Sorting trees — work and span revisited

15-150

Lecture 8: September 19, 2024

Stephanie Balzer
Carnegie Mellon University

Midterm I: room update

When and where:

- Thursday, **September 26**, **11:00am—12:20pm**.
- **MM 103** (Sections A—D), **PH 100** (Sections E—L).

Scope:

- Lectures: 1—8.
- Labs: 1—4 and midterm review section of Lab 5.
- Assignments: Basics, Induction, and Datatypes.

What you may have on your desk:

- Writing utensils, we provide paper, something to drink/eat, tissues.
- 8.5" x 11" cheatsheet (back and front), handwritten or typeset.
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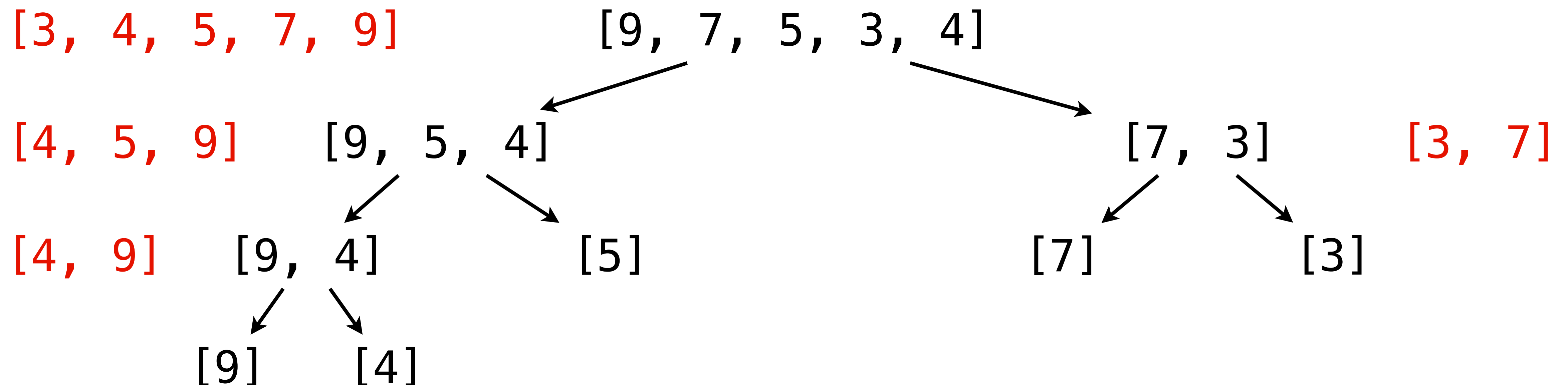

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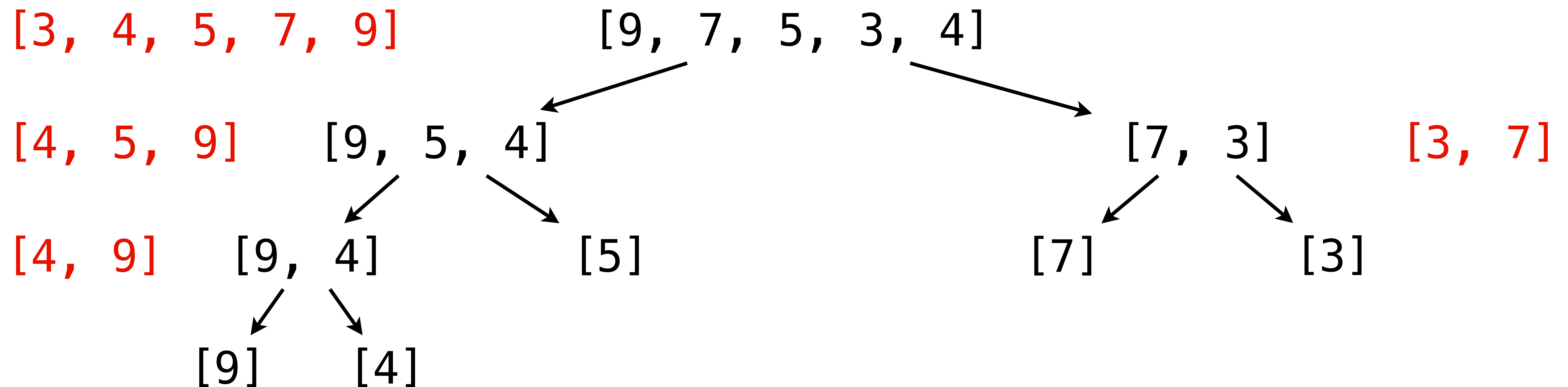

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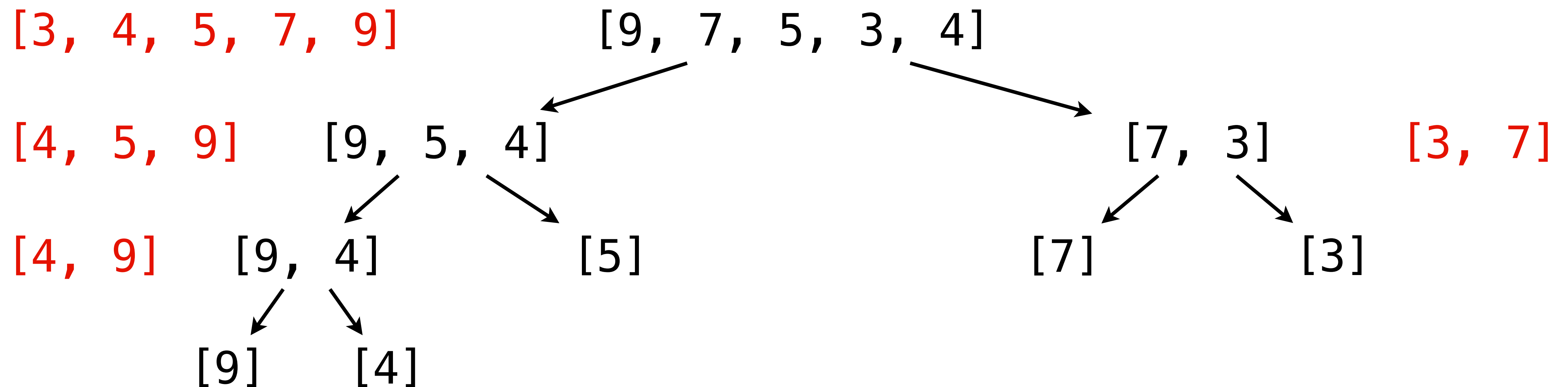
parallelize recursive calls on sub-lists



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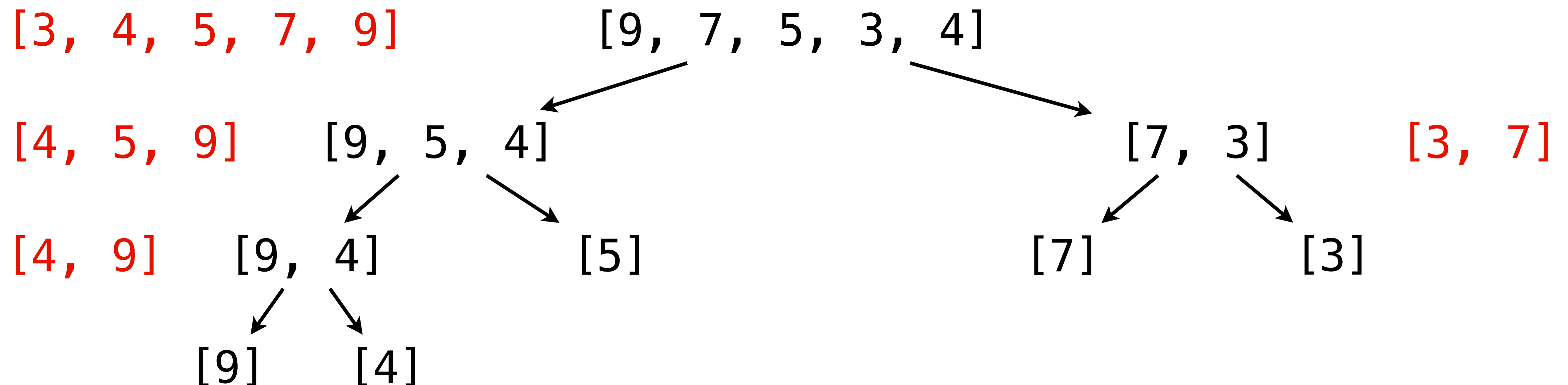

