Delaunay Diagrams for Well-Spaced Point Sets in Fixed Dimension

Gary L. Miller   Dafna Talmor

Department of Computer Science *
Carnegie Mellon University
Pittsburgh, Pennsylvania 15213

Shang-Hua Teng

Department of Computer Science†
University of Minnesota
Minneapolis, Minnesota 55455

Abstract

We present new structure theorems for the Delaunay diagram of point sets in $\mathbf{R}^d$ for fixed d where the point sets arise naturally in the construction of finite element meshes. We then present a new parallel algorithm for computing the Delaunay diagram in any fixed dimension in $O(\log n)$ random parallel time and n processors. In particular, we show that if the largest ratio of the circum-radius to the length of smallest edge over all simplexes in the Delaunay diagram of P, $DT(P)$, is bounded, (called the bounded radius-edge ratio property), then $DT(P)$ is a subgraph of a density graph. Our algorithm then uses the the observation that $DT(P)$ has a separator of size $O(n^{1-1/d})$ that can be efficiently found by the geometric separator algorithm of Miller, Teng, Thurston, and Vavasis. The bounded radius-edge ratio property is desirable for well-shaped triangular meshes for finite element and finite difference methods. Further, we show that the Delaunay Ball system has finite ply based on which we give a linear space point-location structure for these Delaunay diagrams with $O(\log n)$ time per query.

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1 Introduction

The Delaunay diagram (and its dual, the Voronoi Digram) is one of the most fundamental concepts in computational geometry. Geometric properties and algorithms for Delaunay diagrams (DT)[1] have been active topics of research for several years. The 2D Delaunay triangulation has several desired properties that make it very important to applications such as computer graphics, numerical computing and geometric optimization [2]. In general, a higher dimensional DT may have two drawbacks. One, it may have an exponential number of simplices, $\Theta(n^{\lceil d/2 \rceil})$ [18]) and two, there will be simplices which have arbitrarily bad aspect ratio even when the points are place with some care [11]. The goal of this paper is to understand how to place points in space so as to minimize the two drawbacks but yet not lose sight of the application cited above.

Much analytical and experimental work has been applied to DT for points placed uniformly and randomly in fixed dimension, the Poisson distribution [14, 5, 16]. As far as we can determine the main importance of the uniform Poisson distribution for DT is that the distribution is easy to generate and thus useful for running experiments. One drawback is that many implemented parallel algorithms are tuned to work most efficiently for the uniform distributions [32, 30] but fail to be efficient for nonuniform distributions. Here we define new point distributions for which we can find efficient parallel algorithms and including all the distributions from the applications above. For these distributions we must prove new structure theorems. Our distributions will allow singularities as in Figure 1 which do not occur in the uniform case but do occur in mesh generation.

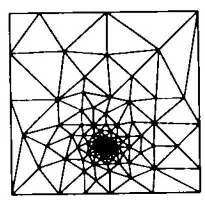

Figure 1: Triangulation of well-spaced point set around a singularity

Algorithmically, by a well-known reduction[6, 7], *DT* in d dimensions can be obtained from a projection of a $d + 1$ dimensional convex-hull. A desired property of convex-hull algorithms is output-sensitivity. The complexity of an output sensitive convex-hull algorithm depends on the number of faces, F, in the convex-hull. However, optimal parallel output sensitive Delaunay diagram construction in high dimensions is still an open problem. Chazelle [8], gave the first optimal deterministic convex-hull algorithm. This algorithm is not output sensitive and thus is optimal only in the worst-case sense, i.e., it runs in $O(n \log n + n^{\lfloor d/2 \rfloor})$ which is the worst-case number of faces possible in $d > 3$ dimensions. Recently, Amato, Goodrich and Ramos [1] gave an optimal randomized parallel algorithm for higher dimensions. They provide a 3-d output-sensitive algorithm, but their algorithm is not output sensitive for $d > 3$. There are no output sensitive parallel algorithms for $d > 3$, all the sequential methods known seem to be hard to parallelize, like the randomized incremental construction algorithm, [10]. Seidel's sequential output sensitive algorithm [29], runs in $O(n^2 + F \log n)$.

In this paper, We present new structure theorems for the Delaunay diagram of point sets in $\mathbf{R}^d$ for fixed d where the point sets arise naturally in the construction of finite element meshes. Our structure theorems lead to the first optimal parallel output sensitive algorithm for this important class of point sets. Furthermore, by exploring theses structure properties, we show that the Delaunay diagrams of this class of points has a linear space, $O(\log n)$ time query structure that can be found in parallel $O(\log n)$ time with n of processors.

In the following paragraphs, we provide some basic background and motivation of our work and give a high level summary of our results.

1.1 Background

Our work is motivated by the following two directions of research concerning Delaunay diagrams.

[1] We use DT to abbreviate Delaunay diagram for historic reasons.

Mesh Generation: An essential step in scientific computing is to find a proper discretization of a continuous domain with a mesh of simple elements such as triangular elements. This is the problem of *mesh generation*. However, not all meshes have equal performance in the subsequent numerical solution. The numerical and discretization error depends on the geometric shape and size of such as the angles and the aspect-ratio of its triangular elements.

