

Recursion and Induction

15-150

Lecture 3: September 3, 2024

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Recap of week 1

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Functional programming

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→ evaluation of expressions (no mutation!)

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- evaluation of expressions (no mutation!)
- facilitates specification and reasoning about program

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- correctness proof (today's topic!)

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Types, expressions, values

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Types, expressions, values

- types as specifications

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Types, expressions, values

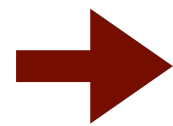
- types as specifications
- observation: once your program type checks, it works!

Recap of week 1

Extensional equivalence (\cong)

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Extensional equivalence (\cong)



“Two things are equal if they behave the same”

Recap of week 1

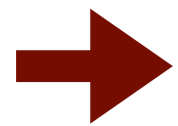
we'll revisit
exact definition

Extensional equivalence (\cong)

→ “Two things are equal if they behave the same”

Recap of week 1

Extensional equivalence (\cong)



“Two things are equal if they behave the same”

Recap of week 1

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Declarations, binding and scope

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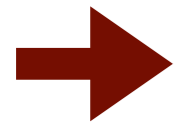
- shadowing of bindings
- function declarations bind a closure to the function identifier
- closure comprises lambda expression and environment with bindings existing at declaration time

Recap of week 1

Pattern matching

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Pattern matching



patterns are used at binding sites of values

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Pattern matching

- patterns are used at binding sites of values
- eg, val bindings, function arguments, case expression
- allow us to match against an expected value
- allow us to decompose a value in its constituent parts, introducing appropriate bindings for parts

Recap of week 1

5-step methodology of function declaration

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1 function name and type

Recap of week 1

5-step methodology of function declaration

- 1 function name and type
- 2 **REQUIRES:** precondition

Recap of week 1

5-step methodology of function declaration

- 1 function name and type
- 2 REQUIRES: precondition
- 3 ENSURES: postcondition

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5-step methodology of function declaration

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- 4 function body

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- 5 tests

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Today, we add a 6th step:

Recap of week 1

5-step methodology of function declaration

- 1 function name and type
- 2 REQUIRES: precondition
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- 4 function body
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Today, we add a 6th step:

- 6 correctness proof

Today's topic: functional correctness

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Let's prove our programs correct, one function at a time!

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→ we consider how expressions are evaluated

Today's topic: functional correctness

Let's prove our programs correct, one function at a time!

- we will use three kinds of induction:
 - mathematical induction
 - strong induction
 - structural induction
- we consider how expressions are evaluated
- we may appeal to mathematical properties and assume that SML implements them correctly

An example: compute n^k

```
(* power : (int * int) -> int
   REQUIRES: k >= 0
   ENSURES: power(n,k) ==> n^k, with 0^0 = 1.
*)
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recursive call

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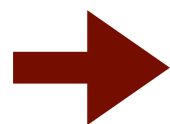


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$$\text{eg, } 3^7 = 3 * 3 * 3 * 3 * 3 * 3 * 3$$

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➔ Number of recursive calls: $O(k)$

A more efficient version of power

```
(* even : int -> bool
   REQUIRES: true
   ENSURES: even(k) evaluates to true if k is even
             evaluates to false if k is odd.
*)
```

```
fun even (k:int) : bool = ((k mod 2) = 0)
```

```
(* square : int -> int
   REQUIRES: true
   ENSURES: square(n) ==> n^2
*)
```