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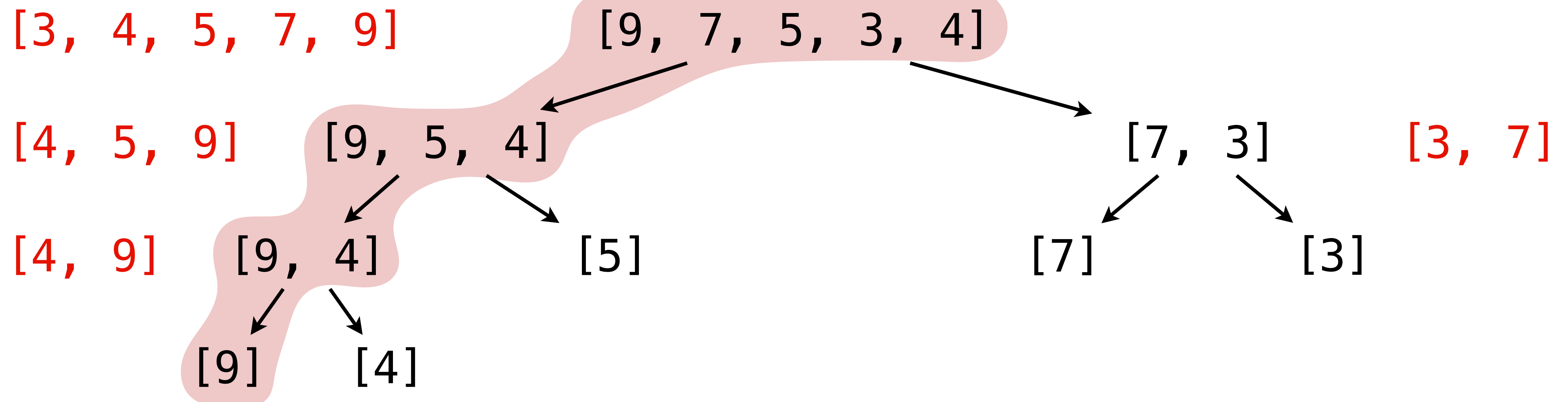

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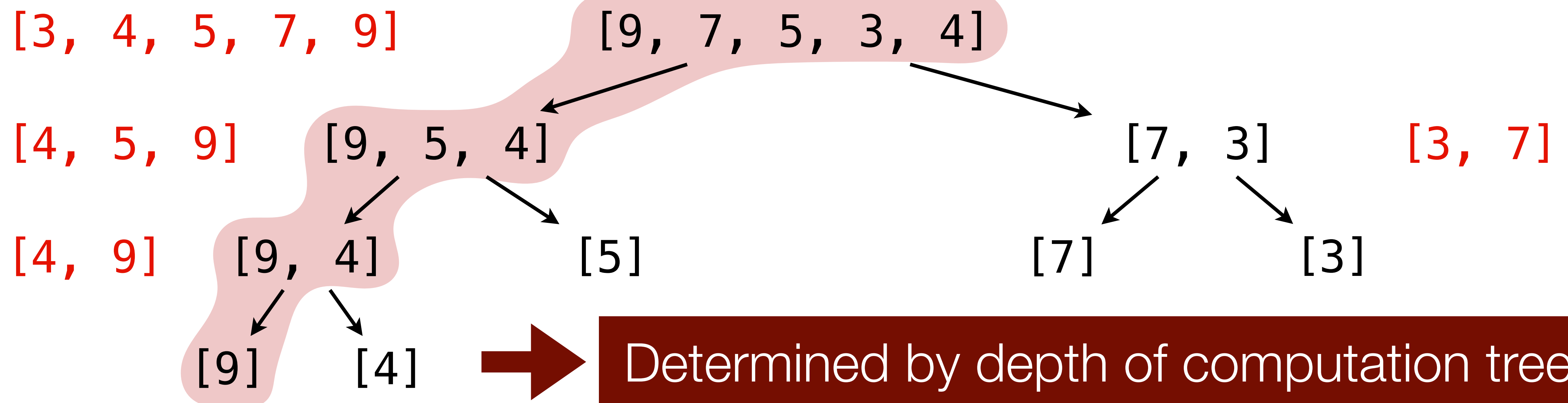

Span for mergesort for lists

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Recall work: $W_{msort}(n)$ with n the list length.

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Equations:

$$W_{\texttt{msort}}(0) = c_0$$
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$$W_{\texttt{msort}}(n) = c_2 + W_{\texttt{split}}(n) + W_{\texttt{msort}}(n_a) + W_{\texttt{msort}}(n_b)$$
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→ Let's look at the tree method to find a closed form.

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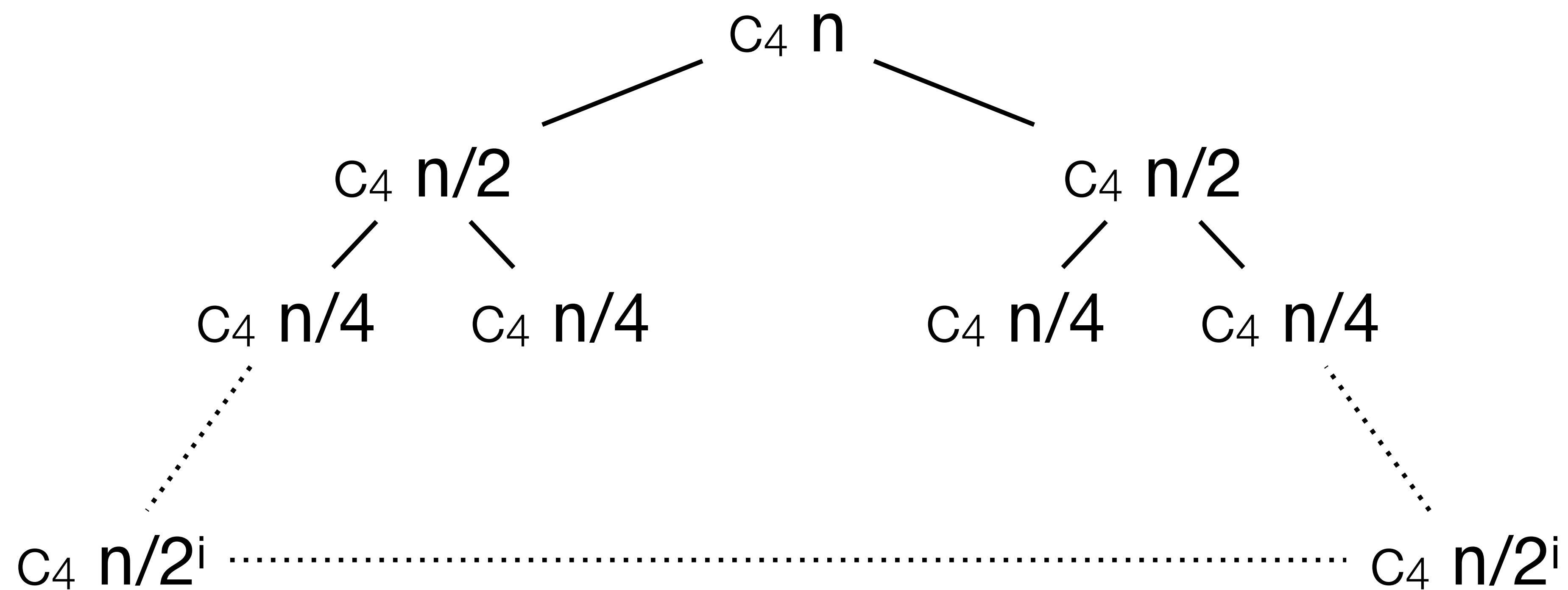