The DT has some desired properties for mesh generation. For example, among all triangulations of a point set in 2D, the DT maximize the smallest angle. In fact, Ruppert [27] and Chew [9] have developed Delaunay refinement algorithms that generate a provably good mesh for 2D domains. DT based methods have also been used for coarsening and refinement in domain decomposition and multi-grid methods. In addition to Ruppert's Delaunay refinement algorithm, Bern, Eppstein, Gilbert gave a provably good mesh generators using quad-trees [3, 28]. 3D mesh generation is much harder than 2D; only one provably good mesh generator exist and it was developed by Mitchell and Vavasis [24] and uses oct-trees. On the other hand, various parallel algorithms have been developed in recent years for finite element methods but parallel mesh generation is still less common. Theoretically, Bern, Eppstein, and Teng [4] developed the first parallel algorithm for quality mesh generation in 2D. Although, their approach can be extended to 3D by parallelizing Mitchell and Vavasis oct-tree algorithm [24], the constant in mesh size may be fairly large. Its performance in practice still needs to be seen. It is desirable to have a practical parallel mesh generator especially for 3D.

The expected extremes in Delaunay Triangulations: Bern, Eppstein, and Yao [5] studied various expected properties of the DT of points from uniform distribution. They showed that the expected maximum vertex degree of a Delaunay triangulation is $\Theta(\log n/\log\log n)$ in any fixed dimension and the smallest angle is $\Theta(1/\sqrt{n})$ in 2D.

1.2 Our Contributions

By weakening the condition of bounded aspect ratio of well-shaped meshes, we introduce a geometric condition, called *bounded radius-edge ratio*. The radius-edge ratio is equivalent to aspect-ratio in 2D (See Section 2 for details). We present the following structure results for points with bounded radius-edge ratio.

- The Voronoi polytopes has bounded aspect ratio, namely, the ratio of the circum-scribed radius to the inscribed radius is bounded.
- The Delaunay diagram is a bounded density embedding [22, 21]. The density of a Delaunay diagram is the largest ratio of the longest edge incident to a point to the distance to its nearest neighbor.
- The set of Delaunay balls has finite ply, where the ply of a collection of balls is the maximum depth of overlap among balls.

The bounded density property implies that the DT of a point set of bounded radius-edge ratio has a bounded degree. A density graph is a special case of the overlap graph defined by the nearest neighborhood system [23]. Therefore, we can use the geometric sphere separator decomposition of Miller, Teng, Thurston and Vavasis to develop a divide-and-conquer algorithm that reduces the Delaunay diagram problem of n points in $\mathbf{R}^d$ to a collection of n independent convex hull problems of constant size in $\mathbf{R}^d$. The resulting algorithm finds the DT of a point set of bounded radius-edge ratio in random $O(\log n)$ parallel time using n processors in any dimension.

The finite ply property of Delaunay spheres enables us to use the geometric sphere separator decomposition to develop a linear space, $O(\log n)$-query time structure. We can also find such a structure in $O(\log n)$ parallel time using n processors.

The result of Bern *et al* that the expected smallest angle of the DT of a random point set in two dimensions is $\Theta(1/\sqrt{n})$ implies that with high probability, a random point set does not have the bounded radius-edge ratio property. Their result implies that, numerically, random point sets are not desirable for numerical discretization. This result may be surprising, as the regular grids used for the finite difference method had point density very similar to uniform distribution. But the regular spaced points set give great numerical stability. We introduce an algorithmically efficient "smoothing" technique to make uniformly distributed point sets close to well-spaced.

- We show that the Delaunay balls of random point set from a uniform distribution has ply $O((\log n/\log\log n)^2)$ with high probability.
- We present a smoothing technique, when applying to point set from random distribution, it gives a point set that satisfies the bounded radius-edge ratio.

- We define a random point distribution called *Lipschitz distribution* and show that the application of our smoothing technique to a point set from Lipschitz distribution gives a point set of bounded radius-edge ratio.
- We give an efficient parallel algorithm to approximate the Lipschitz distribution of an input domain and generate a point set of bounded radius-edge ratio that respect the Lipschitz condition.

Our work has important consequences for computational geometry and numerical analysis. For the computational geometry practitioner, this opens a new approach to devising higher dimensional mesh generation algorithms. For the numerical analyst, this supplies tools for experimentation: the basic mesh can be set up quickly and efficiently, and then heuristics can be used to remove the few special slivers that are still in the mesh. Combining our algorithms for DT and Lipschitz distribution point set generation, we obtain a new scheme for parallel 3D mesh generation. Furthermore, our parallel algorithms for DT and point location lead to parallel algorithm of adaptive multi-grid implementations.

2 Definitions

Suppose $P = \{p_1, \ldots, p_n\}$ is a point set in d dimensions. The convex hull of $d+1$ affine independent points from P forms a *Delaunay simplex* if the circumscribed ball of the simplex contains no point from P in its interior. The union of all Delaunay simplices forms the *Delaunay diagram*, *DT(P)*. If the set P is not degenerate then the *DT(P)* is a simplex decomposition of the convex hull of P.

Associated with *DT(P)* is a collection of balls, called *Delaunay balls*, one for each cell in *DT(P)*. The Delaunay ball circumscribes its cell. We denote the set of all Delaunay balls of P by *DB(P)*.

The geometric dual of Delaunay Diagram is the *Voronoi Diagram*, which of consists a set of polyhedra $V_1, \ldots, V_n$, one for each point in P, called the *Voronoi Polyhedra*. Geometrically, V_i the set of points $p \in \mathbf{R}^d$ whose Euclidean distance to p_i is less than or equal to that of any other point in P. We call p_i the *center* of polyhedra V_i. For more discussion, see [26, 13].

Following [21], we call a collection of balls in $\mathbf{R}^d$ a *neighborhood system*. For this reason, we refer the set *DB(P)* the *Delaunay neighborhood system* of P. The *ply* of a point $p \in \mathbf{R}^d$ with respect to a neighborhood system $B = \{B_1, \ldots, B_n\}$ is the number of balls from B that contains p. The ply of a neighborhood system B is the largest ply among all points in $\mathbf{R}^d$. Given a neighborhood system $B = \{B_1, \ldots, B_n\}$, we define a family of geometric graphs called *overlap graphs*.

