(A+ta)^{1/2}\right)_{t=1} \le 2\left((A+a)^{1/2} - A^{1/2}\right)$.

Then, taking $A = \sum_{k=i+1}^{n} \mu_k$ and $a = \mu_i$ in this inequality, we get

$$
\begin{aligned}
\sum_{i=j}^{n} \mu_i y_i &\le 2H^{1/2} \left(\sum_{i=j}^{n-1} \left(\mu(A_i)^{1/2} - \mu(A_{i+1})^{1/2} \right) + \mu(A_n)^{1/2} \right) \\
&= 2H^{1/2} \mu(A_j)^{1/2}.
\end{aligned}
$$

We will use the inequality once more. This time, take $A = \sum_{k=1}^{j-1} \frac{1}{\kappa_k}$ and $a = \frac{1}{\kappa_j}$. Then,

$$
\begin{aligned}
\alpha_j &= \kappa_j (y_j - y_{j-1}) \\
&= \kappa_j \left((A+a)^{1/2} - A^{1/2} \right) \\
&\ge \kappa_j \left(\frac{a}{2} (A+a)^{-1/2} \right) \\
&= \frac{1}{2} R(v_0, A_j)^{-1/2}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\max_{1\le j\le n} \frac{1}{\alpha_j} \sum_{i=j}^{n} \mu_i y_i &\le 4H^{1/2} \max_{1\le j\le n} \left(\mu(A_j) R(v_0, A_j) \right)^{1/2} \\
&\le 4H.
\end{aligned}
$$

Rearranging completes the proof. ◀

The following theorem follows as a corollary.

▶ **Theorem 27.** *Let (P, v_0) be a nondegenerate weighted path graph with boundary. Let $\lambda(P, v_0)$ be the Dirichlet eigenvalue and let $\Psi(P, v_0)$ be the Dirichlet content of (P, v_0). Then,*

$$
\frac{\Psi}{4} \le \lambda \le \Psi.
$$

5 The Dirichlet problem on general graphs

5.1 Bounding the Dirichlet eigenvalue

▶ **Theorem 28.** *Let (G, S) be a nondegenerate weighted graph with boundary. Let $\lambda(G, S)$ be the Dirichlet eigenvalue and let $H(G, S)$ be the Hardy quantity of G. Then*

$$\frac{1}{4H} \leq \lambda \leq \frac{1}{H}.$$

The proof of the upper bound in the graph case is the same as the proof of the upper bound in the path case. To prove the lower bound, we split edges by inserting zero mass vertices. We then treat the new graph as a path graph.

Proof. The upper bound follows immediately.

$$\begin{aligned}
\lambda &= \min_{x} \left\{ \frac{x^\top L x}{x^\top M x} \,\middle|\, x\restriction_S = 0,\, x \neq 0 \right\} \\
&\leq \min_{A \subseteq V,\, x} \left\{ \frac{x^\top L x}{x^\top M x} \,\middle|\, A \neq \varnothing,\, A \cap S = \varnothing,\, x\restriction_S = 0,\, x\restriction_{A_k} = 1 \right\} \\
&\leq \min_{A \subseteq V} \left\{ \frac{\kappa(S, A)}{\mu(A)} \,\middle|\, A \neq \varnothing,\, A \cap S = \varnothing \right\} \\
&= H^{-1}.
\end{aligned}$$

We turn to the lower bound. We construct a new weighted graph $G' = (V', E', \mu', \kappa')$ from G as follows. Let x be a Dirichlet eigenvector corresponding to $\lambda(G, S)$. Without loss of generality x is nonnegative. Let $0 = l_0 < \cdots < l_N$ be the distinct values of x. For each edge $(a, b) \in E$ such that $x_a = l_i < l_{i+1} < l_j = x_b$, split e into $j - i$ segments such that in the minimum energy extension of x, the new vertices on e take on all intermediate values $l_{i+1}, \ldots, l_{j-1}$. This is possible by Lemma 17 and the discussion following it. Let y be the minimum energy extension of x.