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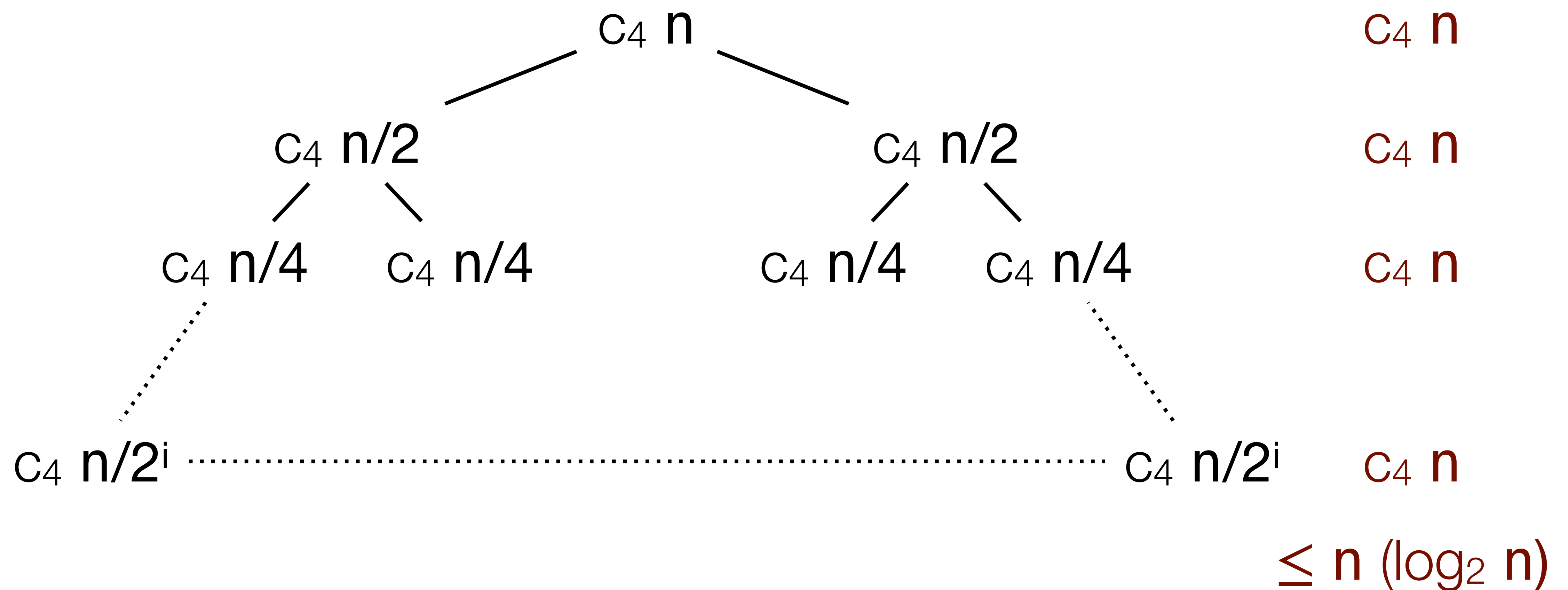

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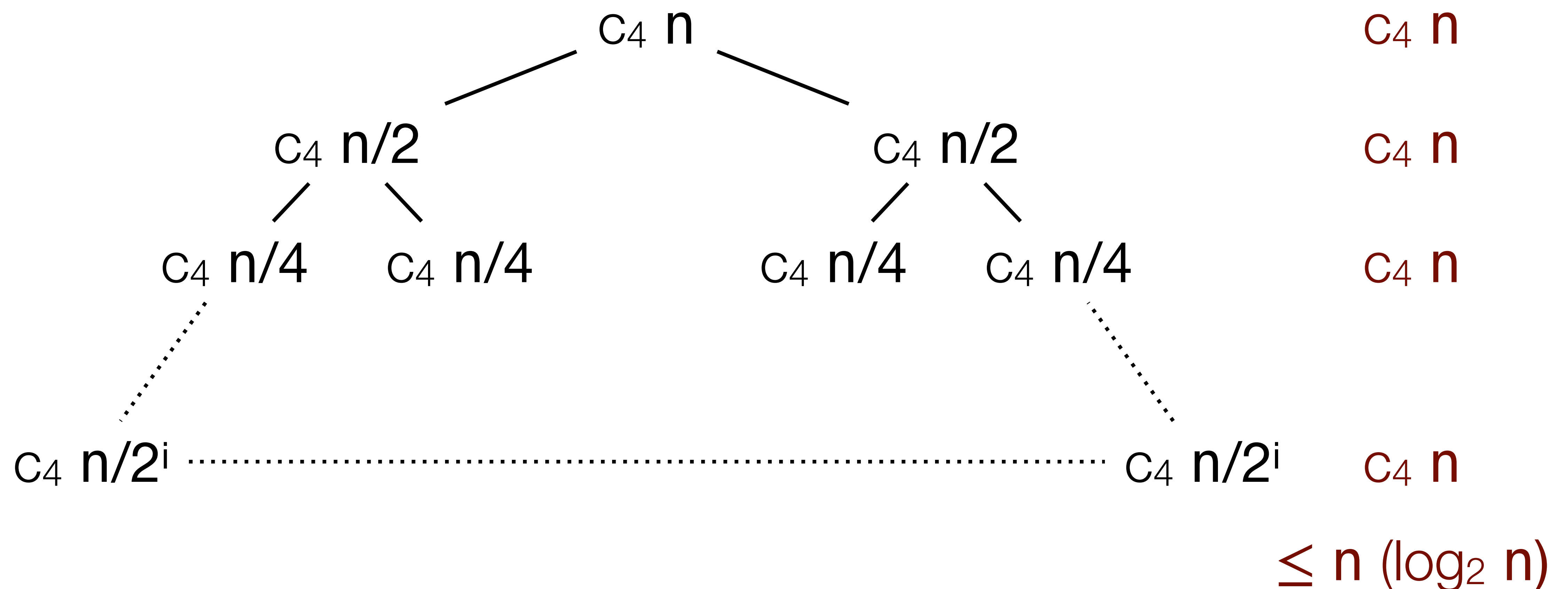

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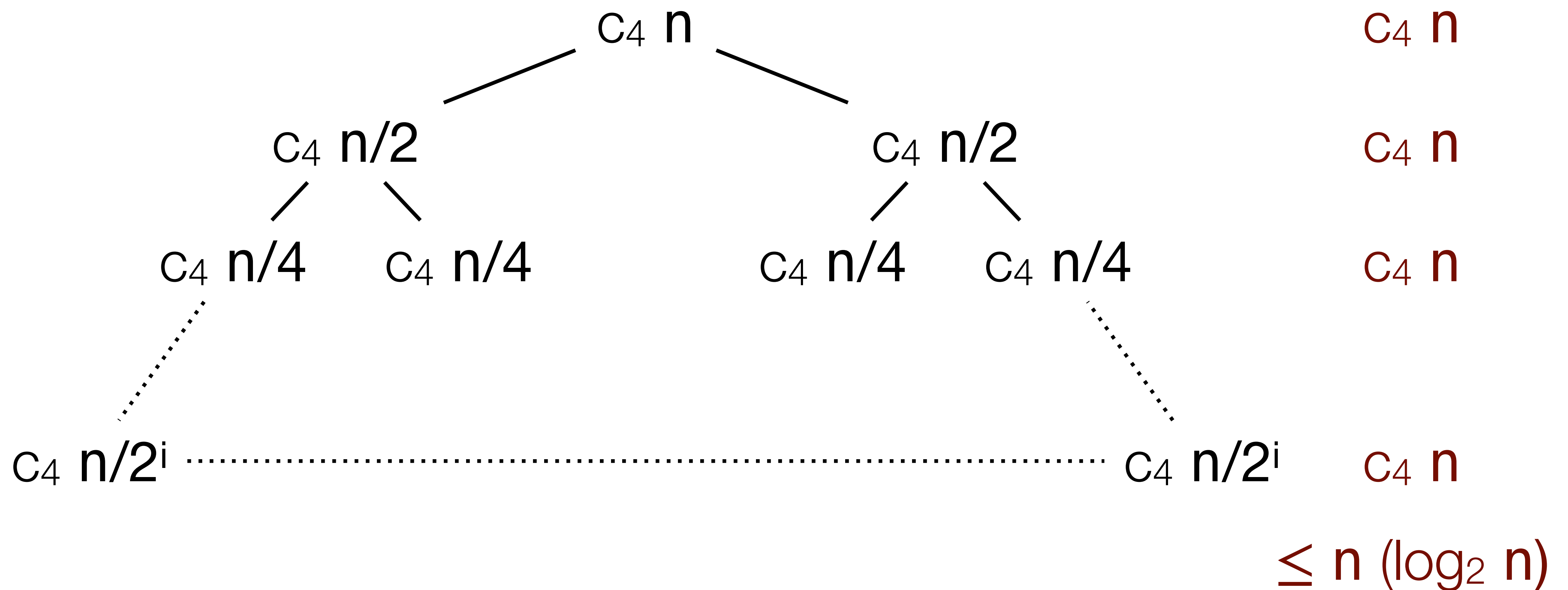