Definition 2.1 (Overlap Graph) *Let $\alpha \geq 1$ and let $B = \{B_1, \ldots, B_n\}$ be a k-ply neighborhood system. The* **(α, k)-overlap graph** *of B is the undirected graph with vertices $V = \{1, \ldots, n\}$ and edges $E = \{(i,j) : (B_i \cap (\alpha \cdot B_j) \neq \emptyset)$* **and** *$((\alpha \cdot B_i) \cap B_j \neq \emptyset)\}$.*

An important property of overlap graphs, as shown by Miller, Teng, Thurston and Vavasis [23] is that they have small separator. The following is a definition of separators.

Theorem 2.2 (Sphere Separators) *Suppose $B = \{B_1, \ldots, B_n\}$ is a k-ply neighborhood system in $\mathbf{R}^d$. Then for each $\alpha \geq 1$, there is a sphere S that divides B into three subsets: B_I, B_E and B_O such that (1) balls from B_I are completely in the interior of S and balls from B_E are in the exterior of S, (2) there exists a constant $1/2 < \delta < 1$ depending only on d such that $|B_I|$, $|B_E| \leq \delta n$; (3) there is no edge in the (α, k)-overlap graph of B that connect any ball from B_I with any ball in B_E. (4) $|B_O| = O(\alpha k^{1/d} n^{1-1/d})$. Furthermore, such a separator can be found in random linear time sequentially and in random constant time, using n processors.*

A special case of the overlap graph is the *density graph* (first introduced by Miller and Vavasis [22]). The density condition of an embedding is important for finite difference methods. Let G be an undirected graph and let π be an embedding of its nodes in $\mathbf{R}^d$. We say π is an embedding of G of *density* α if the following inequality holds for all vertices v in G. Let u be the closest node to v. Let w be the farthest node from v that is connected to v by an edge. Then

$$\frac{||\pi(w) - \pi(v)||}{||\pi(u) - \pi(v)||} \leq \alpha.$$

In general, G is an *α-density graph* in $\mathbf{R}^d$ if there exist an embedding of G in $\mathbf{R}^d$ with density α. We will show later that there is a $\Delta(\alpha, d)$ depending only on α and d such that the maximum degree of an α-density graph is bounded by $\Delta(\alpha, d)$.

3 Well Spaced Point Sets and their Structures

Numerically, the bounded aspect-ratio is a very desired property for mesh discretization. Computationally, it is important to generate the mesh and to perform point location in the mesh efficiently. Geometrically, various fundamental questions about DT need to be answered: Does the Delaunay neighborhood system of a bounded aspect-ratio DT have bounded ply? What is the "weakest" local condition that point sets need to satisfy to ensure linear size DT and bounded ply Delaunay neighborhood system? How can we efficiently generate a point set with these desired conditions?

We give a characterization of well-spaced point sets based on a weakened aspect-ratio property, called the *bounded radius-edge ratio* property. We show that this local condition implies

- The Delaunay diagram is a bounded density embedding and has linear size.
- The Delaunay neighborhood system has bounded ply.
- The Voronoi polyhedra have bounded aspect ratio.

3.1 Bounded Radius-Edge Ratio of Well-Spaced Point Sets

Definition 3.1 (Bounded radius-edge ratio) *A DT has the* **radius-edge ratio** *bounded by $C \geq 1$ if the largest ratio of the Delaunay sphere radius to the smallest edge over all of its simplices is bounded by β.*

In 2D, if a DT has radius-edge ratio bounded by C then its smallest angle is at least $sin^{-1}(1/(2C))$. Thus bounded radius-edge ratio implies bounded aspect ratio and vice verse. In 3D and higher, the bounded radius-edge ratio does not guarantee that the minimal dihedral angle is bounded. A notorious example is a sliver in 3D which is a simplex whose four nodes are placed almost in a square along the equator of their circumscribing Delaunay sphere. The radius-edge ratio in that case is about $\sqrt{2}$, but the area of the sliver can approach zero. Thus, the radius-edge ratio condition is weaker than the aspect-ratio condition and hence all the structure theorems and algorithms presented in this paper apply to the DT with bounded aspect ratio. An important application is mesh generation, as mesh generators all implicitly maintain bounded radius-edge ratio.

3.2 Density of Delaunay Diagrams

In this subsection, we show that if $DT(P)$ has a bounded radius-edge ratio, then its 1-dimensional skeleton is a density graph, and hence has a bounded degree and a small sphere separator. We will use this result in section 5 to develop an $O(\log n)$ parallel time n processor parallel algorithm for constructing the Delaunay diagram. In all the lemmas and theorems in this section, let P be a point set in $\mathbf{R}^d$ such that $DT(P)$ has ratio bounded by $C > 1$.

Theorem 3.2 (Density Embedding) *There is a constant α dependent only on d and C such that P is an α-density embedding of $DT(P)$.*

It is worthwhile to point out that we can not use the standard volume argument to prove Theorem 3.2 because the bounded radius-edge ratio allows slivers which have close to zero volume. We introduce some notations. For each point $p \in P$, let $N(p)$ be the set of all Delaunay simplices incident to p. For each Delaunay simplex $T \in N(p)$, we refer to the vector from p to the center of the Delaunay sphere of T as the *radius vector* of T. Two simplices are *neighboring* if they share a common edge. The following set of lemmas will be used to prove Theorem 3.2.