```
fun square (n:int) : int = n * n
```

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(* powere : (int * int) -> int
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  REQUIRES: k >= 0
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  ENSURES: powere(n,k) ==> n^k, with 0^0 = 1.
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  powere computes n^k using O(log(k)) multiplies.
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    then square(powere(n, k div 2))
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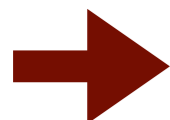
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Number of recursive calls: $O(\log(k))$

Let's verify our naive version of power

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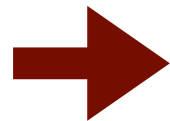
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Let's use mathematical induction!

Mathematical (simple, weak) induction

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To prove a property $P(n)$ for every natural number n :

- show that $P(0)$ holds
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base case

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inductive step

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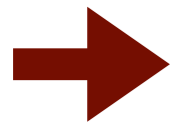
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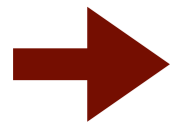


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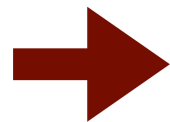
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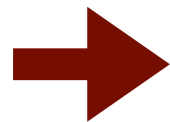
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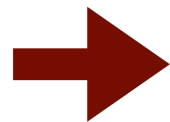
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- $P(2)$ follows from $P(1)$.
- etc...

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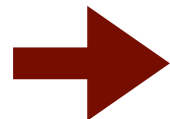


Proof by mathematical induction on ???

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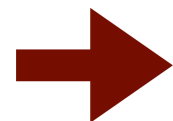


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Proof by mathematical induction on k .

k is the integer that gets smaller!

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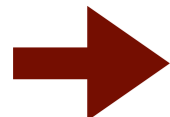
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need for applying IH!

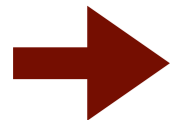
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Proof by mathematical induction on k .



Let's do the proof together!

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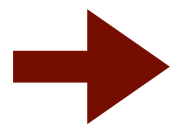
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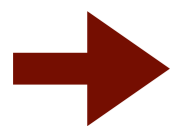


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Proof by strong induction on k .

Strong induction

To prove a property $P(n)$ for every natural number n :

- show that $P(0)$ holds
- then, show that for all $k > 0$,
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Strong induction

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base case

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inductive step

Strong induction

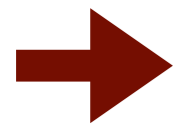
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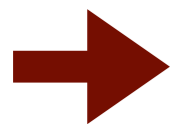
➔ For mathematical induction, IH can only be appealed to for the immediate predecessor!

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fun powere (_:int, 0:int) : int = 1
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Theorem: $\text{power}(n, k)$ evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .



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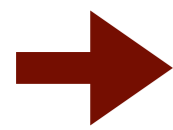
→ Proof by strong induction on k .

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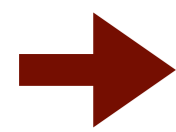
Taking stock

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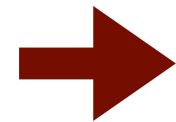


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more on datatypes in
later lectures!

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- today: example using **structural induction** on lists

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Type: `t list` for any type `t`

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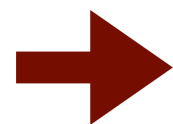
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ie, `1 :: 2 :: 3 :: nil` means `1 :: (2 :: (3 :: nil))`

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➔ cons is right-associative

ie, `1 :: 2 :: 3 :: nil` means `1 :: (2 :: (3 :: nil))`

➔ cons evaluates left to right (for sequential evaluation)

Length function for an int list

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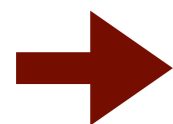
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➔ eg, recursive functions can loop!

Totality

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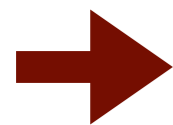
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Partiality complicates extensional equivalence (next time!)

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To prove a property $P(L)$ for every value L of type `int list`:

- show that $P([])$ holds
- show that, if $P(L')$ for some value L' of type `int list`, then it also holds for $v :: L'$ (for any value v of type `int`).

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Length is total: proof

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Theorem: For all values $L : \text{int list}$, $\text{length}(L)$ evaluates to an integer value v .

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Proof: By structural induction on L .

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Need to show: $\text{length}([])$ evaluates to some value $v : \text{int}$.

Showing:

$\text{length } []$
 $\implies 0$ (step, 1st clause of `length`)

Length is total: proof continued

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$\text{length}(x :: xs)$
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Showing:

$\text{length}(x :: xs)$	
$\implies 1 + \text{length}(xs)$	(step, 2nd clause of <code>length</code>)
$\implies 1 + v$	(IH)
$\implies v'$	(some v' , assume + total)

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Inductive case: $L = x :: xs$, for some values $x : \text{int}$ and $xs : \text{int list}$.

IH: $\text{length}(xs)$ evaluates to some value $v : \text{int}$.

Need to show: $\text{length}(x :: xs)$ evaluates to some value $v' : \text{int}$.

Showing:

$\text{length}(x :: xs)$	
$\implies 1 + \text{length}(xs)$	(step, 2nd clause of <code>length</code>)
$\implies 1 + v$	(IH)
$\implies v'$	(some v' , assume + total)

SML addition
must be total!

Correspondence

Correspondence

Data structure

Code

Proof

Correspondence

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Proof

base case(s):

inductive/recursive case(s):

Correspondence

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Proof

base case(s):

0

```
fun power(_, 0) = 1
```

$\text{power}(_, 0) \Leftrightarrow 1$

inductive/recursive case(s):

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Proof

base case(s):

0

```
fun power(_, 0) = 1
```

$\text{power}(_, 0) \hookrightarrow 1$

inductive/recursive case(s):

$(k-1)+1$

```
power(n, k) = n * power(n, k-1)
```

IH: $\text{power}(n, k) \hookrightarrow n^k$

NTS: $\text{power}(n, k+1) \hookrightarrow n^{k+1}$

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base case(s):

0

```
fun power(_, 0) = 1
```

$\text{power}(_, 0) \hookrightarrow 1$

[]

```
0
```

$\text{length } [] \hookrightarrow 0$

inductive/recursive case(s):

$(k-1)+1$

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0
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$\text{length } [] \hookrightarrow 0$

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$x :: xs$

```
length(x :: xs) = 1 + length(xs)
```

IH: $\text{length}(xs) \hookrightarrow v$

NTS: $\text{length}(x :: xs) \hookrightarrow v'$

Correspondence

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Proof

base case(s):

\emptyset

```
fun power(_, 0) = 1
```

$\text{power}(_, 0) \hookrightarrow 1$

[]

\emptyset

$\text{length } [] \hookrightarrow 0$

inductive/recursive case(s):

$(k-1)+1$

```
power(n, k) = n * power(n, k-1)
```

IH: $\text{power}(n, k) \hookrightarrow n^k$

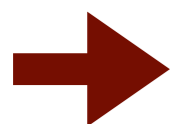
NTS: $\text{power}(n, k+1) \hookrightarrow n^{k+1}$

$x :: xs$

```
length(x::xs) = 1 + length(xs)
```

IH: $\text{length}(xs) \hookrightarrow v$

NTS: $\text{length}(x::xs) \hookrightarrow v'$



possibly several bases and inductive cases!