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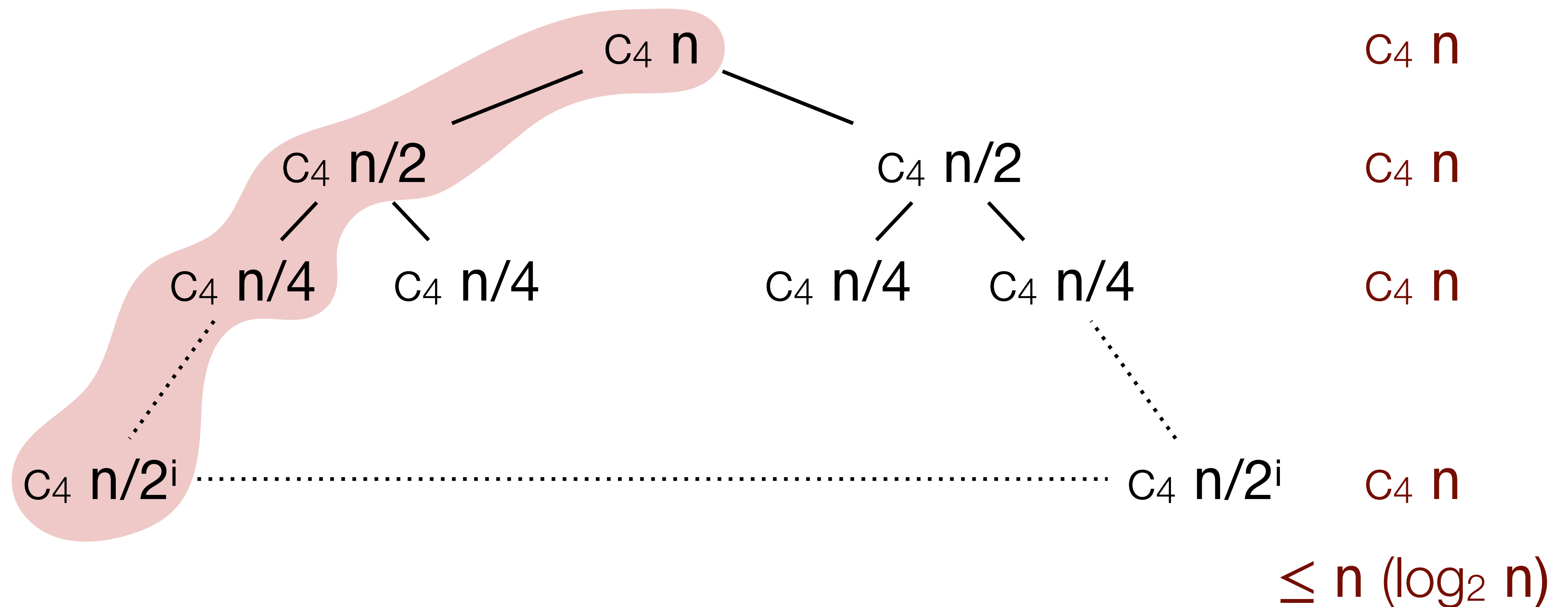

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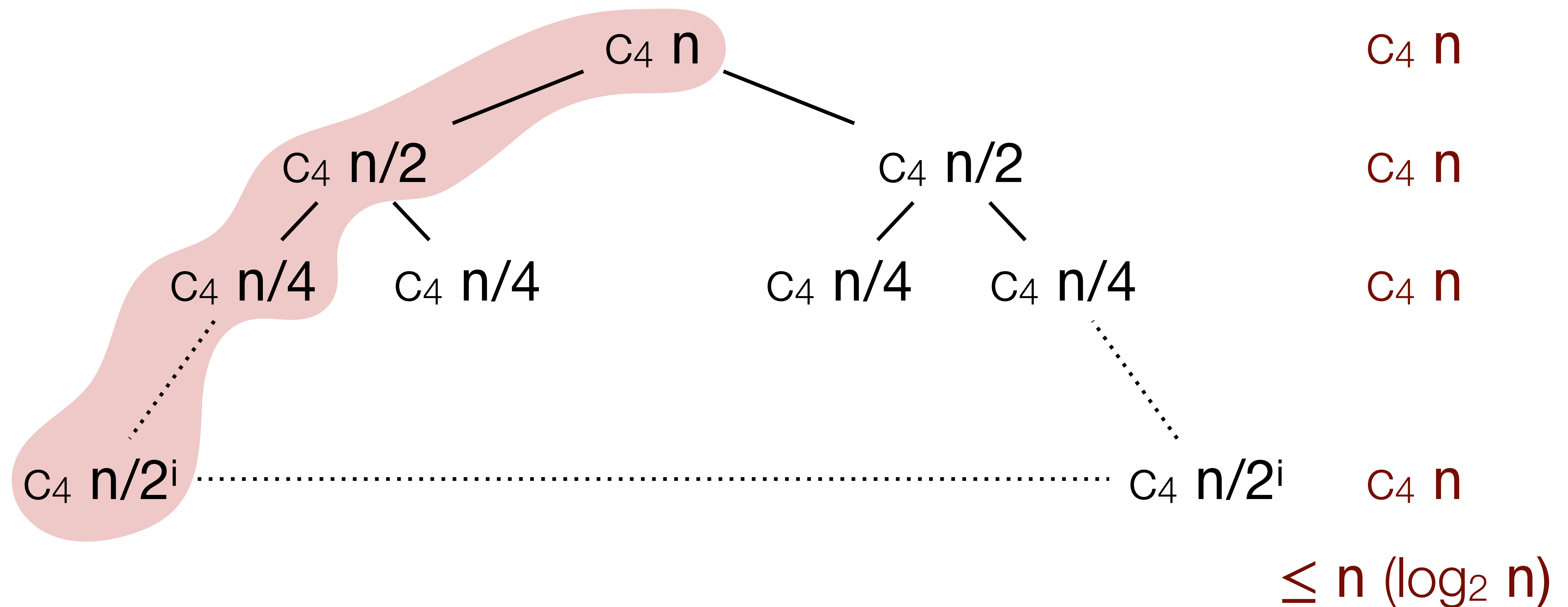

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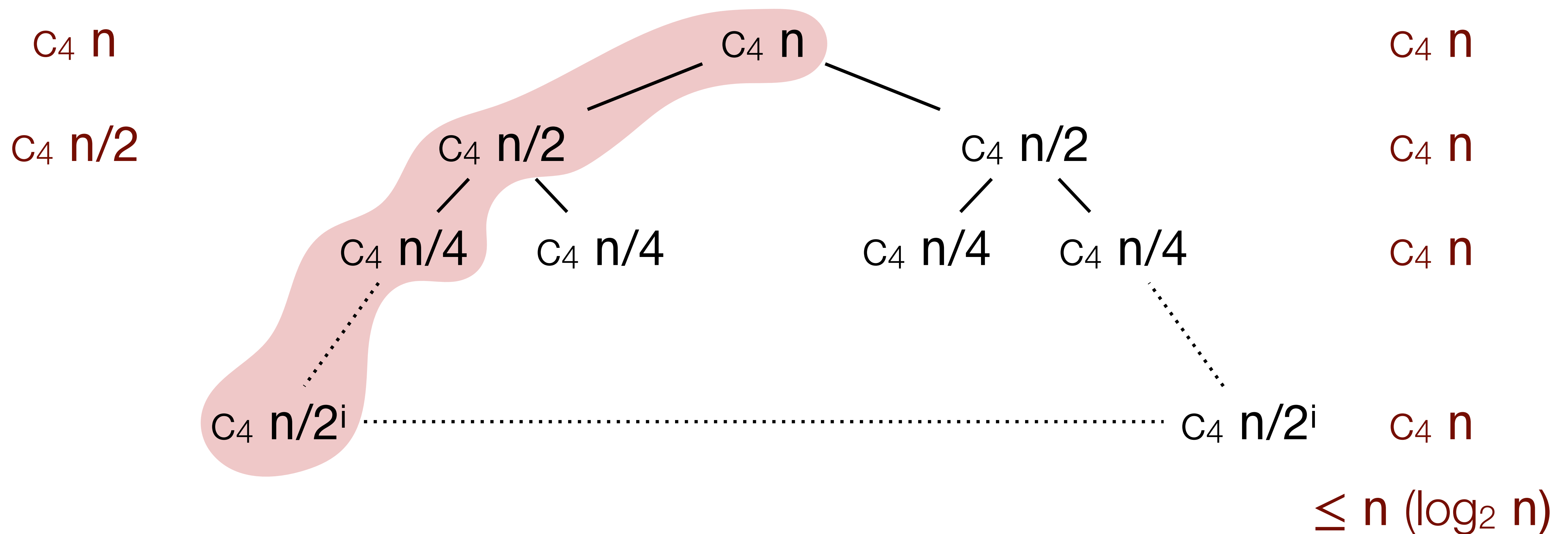