Lemma 3.3 *There are constants α_0 and C_1 depending only on C and d such that for all $p \in P$, for each pair of Delaunay simplices T_1 and T_2 in $N(p)$, if the angle between the two radius vectors, respectively, from p to T_1 and T_2 is smaller than α_0, then $\frac{R}{r} \leq C_1$, where R is the larger radius and r is the smaller radius, of the two Delaunay balls.*

Proof: Let $\alpha_0 = arcsine(1/C)/4$. We depict the case in Figure 3.2, where we assume $\alpha \leq \alpha_0$. We have $R\sin(\beta) = r\sin(\alpha + \beta)$. Because all the vertices of the simplex of the smaller Delaunay sphere must be in the exterior of the larger Delaunay sphere, by the bounded radius-edge ratio property, we have $r/(r\sin(\beta + \alpha)) \leq C$. Hence $\sin(\beta + \alpha) \geq 1/C$ and therefore $\beta + \alpha \geq arcsine(1/C)$. So $\beta \geq 3arcsine(1/C)/4$. Let $C_1 = 1/\sin[3arcsine(1/C)/4]$. We have $R/r = \sin(\beta + \alpha)/\sin(\beta) \leq C_1$. □

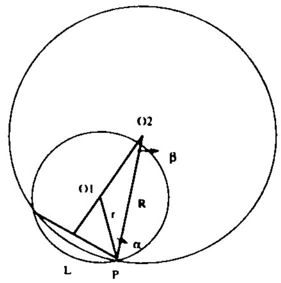

Figure 2: Projection of two intersecting spheres on the plane defined by their radius vectors from P

Lemma 3.4 *If e and E are edges of two neighboring simplices then* $|E|/|e| \leq 4C^2$.

Proof: If ge is an edge common to the two simplices, then $|E|/|e| = (|E|/|ge|)(|ge|/|e|) \leq 4C^2$. □

Lemma 3.5 *If the ratio of the length of radius vectors of two simplices in a Delaunay diagram with radius-edge ratio C is bounded by* C_1, *then the length ratio of the largest edge E to the smallest edge e in the union of two simplices is bounded by* $2C_1C$.

Lemma 3.5 follows directly from the definition of the radius-edge ratio. Let $D = \max(2CC_1, 4C^2)$. To show the DT is a density graph, we cover a very small sphere S centered at a point $p \in P$ by a collection of circular patches with cone angle α_0. The following lemma is a folklore,

Lemma 3.6 *There is a constant K dependent only on* α_0 *and d such that there is a cover of the unit sphere in* $\mathbf{R}^d$ *with no more than K circular patches whose angle is equal to* α_0.

Proof of Theorem 3.2. Let S be a very small sphere centered at $p \in P$. We cover S according to Lemma 3.6. Each radius vector from p intersects sphere S in at least one cone patch (the patches are not necessarily disjoint, so it could intersect more than one patch). Assign to each radius vector a label which corresponds to one of the patches it intersects. If two radius vectors have the same label, then by Lemmas 3.5 and 3.4, the maximal ratio of the edges belonging to the two simplices is bounded by D.

Let e and E be the shortest and the longest Delaunay edges, respectively, incident to p. There is a path between e and E through edges that belong to neighboring simplices incident to p. In each transition of the path, the edge lengths can grow by at most a factor of D.

We assign a label to each edge in the path. The label indicates the patch that the edge's radius vector intersects. If a label appears more than once in the path, we can "erase" all labels between last and first appearance of the label, and instead use the ratio information forced by the label, which is D. This "erasing" process reduce the number of labels to a constant because no label can repeat, and therefore the ratio of $|E|$ to $|e|$ is bounded by a constant and hence P is a density embedding of $DT(P)$. □

Lemma 3.7 *There is a constant* D_G *dependent only on* α *and d such that the vertex degree of each* α*-density graph is bounded by* D_G.

Proof: For each $p \in P$, the neighboring nodes of p are contained in the sphere with radius $\alpha|e|$ centered at p, where e is the smallest edge incident to p. Let q be one of q's neighbors, then q has an edge of length at least $|e|$, so q's nearest neighbor is no closer than $|e|/\alpha$. Therefore, the sphere centered at q of radius $|e|/(2\alpha)$ does not intersect with the sphere centered at any other neighboring node of p of radius $|e|/(2\alpha)$. A simple volume argument shows that p can have bounded number of neighbors. □

3.3 The Ply of Delaunay Neighborhood Systems

In this section, we show that bounded radius-edge ratio, a local geometric condition, implies the bounded ply of the Delaunay neighborhood system, a global condition. We will use this result in the next section to design a space and time optimal query structure of Delaunay diagram of well-spaced point sets.

Theorem 3.8 (Bounded Ply) *There is a constant k depending only on C and d such that the ply of the Delaunay neighborhood system of P is bounded by k.*

For ease of exposition, we present the 2D version of the proof and will point out how to extend it to higher dimensions. We will use different techniques to bound the ply of points inside and outside the convex hull.

Lemma 3.9 (Voronoi is Well-Shaped) *There is a constant C_1 depending only on C and d such that the Voronoi diagram of P has the property that for each Voronoi polytope, the ratio of its circumscribed sphere to its inscribed sphere is bounded by C_1. For each Voronoi polyhedra, we define a truncated form which is aspect-ratio bounded.*

Proof: We first consider the finite Voronoi polytope of point $p \in P$. Let R be the radius of the largest Delaunay ball touching p, and let ρ be half the distance to p's nearest neighbour. By definition, the ball of radius ρ is contained in the polytope, and the ball of radius R contains the polytope. By theorem 3.2 the ratio of ρ and R is bounded. For the Voronoi polyhedra, which is an infinite region, we consider the region which is the intersection of the Voronoi polyhedra and the ball of radius R defined above. □

Lemma 3.10 *Each Delaunay ball can intersect at most a constant number of Voronoi cells within the convex hull. Moreover, the ratio between the Delaunay ball to the smallest inscribed radius of the Voronoi cell it intersects within the convex hull is bounded.*

Proof: By scaling, we assume the radius of the Delaunay ball B is equal to 1. No Voronoi cell can be contained completely in a Delaunay ball. We choose a small constant $\epsilon < 1$. Let B' be the ball of radius $1 - \epsilon$ that is concentric to B.