Let $\tilde{v}_i = \{v \in V' \,|\, y_v = l_i\}$, let $\tilde{A}_k = \{v \in V' \,|\, y_v \geq l_k\}$. Let $\tilde{\kappa}_i = \sum_{u \in \tilde{v}_i,\, v \in \tilde{v}_{i-1}} \kappa'_{(u,v)}$ be the conductance between $\tilde{v}_i$ and $\tilde{v}_{i-1}$. Let $\tilde{\mu}_i = \mu'(\tilde{v}_i)$. Note that $S \subseteq \tilde{v}_0$. Then,

$$\begin{aligned}
\lambda(G, S) &= \lambda(G', S) \\
&= \min_{y \in \mathbb{R}^{V'}} \left\{ \frac{y^\top L' y}{y^\top M' y} \,\middle|\, y\restriction_S = 0,\, x \neq 0 \right\} \\
&= \min_{z \in \mathbb{R}^N} \left\{ \frac{\sum_{i=1}^N \tilde{\kappa}_i (z_i - z_{i-1})^2}{\sum_{i=1}^N \tilde{\mu}_i z_i^2} \,\middle|\, z_0 = 0,\, z \neq 0 \right\}.
\end{aligned}$$

Equality in the last line follows by taking $z_i = l_i$. Then note that the objective function in the final optimization problem is the Rayleigh quotient of a nondegenerate weighted path graph with boundary with vertices $\tilde{v}_i$, conductances $\tilde{\kappa}_i$, and boundary set $\tilde{v}_0$. Then applying the lower bound of Theorem 26.

$$\begin{aligned}
\lambda(G,S) &\geq \frac{1}{4}\min_{1\leq k\leq N}\frac{\min_{z\in\mathbb{R}^N}\left\{\sum_{i=1}^{n}\tilde{\kappa}_i(z_i-z_{i-1})^2 \,\middle|\, z_0=0,\ z\restriction_{\{k,\dots,N\}}=1\right\}}{\sum_{i=k}^{N}\tilde{\mu}_i}\\
&\geq \frac{1}{4}\min_{1\leq k\leq N}\frac{\min_{y\in\mathbb{R}^{V'}}\left\{y^\top L' y \,\middle|\, y\restriction_S=0,\ y\restriction_{\tilde{A}_k}=1\right\}}{\mu'\left(\tilde{A}_k\right)}\\
&= \frac{1}{4}\min_{1\leq k\leq N}\frac{\kappa'\left(S,\tilde{A}_k\right)}{\mu'\left(\tilde{A}_k\right)}\\
&\geq \frac{1}{4}\min_{A'\subseteq V'}\left\{\frac{\kappa'(S,A')}{\mu'(A')} \,\middle|\, A'\neq\varnothing,\ A'\cap S=\varnothing\right\}.
\end{aligned}$$

Note that for any $A' \subseteq V'$, we can take $A = A' \cap V$. For this choice of A, we have $\mu(A) = \mu'(A')$ and $\kappa'(S, A') \geq \kappa(S, A)$. Thus,

$$\begin{aligned}
\lambda(G,S) &\geq \frac{1}{4}\min_{A\subseteq V}\left\{\frac{\kappa(S,A)}{\mu(A)} \,\middle|\, A\neq\varnothing,\ A\cap S=\varnothing\right\}\\
&= \frac{1}{4H}.
\end{aligned}$$ ◀

The following theorem follows as a corollary.

▶ **Theorem 29.** *Let (G, S) be a nondegenerate weighted graph with boundary. Let $\lambda(G, S)$ be the Dirichlet eigenvalue and let $\Psi(G, S)$ be the Dirichlet content of G. Then*

$$\frac{\Psi}{4} \leq \lambda \leq \Psi.$$

6 The Neumann problem on general graphs

Throughout this section, let $G = (V, E, \mu, \kappa)$ be a nondegenerate weighted graph.

6.1 The two-sided Hardy quantity and the Neumann content

Let $A, B \subseteq V$ be disjoint nonempty sets. Consider the graph consisting of two vertices v_A, v_B where vertex v_A has mass $\mu(A)$, vertex v_B has mass $\mu(B)$ and the edge (v_A, v_B) has conductance $\kappa(A, B) > 0$. Then the Neumann eigenvalue of this two node system is given by $\frac{\kappa(A,B)}{(\mu(A)^{-1}+\mu(B)^{-1})^{-1}}$. We will define the Neumann content of G, $\Psi_2(G)$, to be the minimum such quantity over choices of A and B. For historical reasons, we will define the two-sided Hardy quantity to be $H_2 = \Psi_2^{-1}$.