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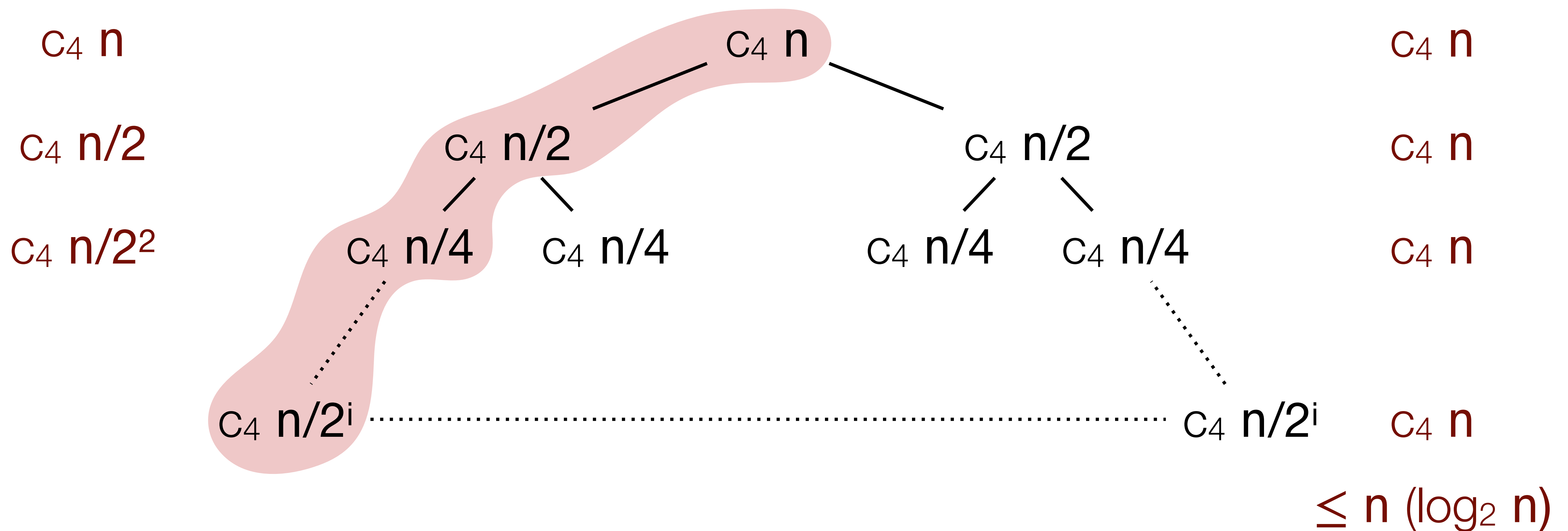

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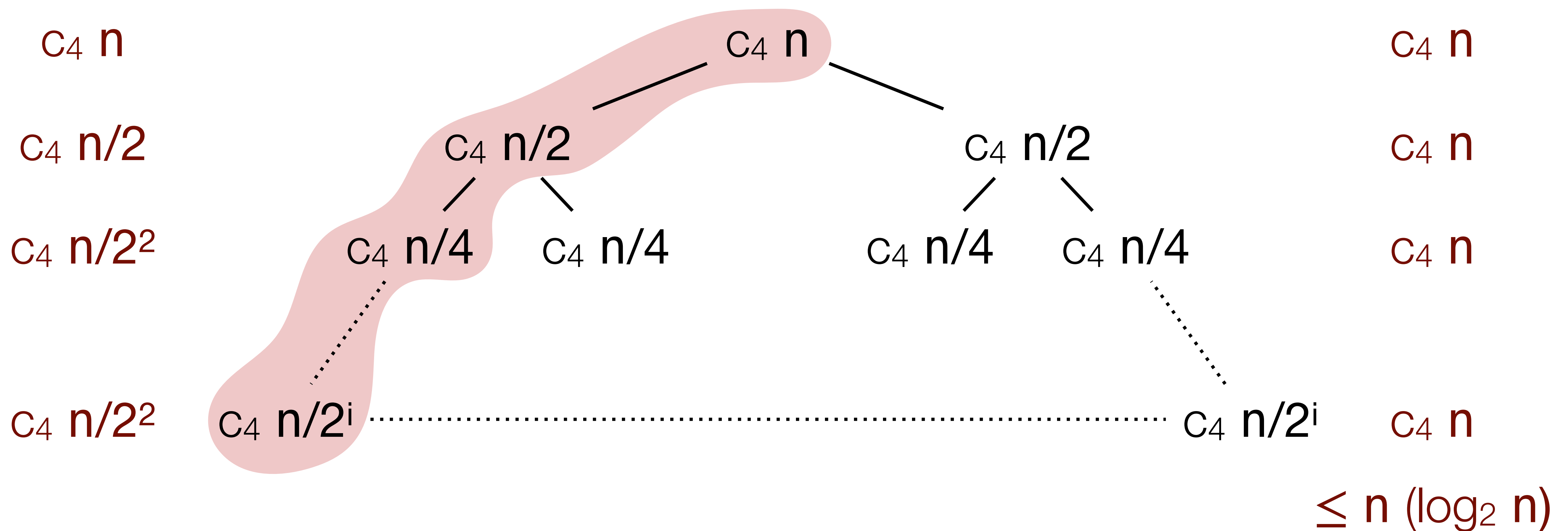

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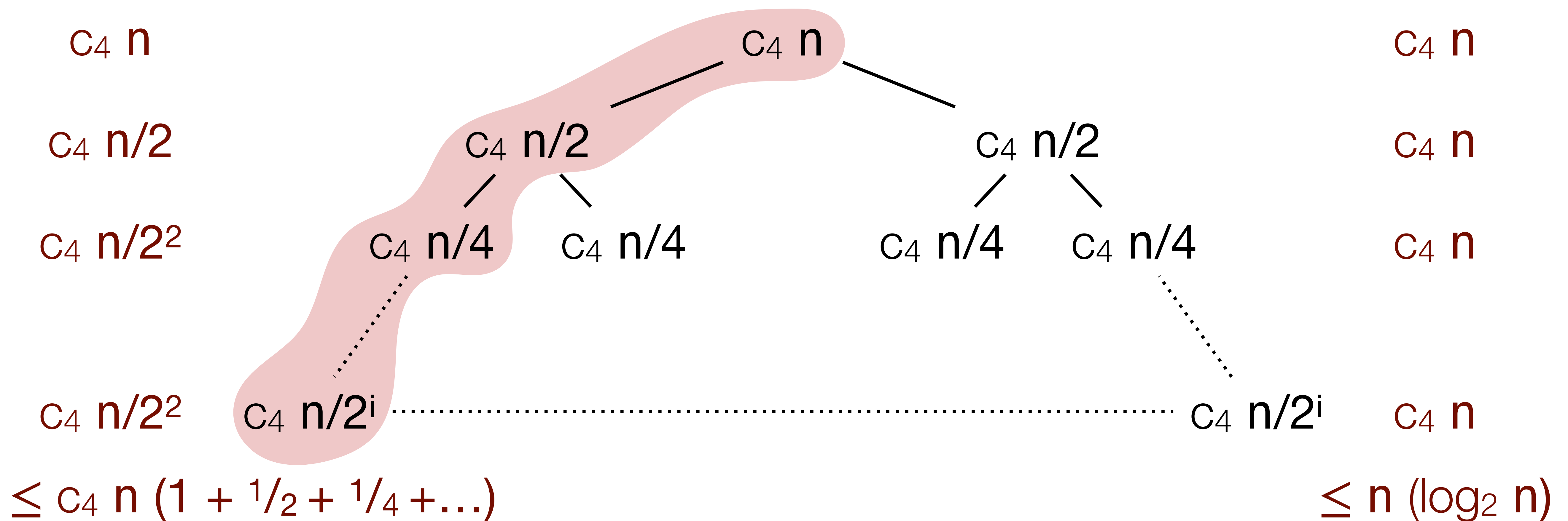

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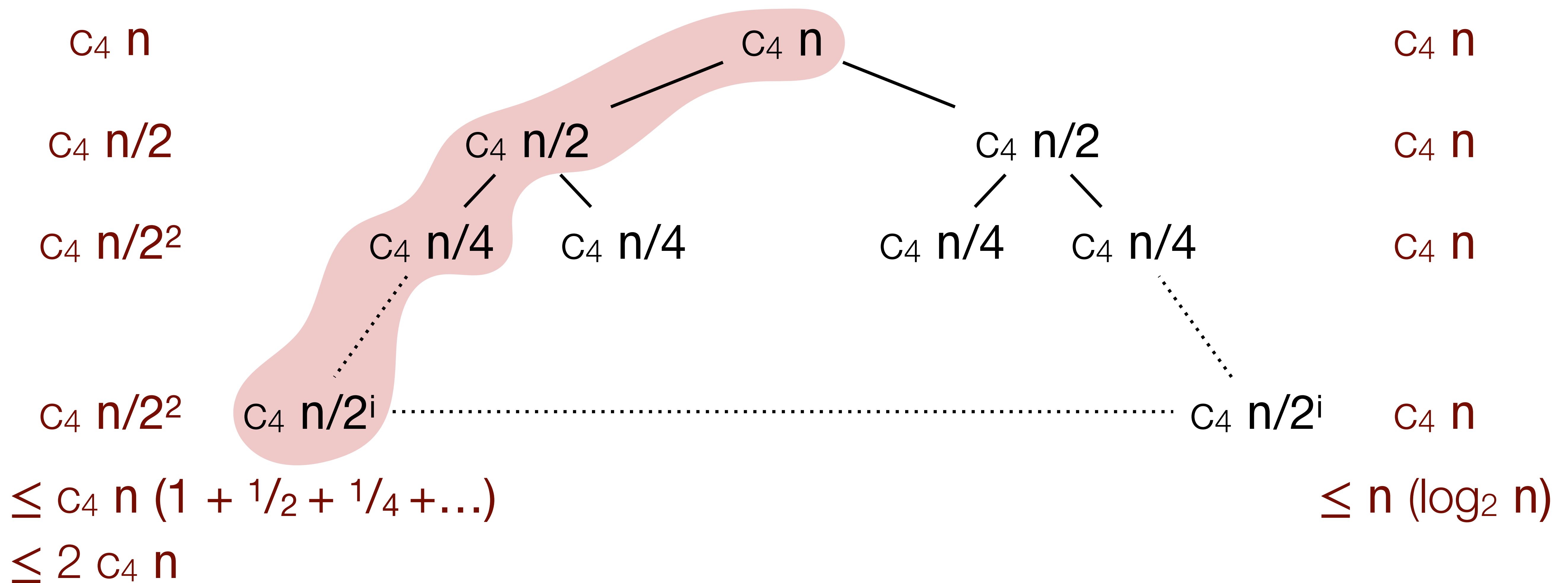