We first bound the number of Voronoi cells that intersect B'. Let α be the aspect-ratio of the Voronoi diagram. By Lemma 3.9, α is a constant. For each Voronoi cell that intersect B', one of its dimension must be at least ϵ. Therefore, the area of the intersection of the Voronoi cell with the B is at least $\Theta(\epsilon^2/\alpha)$. Since Voronoi cells are disjoint in their interior, the number of such cells is bounded by $O(\alpha\pi/\epsilon^2)$.

We now define a "protective ring" around the outside of B and show that the number of Voronoi cells whose input point lies in this ring is bounded. To define such a protective ring, we first examine a point q on the boundary of B'. Suppose q is in the Voronoi cell for point p. Let Γ be a ball centered at q of radius $\epsilon + \mu$ for a small constant $\mu < \epsilon$ such that for each point in the "crescent" defined by the section of Γ outside B, a ball of radius ϵ/α centered at the point completely contains the crescent. If p is not in the crescent, then clearly the crescent can not contain any other input point because p is its closest point. Now assume p is in the crescent, then because its Voronoi cell has one dimension of size at least ϵ, the ball centered at p of radius ϵ/α must be free of other input points. Therefore, the crescent contain at most one input point. We can choose a finite number of points q_i on the boundary of B' such that the union of their crescent cover a ring of some small constant width $\epsilon_1 < \mu$ outside B. Moreover, the number of Voronoi cells whose input point is in this ring is bounded.

To complete the proof, it suffices to bound the number of Voronoi cells whose point from P lies outside the protection ring around B. All such Voronoi cells must have one dimension at least ϵ_1 and the area of intersection with the protective ring is at least $(\epsilon_1)^2/\alpha$. Therefore, there can be at most finite of them.

If B is not completely contained in the convex hull, we argue only for the section that is in the convex hull. The argument generalizes by showing we can find some such B' of radius δ which is contained in a union of truncated polyhedra as defined in the previous lemma. Our proof can be directly extended to higher dimensions. □

It is interesting to point out that the part of a Delaunay ball B outside the convex hull may intersect unbounded number of Voronoi cells. To bound the ply outside the convex hull, we need a different argument.

Lemma 3.11 *There is a constant C_1 depending only on C and d such that if two Delaunay balls B_1 and B_2 intersect outside the convex hull of P then the ratio between their radiuses is bounded C.*

Proof: We give a sketch of the proof. By scaling assume the radius of B_1 is 1. We focus on the intersection of B_1 with the boundary of the convex hull of P. The Delaunay simplices neighboring the intersection of B_1 with the convex hull must be greater than some constant fraction α, by previous theorems. Therefore, the union of these neighboring cells and their Delaunay balls forms a "protective region" near the boundary of the convex hull where B_1 intersect. If B_2 is too small, it must originate in some simplex away from the protective region, and therefore its boundary is some fraction away from the boundary of B_1 as they both intersect the convex hull. Some constant fraction of B_1 is contained in the convex hull, and therefore its slope as it intersect the convex hull is such that if B_2 is to intersect it outside the convex hull, the radius of B_2 must be greater than some constant fraction. □

Proof of Theorem 3.8: In Lemma 3.10, we show that each Delaunay ball intersects at most a constant N_1 number of Voronoi cells inside the convex hull. Because $DT(P)$ is a bounded density embedding, each Voronoi cell neighbors at most N_2 Voronoi cells. When two Delaunay balls intersect within the convex hull, they intersect in some Voronoi cell. Therefore the set of all Voronoi cells intersected by the union of the two Delaunay balls can be reached from each other by a N_1 steps of neighboring relation among Voronoi cells. There are at most $N_1^{N_2}$ Voronoi cells that are within N_1 steps of neighboring relation among Voronoi cells. This implies that a Delaunay ball can only intersect a constant number of other Delaunay balls within the convex hull of P.

For each point q in the exterior of the convex hull of P, all Delaunay balls that contain q must intersect among themselves. Therefore the ratio of their radius is bounded. Assume the largest radius is 1. For each point $p \in P$ that is on one of the Delaunay balls, its nearest neighborhood ball has radius at least some constant fraction (from the bounded radius-edge ratio) and all these balls are disjoint. By volume argument, there can be constant number of them. Because the $DT(P)$ has bounded degree, there are only constant number of Delaunay balls that touch them. Thus the ply of p is bounded. □

3.4 Optimal Query Structure for Delaunay Balls

Given the Delaunay Diagram $DT(P)$ of a point set P, there are two types of queries that are important to numerical and geometric computing (1) Return the Delaunay simplex in $DT(P)$ that contains a given query point $q \in \mathbf{R}^d$. (2) Return the set of Delaunay Balls of $DT(P)$ that cover a given query point $q \in \mathbf{R}^d$.

An important corollary Theorem 3.8 is the following theorem:

Theorem 3.12 *Both $DT(P)$ and the Delaunay neighborhood system of P has a query structure of linear size and $O(\log n)$ query time that can be found in random $O(\log n)$ parallel time using n processors.*

To our knowledge, this is the first linear time structure for Delaunay diagrams in higher dimensions that achieves both linear size and $O(\log n)$ query time.