▶ **Definition 30.** *Let G be a nondegenerate weighted graph. The Neumann content of G, denoted $\Psi_2(G)$, is*

$$\Psi_2(G) = \min_{A,B\subseteq V}\left\{\frac{\kappa(A,B)}{(\mu(A)^{-1}+\mu(B)^{-1})^{-1}} \,\middle|\, A,B\neq\varnothing,\ A\cap B=\varnothing\right\}.$$

▶ **Definition 31.** *Let G be a nondegenerate weighted graph. The two-sided Hardy quantity of G, denoted $H_2(G)$, is $H_2(G) = \Psi_2(G)^{-1}$, i.e.,*

$$H_2 = \max_{A,B\subseteq V}\left\{\frac{R(A,B)}{\mu(A)^{-1}+\mu(B)^{-1}} \,\middle|\, A,B\neq\varnothing,\ A\cap B=\varnothing\right\}.$$

6.2 Bounding the Neumann eigenvalue

In this section we show how to extend the bounds on the Dirichlet eigenvalue to the Neumann eigenvalue.

We will bound the Neumann eigenvalue by applying Courant-Fischer to a carefully chosen two-dimensional subspace. In particular, we will split our graph into two parts sharing a common boundary. We will then take our two-dimensional subspace to be the linear span of solutions to the Dirichlet problem on either side of this boundary.

Let $f \in \mathbb{R}^V$ such that f takes both positive and negative values. We will write this concisely as $\pm f \notin \mathbb{R}^V_{\geq 0}$. We will "pinch" the graph at the zero level set of f to create a new weighted graph $G' = (V', E', \mu', \kappa')$: for every edge $(u, v) \in E$ such that $f_u < 0 < f_v$, insert a new vertex s such that the minimum energy extension of f assigns $f(s) = 0$. Let $\mu'(s) = 0$.

Abusing notation we will also let $f \in \mathbb{R}^{V'}$ be the minimum energy extension of f to V'. Let $F_0 = \{v \in V' \,|\, f_v = 0\}$, let $F_{\geq 0} = \{v \in V' \,|\, f_v \geq 0\}$ and $F_{\leq 0} = \{v \in V' \,|\, f_v \leq 0\}$. Similarly define $F_{>0}, F_{<0}$ and note that G' has no edges between $F_{>0}$ and $F_{<0}$. For $A, B \subseteq V$, let $A <_f B$ if $f_a < f_b$ for all $a \in A$, $b \in B$.

We have the following lemma regarding the optimal "pinch."

▶ **Lemma 32.** *Let G be a nondegenerate weighted graph. Let $f \in \mathbb{R}^V$ take both positive and negative values. Then $(G', F_{\geq 0})$ and $(G', F_{\leq 0})$ are both nondegenerate weighted graphs with boundary and*

$$\lambda_2(G) = \min_f \left\{ \max\left(\lambda(G', F_{\geq 0}), \lambda(G', F_{\leq 0})\right) \,\middle|\, \pm f \notin \mathbb{R}^V_{\geq 0} \right\}.$$

Proof. Let $\mathcal{R}$ denote the quantity on the right hand side.

We begin by showing that $\lambda_2(G) \leq \mathcal{R}$. Let $f \in \mathbb{R}^V$ take both positive and negative values. Note that $\lambda_2(G) = \lambda_2(G')$. It is easy to see that $(G', F_{\geq 0})$ and $(G', F_{\leq 0})$ are both nondegenerate weighted graphs with boundaries. Let $y, z \in \mathbb{R}^{V'}$ be Dirichlet eigenvectors with Dirichlet eigenvalues $\lambda(G', F_{\geq 0})$ and $\lambda(G', F_{\leq 0})$ respectively. Note that $\operatorname{supp}(L'y) \subseteq F_{\leq 0}$ and that $z\restriction_{F_{\leq 0}} = 0$, thus $z^\top L' y = 0$. Noting that there exists some nonzero $x \in \operatorname{span}(y, z)$ such that $x^\top M' 1 = 0$,

$$\begin{aligned}
\lambda_2(G) &= \lambda_2(G') \\
&= \min_{x \in \mathbb{R}^{V'}} \left\{ \frac{x^\top L' x}{x^\top M' x} \,\middle|\, x^\top M' 1 = 0,\, x \neq 0 \right\} \\
&\leq \max_{x \in \operatorname{span}(y,z)} \frac{x^\top L' x}{x^\top M' x} \\
&= \max_{(\alpha,\beta) \neq 0} \frac{\alpha^2 y^\top L' y + \beta^2 z^\top L' z}{\alpha^2 y^\top M' y + \beta^2 z^\top M' z} \\
&= \max\left(\lambda(G', F_{\geq 0}), \lambda(G', F_{\leq 0})\right).
\end{aligned}$$

Next we show that $\mathcal{R} \leq \lambda_2(G)$. We will exhibit a choice of f taking both positive and negative values such that $\lambda(G', F_{\geq 0}), \lambda(G', F_{\leq 0}) \leq \lambda_2(G)$. This will additionally imply that the minimum is achieved.