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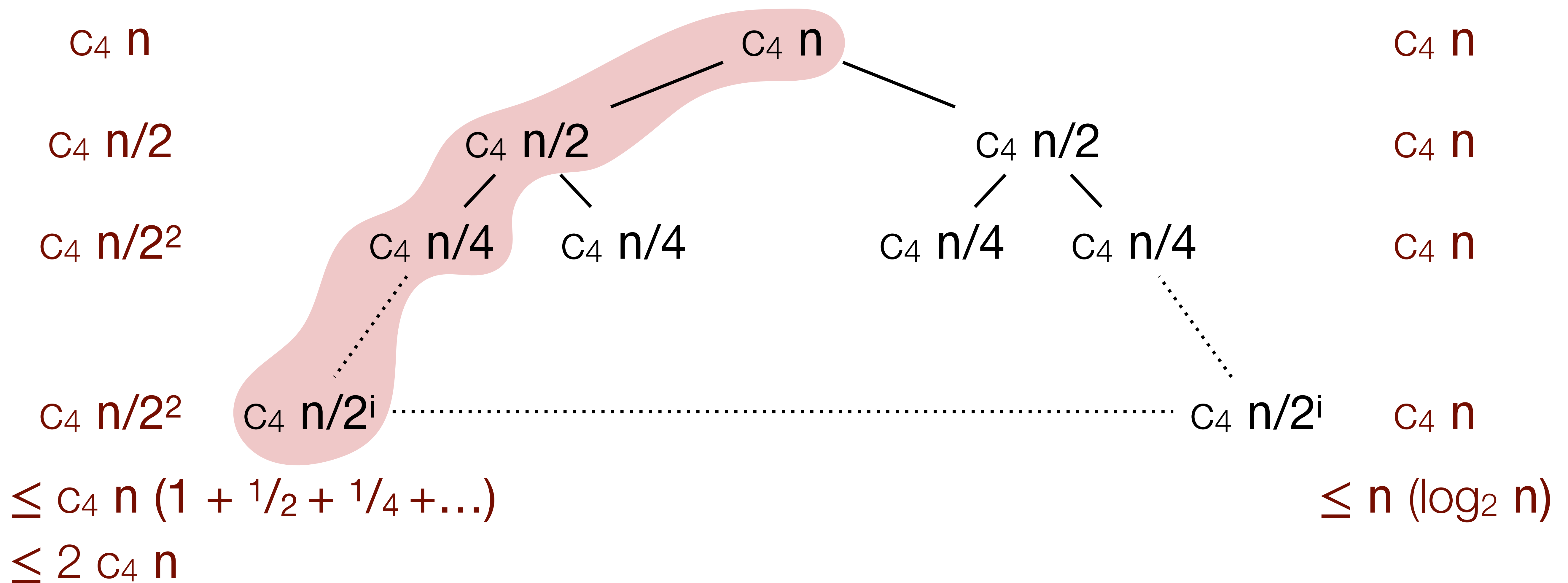

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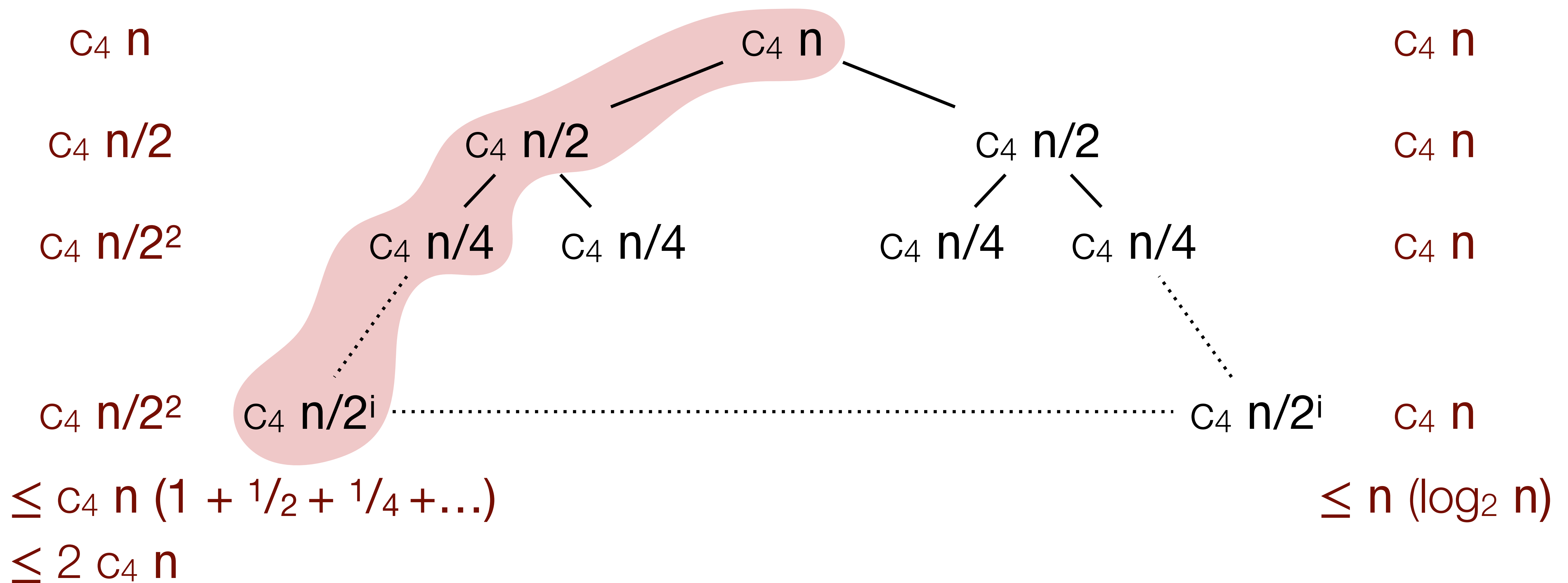

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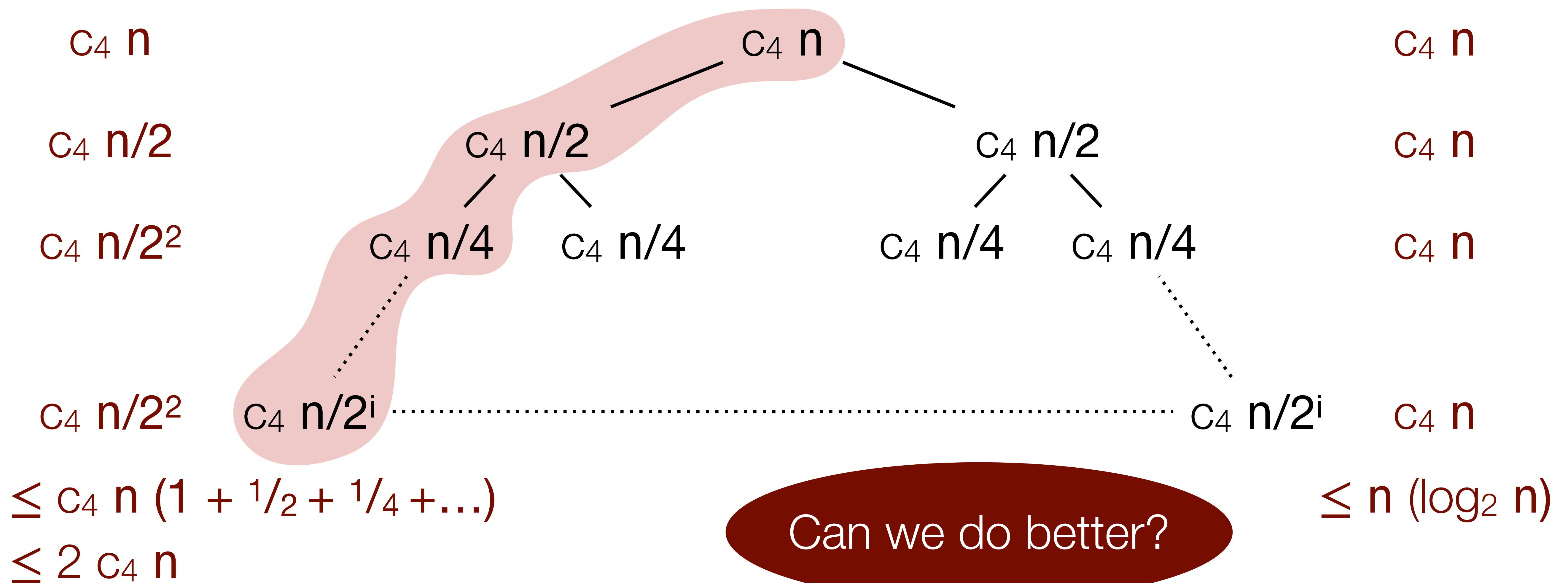