Our query structure uses the sphere separator theorem of Miller *et al* [21], showing that for each k-ply neighborhood system $B = \{B_1, \dots, B_n\}$ in $\mathbf{R}^d$, there is a sphere S that divides B into three subsets: B_I, B_E and B_O such that (1) balls from B_I are completely in the interior of S and balls from B_E are in the exterior of S, (2) there exists a constant $1/2 < \delta < 1$ depending only on d such that $|B_I|, |B_E| \leq \delta n$; (3) $|B_O| = O(k^{1/d} n^{1-1/d})$.

By Theorem 3.8, we have that the ply of the Delaunay neighborhood system of P is bounded by a constant. We can apply the sphere separator theorem recursively to build a binary tree of separating spheres. The root of the tree contains the top level sphere separator S and its left subtree and right subtree are recursively generated for $B_I \cup B_O$ and $B_E \cup B_O$. From Condition 3 of the sphere separator theorem, we can show that the query structure has size $O(n)$. From Condition 2, we can show that the height of the structure is $O(\log n)$. To answer a query for point q, we step down the structure by inclusion and exclusion tests against the separating sphere of the structure nodes.

4 Well-Space Point Set Generation

In this section, we show how to use two techniques, *oversampling* and *filtering* to generate a well-spaced point set according to a density function. We also show that even a uniform random point set is ill-spaced, its Delaunay neighborhood system in 3D has ply bounded by $O(\log^2 n)$ and at least as big as $\Omega((\log n/ \log\log n)^2)$.

For uniform distribution, we use the *homogeneous Poisson point process of intensity one* which is characterized by the property that the number of points in a region is a random variable that depends only on the d-dimensional volume of the region [19, 17, 5, 31]. In this model, The probability of exactly k points appearing in any region of volume V is $e^{-V}V^k/k!$ and the conditional distribution of points in any region given that exactly k points fall in the region is joint uniform.

4.1 Point Generation for Lipschitz Distributions

The aspect ratio, degree, and ply of the Delaunay diagram of a uniform random point set generated by the Poisson process are all unbounded, as shown by Bern, Eppstein, and Yao [5]. If instead of using the Poisson point process with intensity one, we oversampled such that with high probability each unit area has at least one point, the degree, aspect ratio and the ply would still be unbounded. Our idea is to selectively remove some of the extra points after oversampling. By carefully using these two techniques, *oversampling* and *filtering*, we can efficiently create a point set whose Delaunay diagram has constant degree and constant ply. Moreover, we can extend the results to probability densities whose inverse, i.e., their spacing function, is Lipschitz.

Definition 4.1 (Lipschitz) *A function f is Lipschitz with constant α if for any two points x, y in the domain $|f(x) - f(y)| \leq \alpha||x - y||$.*

Definition 4.2 *Let f be a Lipschitz function, S be a point set, Ω be the domain. S is f* **-spaced** *if for any two points $P, Q \in S$, $||P - Q|| \geq \min(f(P), f(Q))$. S is* **maximally** *f-spaced if no point from Ω can be added to S without violating the f-spacing. S is* **approximately** *f-spaced if $\exists\delta$ s.t. $\forall x \in \Omega$ the $\delta f(x)$-ball centered at x contains at least one point that can not be added to S without violating the f-spacing.*

We prove the following critical theorem for well-spaced point generation:

Theorem 4.3 *A maximally f-spaced point set, with f α-Lipschitz where $\alpha < 1$, has a bounded radius-edge ratio Delaunay diagram, with constant $1/(1-\alpha)$.*

Proof: Let R be the radius of the Delaunay ball of a Delaunay simplex D. Let l be the length of the smallest edge of D. Therefore, one of the end points of the smallest edge, P, has that $f(P) \leq l$. The value of the function f at the center of the sphere is therefore smaller then $l + \alpha R$. If $l + \alpha R < R$ then the center can be added to S, which contradicts maximality. Therefore, $R \leq l + \alpha R$, or, $\frac{R}{l} \leq \frac{1}{1-\alpha}$. □

A similar theorem is true for an approximately f-spaced point sets, for some δ depending on the Lipschitz constant. The above theorem ignored boundary effects, i.e., it made the assumption that the Delaunay ball is fully contained in the domain. In order to account for boundary effects, the edges will have a g-spaced point set, where g is β-Lipschitz for an appropriate $\beta < \alpha$. In effect, we first place points on the edges (a lower dimensional process) more densely, so that points in the interior will be slightly repelled from the edges, and will not have huge Delaunay circles that are mostly outside of the domain [20].

Theorem 4.4 (Point Generation) *Given a Lipschitz f in a domain Ω, we can generate an approximately f-spaced point set of size $O(K)$ in $O(\log K)$ time, using K processors, where K is the size of optimal f-spaced point set for f-distribution over Ω.*

Our approach is to first oversample a random point set such that each δ-ball, which is of area $\frac{1}{\delta^2}$, is guaranteed to contain a point, and then pick a maximal independent set of the violations graph, namely filtering.

In the context of mesh generation, the optimal point spacing is the **lfs** function, which assigns to each point in the domain its distance to the second closest input feature [24, 3, 27]. This function has a Lipschitz constant 1. Using the above ideas, we can present an efficient parallel algorithm for generating a point set for an optimal mesh. In 2-d, the algorithm is a hybrid of the quad-tree algorithm of Bern, Eppstein and Gilbert [3], and the Delaunay based algorithm by Ruppert [27]. It has the efficiency and parallelization qualities of the quad-tree algorithm, and the output density is closer to Ruppert's, but unfortunately, in 3-d and higher, it introduces slivers.