Let x be a Neumann eigenvector of G. As $x \neq 0$ and $x^\top M 1 = 0$, it is clear that x takes both positive and negative values. We will pick $f = x$. Abusing notation, also let $x \in \mathbb{R}^{V'}$ be the minimum energy extension of x to V'. Note that $x\restriction_{F_0} = 0$. Let $y = \min(x, 0)$ and $z = \max(x, 0)$ where min and max are taken element wise. Note that $L'y$ agrees with $L'x = \lambda_2(G) M' x$ on the support of y and that y agrees with x on the support of y. Thus

$y^\top L'y = \lambda_2(G) y^\top M'x = \lambda_2(G) y^\top M'y$. Then,

$$\begin{aligned}\lambda(G', F_{\geq 0}) &\leq \frac{y^\top L'y}{y^\top M'y} \\ &= \lambda_2(G).\end{aligned}$$

Similarly, $\lambda(G', F_{\leq 0}) \leq \lambda_2(G)$. ◀

We will need the two following technical lemmas regarding summing resistances.

▶ **Lemma 33.** *Let G be a nondegenerate weighted graph. Let $A, B \subseteq V$ be disjoint nonempty subsets. Let $f \in \mathbb{R}^{V'}$ such that $A \subseteq F_{<0}$ and $B \subseteq F_{>0}$. Then*

$$R'(A, F_{\geq 0}) + R'(F_{\leq 0}, B) \leq R'(A, B).$$

Proof. Let

$$\kappa_A = \kappa'(A, F_{\geq 0}) = \min_y \left\{ y^\top L'y \,\middle|\, y\restriction_A = 1,\, y\restriction_{F_{\geq 0}} = 0 \right\}$$

and let y be the minimizer. Similarly define κ_B and let z be its minimizer. Note that $\operatorname{supp}(L'y) \subseteq F_{\leq 0}$ and $z\restriction_{F_{\leq 0}} = 0$, i.e, $z^\top L'y = 0$.

Let $\alpha = \frac{\kappa_A}{\kappa_A + \kappa_B}$. Note that $(1-\alpha)y - \alpha z$ assigns $1-\alpha$ to vertices in A and $-\alpha$ to vertices in B. Thus

$$\begin{aligned}\frac{1}{R'(A,B)} &\leq ((1-\alpha)y - \alpha z)^\top L'((1-\alpha)y - \alpha z) \\ &= (1-\alpha)^2 \kappa_A + \alpha^2 \kappa_B \\ &= \frac{\kappa_A \kappa_B}{\kappa_A + \kappa_B} \\ &= \frac{1}{R'(A, F_{\geq 0}) + R'(F_{\leq 0}, B)}.\end{aligned}$$

Rearranging terms completes the proof. ◀

▶ **Lemma 34.** *Let G be a nondegenerate weighted graph. Let $A, B \subseteq V$ be disjoint nonempty subsets. For any $\alpha \in (0,1)$, there exists some $f \in \mathbb{R}^V$ with $A \subseteq F_{<0}$ and $B \subseteq F_{>0}$ such that*

$$\kappa'(A, F_{\geq 0}) = \frac{\kappa(A,B)}{\alpha} \qquad \text{and} \qquad \kappa'(B, F_{\leq 0}) = \frac{\kappa(A,B)}{1-\alpha}.$$

Proof. Let

$$\kappa(A,B) = \min_x \left\{ x^\top Lx \,\middle|\, x\restriction_A = 0,\, x\restriction_B = 1 \right\}$$

and let x be the minimizer. Define $f = x - \alpha 1$ and take $y = \min(f, 0)$, where the minimum and maximum is element-wise. Note that $L'y$ agrees with Lx on the support of y. By optimality of x, for $v \in A \setminus (A \cup B)$, we have $0 = \frac{\partial}{\partial x_v}(x^\top Lx) = 2\,(Lx)_v$. Then,

$$
\begin{aligned}
\kappa'(A, F_{\geq 0}) &\leq \frac{y^\top L' y}{\alpha^2} \\
&= \frac{\sum_{v \in \mathrm{supp}(y)} y_v (Lx)_v}{\alpha^2} \\
&= \frac{\sum_{v \in A} (-Lx)_v}{\alpha} \\
&= \frac{x^\top L x}{\alpha} \\
&= \frac{\kappa(A, B)}{\alpha}.
\end{aligned}
$$

Similarly, $\kappa'(B, F_{\leq 0}) \leq \frac{\kappa(A,B)}{(1-\alpha)}$. Then both inequalities must hold with equality by Lemma 33. ◀

We are now ready to prove the following theorem.