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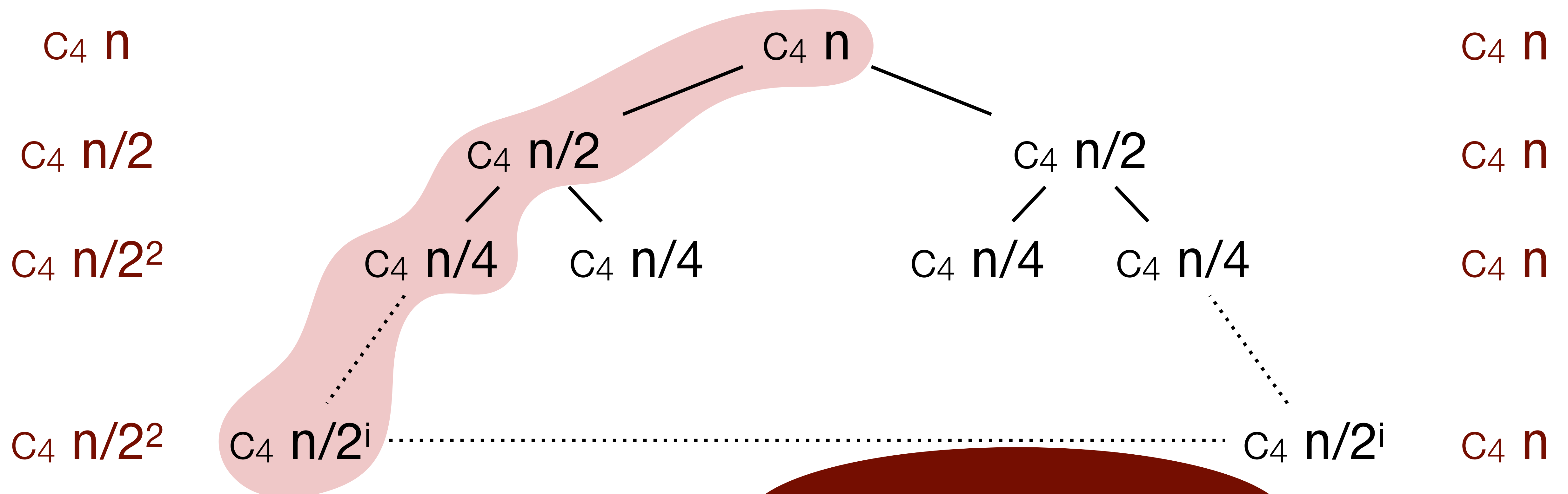

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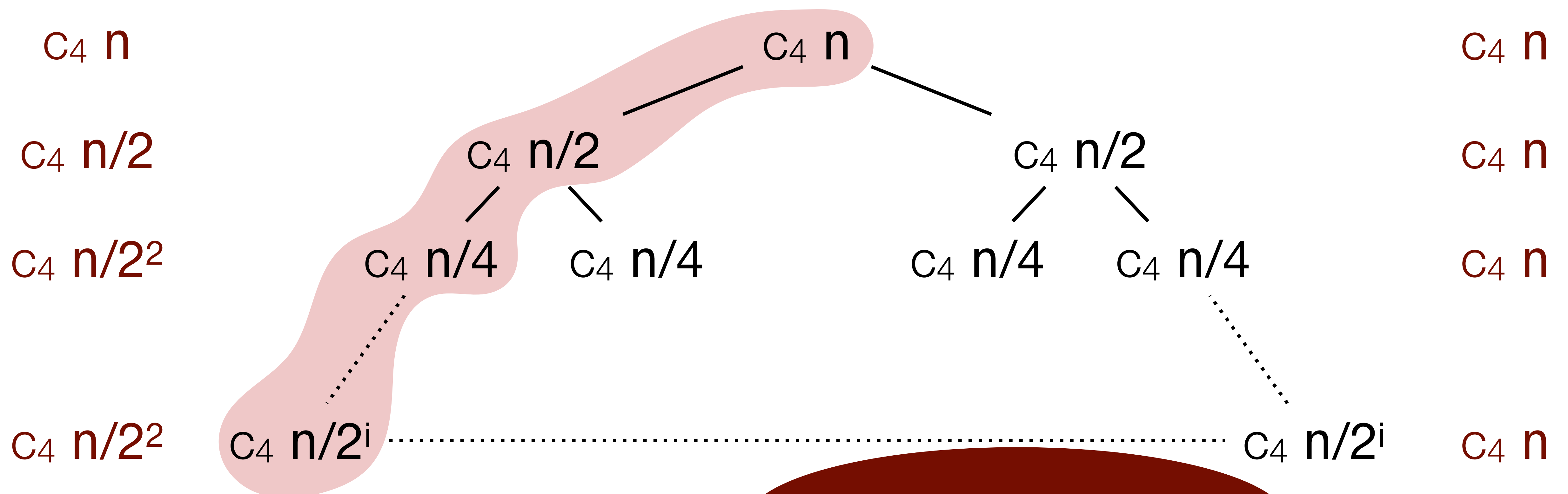

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What if we were given a tree, rather than a list?

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with:

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Int.compare : int * int -> order
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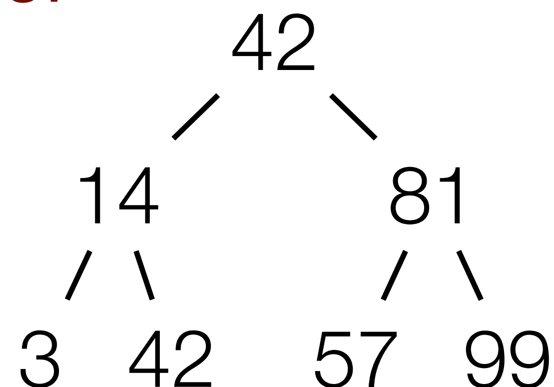

Mergesort for int trees

We define int trees to be sorted by:

- `Empty` is sorted;
- `Node(l,x,r)` is sorted iff
  - `l` is sorted and for every `y:int` in `l`, `Int.compare(y,x)` returns either `LESS` or `EQUAL`, and
  - `r` is sorted and for every `z:int` in `r`, `Int.compare(z,x)` returns either `GREATER` or `EQUAL`.

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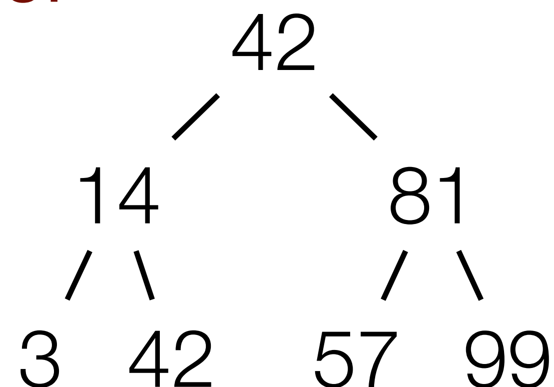

Eg, unsorted tree:

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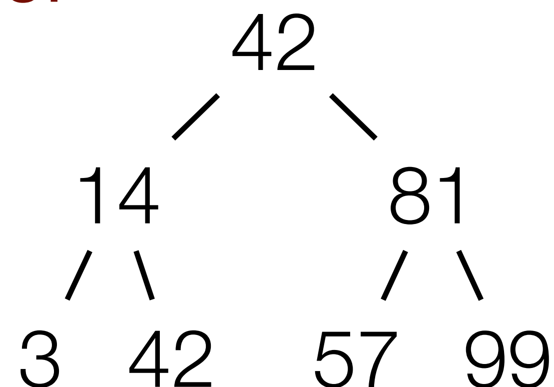
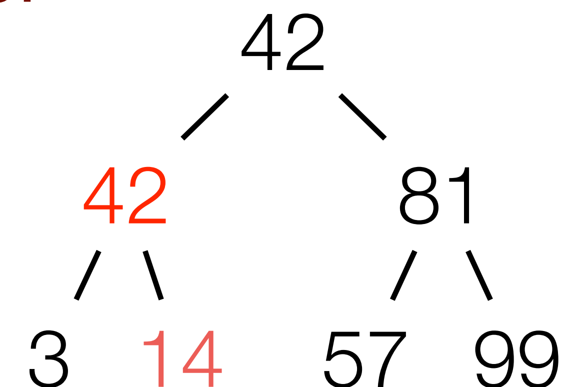


Eg, unsorted tree:



Mergesort for int trees

Mergesort for int trees

Divide and conquer for sorting int trees:

Mergesort for int trees

Divide and conquer for sorting int trees:

- Split the tree into sub-trees;

Mergesort for int trees

Divide and conquer for sorting int trees:

- Split the tree into sub-trees;
- Sort the sub-trees;

Mergesort for int trees

Divide and conquer for sorting int trees:

- Split the tree into sub-trees;
- Sort the sub-trees;
- Merge the results.