We now show how to generate an **lfs**-spaced point set. We will concentrate only on points in the interior, as producing the points on the edges is a straightforward extension. We use Mitchell and Vavasis balanced oct-tree/quad-tree algorithm as our start point as an initial approximation for the **lfs** in the following sense:

Definition 4.5 *Given an oct–tree T, we define the function $f_T(x) = l$ where l is the side length of the box x is in. f_T is said to approximate a function f if $\exists c_1, c_2 > 0$ such that $c_1 f_T \leq f \leq c_2 f_T$*

We use the oct-tree as an initial approximation for the points density, and then perform smoothing and filtering to obtain a density closer to optimal. Of course the oct–tree corners themselves could be well spaced, but by using a rather coarse oct–tree, we get a random point set which is well spaced, rather than boxes which are aligned with the x, y coordinates. By filtering, we also get a point set whose density is closer to optimal, compared with the oct–tree which is experimentally shown to have high constants [27].

Algorithm sketch

- Apply the 3D balanced oct-tree or 2D balanced quad-tree algorithm to approximate the **lfs**.
- In each cell, place a constant number of points, and derive a better bound of the local feature size by searching constant number of nearby cells.
- Create a graph over the points, by connecting two nodes if the distance between them is larger than some constant times the local feature size of either and return a maximal independent set of the graph.

This is a simple sketch of the algorithm. In the presence of boundary faces and edges, a lower dimensional version of the algorithm will have to be run first on the edges and faces of the input.

4.2 DT in Uniform Distribution

We show in the appendix that the ply of the Delaunay balls of points from uniform Poisson process in $\mathbf{R}^d$ is bounded by $O(\log^{\lceil d/2 \rceil} n)$. We also show an almost tight lower bound. This result complements Bern, Eppstein, and Yao's that the DT of uniform random point set over a square of side length $n^{1/d}$ has the expected maximum degree $\Theta(\log n / \log\log n)$ and the 2D aspect-ratio $\sqrt{n}$.

5 Parallel Algorithms

Theorem 5.1 *Let P be a point set in $\mathbf{R}^d$. If DT(P) has a bounded radius-edge ratio, then DT(P) can be found in $O(n \log n)$ time sequentially and in randomized parallel $O(\log n)$ time using n processors.*

Our parallel algorithm uses the structure theorems of Section 3. Using sphere separator decomposition, it first find a supergraph of $DT(P)$ that also has bounded degree. The supergraph reduces the Delaunay diagram problems of n points in $\mathbf{R}^d$ to a collection of n independent convex hull problems of constant size in $\mathbf{R}^d$.

5.1 Density graphs as super-graphs for DT

We first define the *α-density graph* of a point set P, denoted by $DG_\alpha(P)$. Let B_i be the nearest neighbor ball of p_i, i.e., the ball whose center is p_i and whose radius is equal to the distance between p_i to its nearest neighbor in P. The α-density graph of P is the restriction of the α-overlap graph (See section 2) for this neighborhood system to a density graph — that is, all edges that are longer than α times the nearest neighbor are removed from the α-overlap graph.

Notice that the α-density graph of P is the supergraph of any α-density embedding of a graph that uses P as its vertices. Therefore, if $DT(P)$ satisfies the bounded radius-edge ratio property then by Theorem 3.2 there exists a constant α, depending only on d and the radius ratio, such that $DT(P)$ is a subgraph of $DG_\alpha(P)$. Notice also that $DG_\alpha(P)$ has bounded degree as well.

Using a similarly construction as in [15], we can compute the α-density graph in random parallel $O(\log n)$ time using n processors.

5.2 Convex Hulls and Delaunay Triangulations

The following lemma shows how to find the Delaunay diagram when a supergraph of the Delaunay diagram is given.

Lemma 5.2 *Let P be a point set in* $\mathbf{R}^d$ *and assume that DT(P) has degree bounded by* D_1. *Let G be a supergraph of DT(P) of maximum degree* $D_2 \geq D_1$. *Then we can compute DT(P) from G in* $O(T_{CH,d}(D_2))$ *using n processors, where* $T_{CH,d}(m)$ *is the sequential time for finding the convex hull of m points in d dimensions.*

Therefore, given G with a constant degree bound, $DT(P)$ can be found in $O(1)$ time using n processors.

In the proof of Lemma 5.2 we exploit the geometric relationship between Delaunay diagrams and convex hulls. For each point p in $\mathbf{R}^d$, let $lift(p) = (p, ||p||^2)$, where $||p||$ is the norm of vector given by p. Geometrically, *lift* maps point p vertically onto the paraboloid $x_{d+1} = \sum_{i=1}^{d} x_i^2$

Brown [7] and Edelsbrunner and Seidel [12] proved the following result.

Lemma 5.3 *Suppose* $P = \{p_1, \ldots, p_n\}$ *is a point set in* $\mathbf{R}^d$. *Let* $Q = \text{lift}(P)$. *Then DT(P) is isomorphic to the lower convex hull of Q.*

Therefore the problem of finding the Delaunay diagram in d dimensions can be reduced to the problem of finding the convex hull in $(d+1)$ dimensions. Instead of using Lemma 5.3 directly, we use it to relate a Delaunay diagram in d dimensions to a set of small convex hull problems in d dimensions.

From Lemma 5.3, (p_i, p_j) is an edge in $DT(P)$ only if (q_i, q_j) is on the convex hull (lower hull) of Q. One way to recognize the set of edges with endpoint p_i that belong to $DT(P)$ is to recognize the set of edges with endpoint q_i that belong to the convex hull of Q.

Lemma 5.4 *Suppose we take a hyper-plane H in* $\mathbf{R}^{d+1}$ *close enough to* q_i *to separate* q_i *and* $Q - \{q_i\}$. *Let* q'_j *be the intersection of* q_iq_j *and H. Then* q_iq_j *is an edge on the convex hull of Q iff* q'_j *is on the convex hull of* $\{q'_j : j \neq i\}$.