▶ **Theorem 35.** *Let G be a nondegenerate weighted graph. Let $\lambda_2(G)$ be the Neumann eigenvalue and let $H_2(G)$ be the two-sided Hardy quantity of G. Then*

$$\frac{1}{4H_2} \leq \lambda_2 \leq \frac{1}{H_2}.$$

Proof. Both the upper and lower bound will follow the same template: we will apply the pinch point characterization, apply Theorem 28 to each Dirichlet problem, and reorder the minima.

The upper bound is,

$$
\begin{aligned}
\lambda_2(G) &= \min_f \left\{ \max\left(\lambda(G', F_{\geq 0}), \lambda(G', F_{\leq 0})\right) \,\middle|\, \pm f \notin \mathbb{R}^V_{\geq 0} \right\} \\
&\leq \min_f \min_{A,B} \left\{ \max\left(\frac{\kappa'(A, F_{\geq 0})}{\mu(A)}, \frac{\kappa'(B, F_{\leq 0})}{\mu(B)} \right) \,\middle|\, A <_f 0 <_f B,\ A, B \neq \varnothing \right\} \\
&= \min_{A,B} \min_f \left\{ \max\left(\frac{\kappa'(A, F_{\geq 0})}{\mu(A)}, \frac{\kappa'(B, F_{\leq 0})}{\mu(B)} \right) \,\middle|\, A, B \neq \varnothing,\ A <_f 0 <_f B \right\}.
\end{aligned}
$$

The lower bound is,

$$
\begin{aligned}
\lambda_2(G) &= \min_f \left\{ \max\left(\lambda(G', F_{\geq 0}), \lambda(G', F_{\leq 0})\right) \,\middle|\, \pm f \notin \mathbb{R}^V_{\geq 0} \right\} \\
&\geq \frac{1}{4} \min_f \min_{A,B} \left\{ \max\left(\frac{\kappa'(A, F_{\geq 0})}{\mu(A)}, \frac{\kappa'(B, F_{\leq 0})}{\mu(B)} \right) \,\middle|\, A <_f 0 <_f B,\ A, B \neq \varnothing \right\} \\
&= \frac{1}{4} \min_{A,B} \min_f \left\{ \max\left(\frac{\kappa'(A, F_{\geq 0})}{\mu(A)}, \frac{\kappa'(B, F_{\leq 0})}{\mu(B)} \right) \,\middle|\, A, B \neq \varnothing,\ A <_f 0 <_f B \right\}.
\end{aligned}
$$

It remains to understand the following quantity for disjoint nonempty $A, B \subseteq V$.

$$\min_f \left\{ \max\left(\frac{\kappa'(A, F_0)}{\mu(A)}, \frac{\kappa'(B, F_0)}{\mu(B)} \right) \,\middle|\, \pm f \notin \mathbb{R}^V_{\geq 0},\ A <_f 0 <_f B,\ A, B \neq \varnothing \right\} \tag{1}$$

Let $\alpha = \frac{\kappa'(A,B)}{\kappa(A,F_{\geq 0})}$. Then by lemma 33, for all f, we have $\kappa'(B, F_{\leq 0}) \geq \kappa(A, B)/(1-\alpha)$. On the other hand, by lemma 34, there exists some f for which we get equality. Thus,

$$
\begin{aligned}
(1) &= \kappa(A, B) \min_{\alpha \in (0,1)} \max\left(\frac{1}{\mu(A)\alpha}, \frac{1}{\mu(B)(1-\alpha)} \right) \\
&= \frac{\kappa(A, B)}{(\mu(A)^{-1} + \mu(B)^{-1})^{-1}}.
\end{aligned}
$$

Taking the minimum over A, B completes the proof. ◀

The following theorem follows as a corollary.

▶ **Theorem 36.** *Let G be a nondegenerate weighted graph. Let $\lambda_2(G)$ be the Neumann eigenvalue and let $\Psi_2(G)$ be the Neumann content of G. Then,*

$$\frac{\Psi_2}{4} \leq \lambda_2 \leq \Psi_2.$$

7 Conclusion and future work

In this paper we introduced the Dirichlet and Neumann contents for nondegenerate weighted graphs (with and without boundary) and showed that these quantities can be related to the Dirichlet and Neumann eigenvalues (Theorems 28 and 35). We believe that these quantities are natural as evidenced by the simplicity of the corresponding proofs. An open question is whether it is possible to develop approximation algorithms based on these new inequalities as opposed to Cheeger's inequality. Such algorithms would be able to exploit the tighter bounds provided by our theorems under a more general setting of weights. We are hopeful that this open question will be answered affirmatively.

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A Constants in Theorems 28 and 35 are sharp

In this appendix we give constructions of nondegenerate weighted graphs (with and without boundary) that show that the constants in our Theorems 28 and 35 are optimal.

We begin with the Dirichlet case. It is clear that the upper bound is achieved by any nondegenerate weighted graph with boundary (G, S) such that $|\overline{S}| = 1$. In this case

$$\Psi(G, S) = \frac{\kappa(\overline{S}, S)}{\mu(\overline{S})} = \lambda(G, S).$$

We turn to the lower bound. Let $n \in \mathbb{N}$ and let $N = ne^n$. Consider the path graph P with vertices $V = \{v_0, v_1, \ldots, v_N\}$ where v_i has mass

$$\mu_i = \begin{cases} \frac{1}{i(i+1)} & 1 \leq i \leq N-1 \\ \frac{1}{N} & i = N \end{cases}$$

and the usual path edges. Let $\kappa = 1$ for every edge and let v_0 be the boundary set. We compute the Dirichlet content of (P, v_0).

$$\begin{aligned} \Psi(P, v_0) &= \min_{1 \leq k \leq N} \frac{\kappa(v_0, v_k)}{\sum_{i=k}^{N} \mu_i} \\ &= \min_{1 \leq k \leq N} \frac{1/k}{1/k} \\ &= 1. \end{aligned}$$

We next show that $\lambda(P, v_0) \leq \frac{1}{4} + o(1)$. Consider the assignment

$$x_i = \begin{cases} 0 & 0 \leq i \leq n-1, \\ \sqrt{i} - \sqrt{n-1} & n-1 \leq i. \end{cases}$$

Then

$$\lambda(P, v_0) \leq \frac{x^\top L x}{x^\top M x}.$$

We can bound the numerator above by

$$\begin{aligned}
x^\top L x &= \sum_{i=1}^{N} (x_i - x_{i-1})^2 \\
&= \sum_{i=n}^{N} (\sqrt{i} - \sqrt{i-1})^2 \\
&\leq \sum_{i=n}^{N} \left(\frac{1}{2\sqrt{i-1}}\right)^2 \\
&= \frac{1}{4} \sum_{i=n}^{N} \frac{1}{i-1} \\
&= \frac{1}{4} \left(\sum_{i=n}^{N} \frac{1}{i}\right) + O(1).
\end{aligned}$$

We can bound the denominator below by

$$\begin{aligned}
x^\top M x &= \sum_{i=1}^{N} \mu_i x_i^2 \\
&= \sum_{i=n}^{N} \mu_i \left(\sqrt{i} - \sqrt{n-1}\right)^2 \\
&= \sum_{i=n}^{N-1} \frac{1}{i(i+1)} \left(i + (n-1) - 2\sqrt{i(n-1)}\right) + \frac{1}{N} \left(\sqrt{N} - \sqrt{n-1}\right)^2 \\
&= \sum_{i=n}^{N-1} \frac{1}{i(i+1)} \left(i + (n-1) - 2\sqrt{i(n-1)}\right) + O(1) \\
&= \left(\sum_{i=n}^{N} \frac{1}{i}\right) + \left(\sum_{i=n}^{N-1} \frac{(n-1)}{i(i+1)}\right) - 2\left(\sum_{i=n}^{N-1} \frac{\sqrt{n-1}}{\sqrt{i}(i+1)}\right) + O(1) \\
&= \left(\sum_{i=n}^{N} \frac{1}{i}\right) + O(1).
\end{aligned}$$

Finally, noting that $\sum_{i=n}^{N} 1/i \geq \int_n^N t^{-1}\, dt = \ln(ne^n/n) = n$ diverges to infinity with n, we have that $\lambda(P, v_0) = \frac{1}{4} + o(1)$. We conclude that the constants in Theorem 28 are optimal.

The same construction and a simple symmetry argument shows that the constants in Theorem 35 are optimal.