Lemma 5.4 yields another way to find $DT(P)$: Lift P to the paraboloid to obtain Q and solve the n convex hull problems (one for each point in Q) in d dimensions. The convex hull problem for q_i determines the set of convex hull edges of Q with q_i as an endpoint, and hence, the set of edges of the Delaunay Triangulation of P with p_i as an endpoint.

Now suppose G is a supergraph of $DT(p)$. To determine the set of edges with endpoint p_i of $DT(P)$, we simply lift the graph neighbors of p_i and perform a d dimensional convex hull construction (as in Lemma 5.4). We can perform such local operation independently for all points in parallel. Therefore, if the maximum degree of G is D_2, we can compute $DT(P)$ from G in $O(T_{CH,d}(D_2))$ using n processors, completing the proof of Lemma 5.2.

6 Final Remarks

We have given a simple and efficient parallel algorithm to compute the Delaunay diagram in d dimensions, for point sets with bounded radius-edge ratio, and for random point sets. This is the first parallel construction of a Delaunay diagram for more than 2 dimension we know of, that is optimal for a linear sized diagram. We have also demonstrated that point sets with these properties play a crucial part in the important problem of mesh generation and therefore that the bounded radius-edge ratio is a natural restriction. We have shown as well that this condition implies that the Delaunay graph is a density graph. Our point generation algorithm and our Delaunay triangulation algorithm provide two important subroutines for parallel three dimensional mesh generation.

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A Appendix

A.1 DT in Uniform Distribution

We show the ply of the Delaunay balls of points from uniform Poisson process in $\mathbf{R}^d$ is bounded by $O(\log^{d/2} n)$. This result complements Bern, Eppstein, and Yao's that the DT of uniform random point set over a square of side length $n^{1/d}$ has the expected maximum degree $\Theta(\log n/\log\log n)$ and the 2D aspect-ratio $\sqrt{n}$. In the following theorems, let P be a random point set by the unit-intensity Poisson process in $\mathbf{R}^d$ over a cube of size length $n^{1/d}$.

Theorem A.1 *The ply of the Delaunay neighborhood system of P is bounded by $O(\log^{\lceil d/2 \rceil} n)$.*

Proof: As shown in [5], with probability at least $(1 - 1/n^k)$, the radius of all Delaunay balls of P is bounded by $c_1 \log^{1/d} n$, where c_1 is a constant depending only on d. Suppose there are m balls covering a point p. With probability at least $(1 - 1/n^k)$, all m Delaunay balls are contained in the ball B_p of radius $2c_1 \log^{1/d}$ centered at p. With high probability the number of points in B_p is bounded by $O(\log n)$. Hence the number of Delaunay simplices that are contained in B_p is at most $O(\log^{\lceil d/2 \rceil} n)$. □

Theorem A.2 *When $d = 3$, with high probability, the ply of the Delaunay neighborhood system of P is least* $O((\log n/\log\log n)^2)$.

Proof: We use Preparata's example: the union of the set of $N/2$ uniformly placed points on a horizontal circle and another set of $N/2$ uniformly placed points on a vertical line passing through the center of the circle [25]. This example has ply of $O(N^2)$ since its $\Theta(N^2)$ Delaunay balls are all within a cube of size N. We show that a construct similar to Preparata's example has high probability of appearing, for $N = (\log n/log \log n)$.

Rather than placing points, we place small spheres of volume $\log n^{-i}$. The horizontal circle is of radius $O(\log^{1/3} n)$ and the vertical line is of the same length. The small spheres are placed such that if a small sphere S_1 placed on the horizontal circle is not empty, then for each nonempty sphere S_2 on the vertical line: $\exists q \in S_1$ and $\exists p \in S_2$ such that q and p are Delaunay neighbors. This is true if for any such S_1 and S_2 there is a larger sphere (of radius $O(\log^{1/3} n)$) that includes S_1 and S_2 but not any of the other S_i's. Since we are placing $\log n / \log\log n$ spheres on a horizontal circle of length $O(\log^{1/3} n)$, there is some i, for example $i = 6$ or greater, for which this is true.

Our geometric construct is therefore a larger sphere of radius $\delta \log^{1/3} n$, which is empty but for the smaller spheres S_i's placed in it, as described above. For a small enough δ, there are more than $n^{0.9}$ such empty independent spheres, with high probability (see [5]). The following Lemma bounds below the probability that some constant fraction of the horizontal spheres, and some fraction of the vertical spheres, are not empty. That fraction, q, is adjusted to compensate for i fixed above. Since we run $n^{0.9}$ independent experiments in one Poisson point process, such high ply is actually expected to appear with high probability. Theorem A.1 then follows.

Lemma A.3 *The probability that $q \log / \log\log n$ of the horizontal and of the vertical spheres are non empty is at least $e^{-2} n^{-2iq}$.*

Proof: Let $l = \log n / \log\log n$ and $k = ql$ for some small constant q to be fixed later. Let a denote the volume of the smaller spheres. Then (1) The probability that a small sphere is not empty is greater than $\frac{0.5}{\log n^i}$. (2) The probability that $k = q \log n / \log\log n$ of the spheres are non empty is at least (we omit the binomial): $a^k (1-a)^(l-k)$ which is: $a^k = 2^{-iq\log n}$, and $(1-a)^{l-k} = e^{-\frac{1-q}{\log\log n \log n^i - 1}} > \frac{1}{e}$ Therefore, the probability of k spheres non empty is at least $e^{-1} n^{-iq}$. (2) The probability of both horizontal and vertical events occuring at the same time is $e^{-2} n^{-2iq}$. □