Let’s implement the mergesort function!

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```
(* Msort : tree -> tree
   REQUIRES: true
   ENSURES: Msort(t) evaluates to a sorted tree containing
            exactly the elements of t (incl duplicates).
*)
```

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fun Msort (Empty : tree) : tree =
```

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  | Msort (Node(left, x, right)) =
      Ins (x,                                  )
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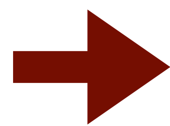

Note: no splitting to create the "computation tree" necessary anymore as with int list! We are already provided with a tree.

Let's implement the insert function!

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```
(* Ins : int * tree -> tree
   REQUIRES: t is a sorted tree.
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fun Ins (x: int, Empty: tree) : tree = Node(Empty, x, Empty)
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        | _         => Node(Ins(x, l), y, r))
```

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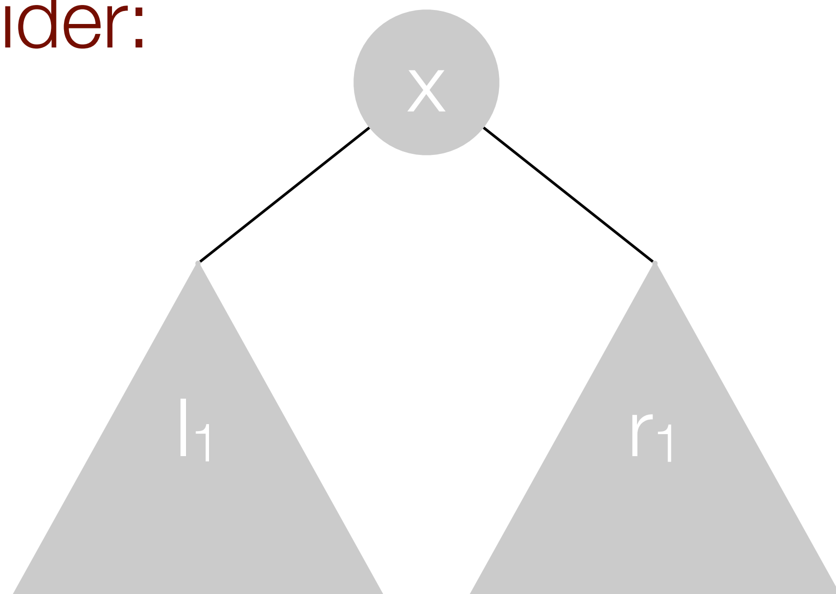

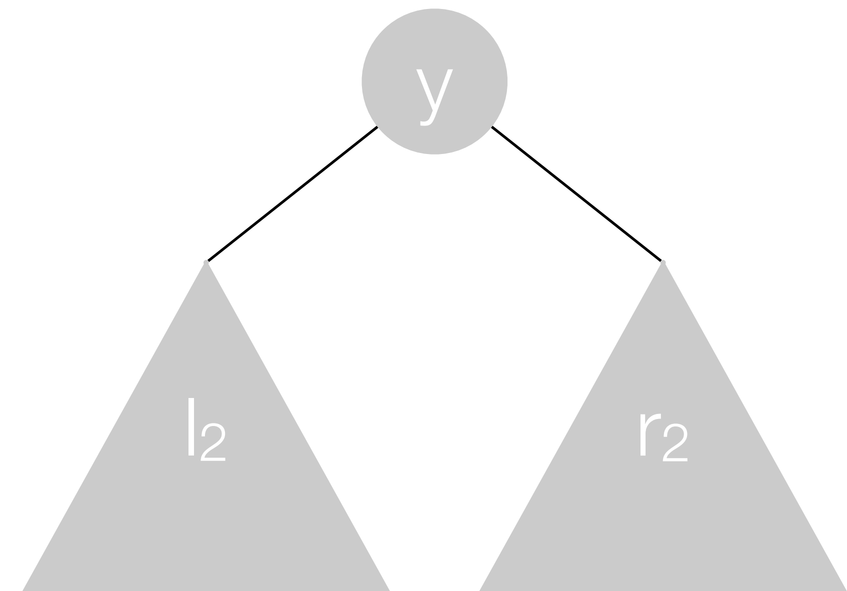

elements in $l_1 \leq x$ and $r_1 \geq x$,
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```
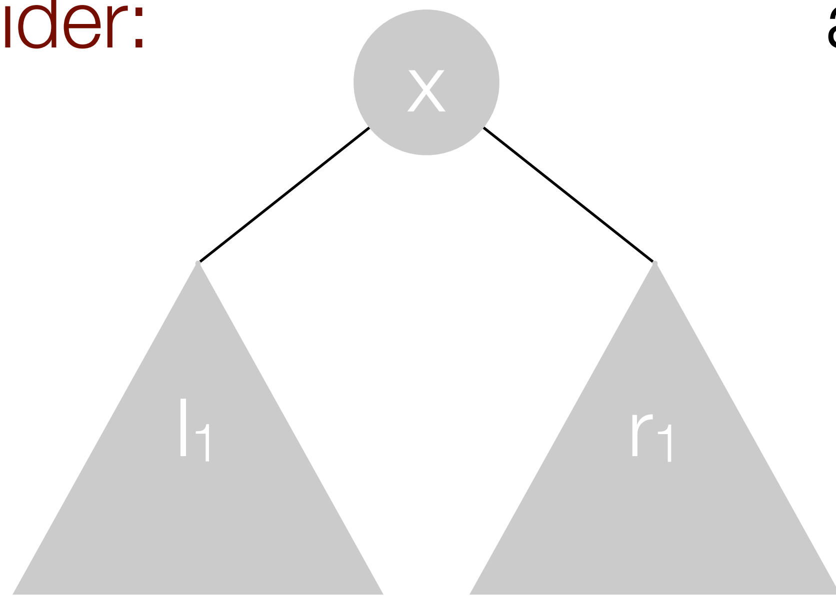
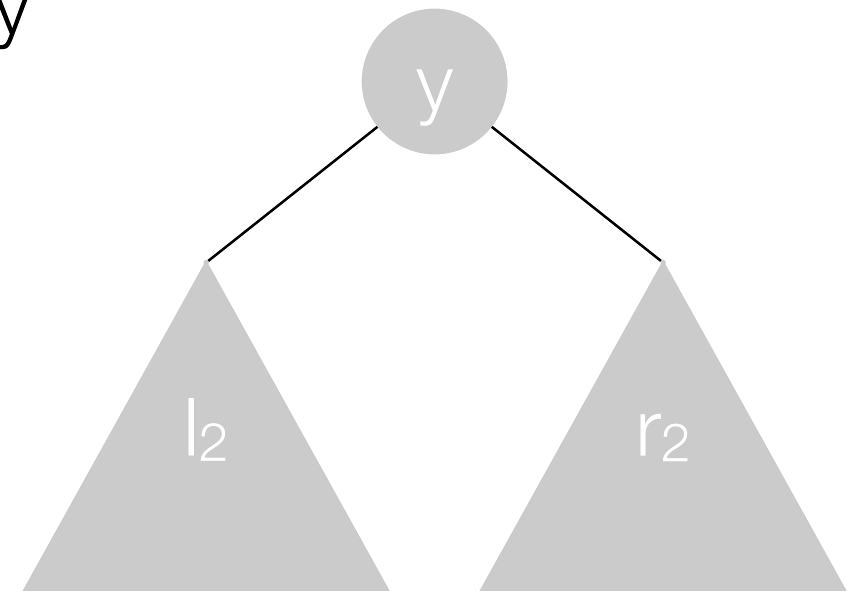

Consider:

assume $x \le y$





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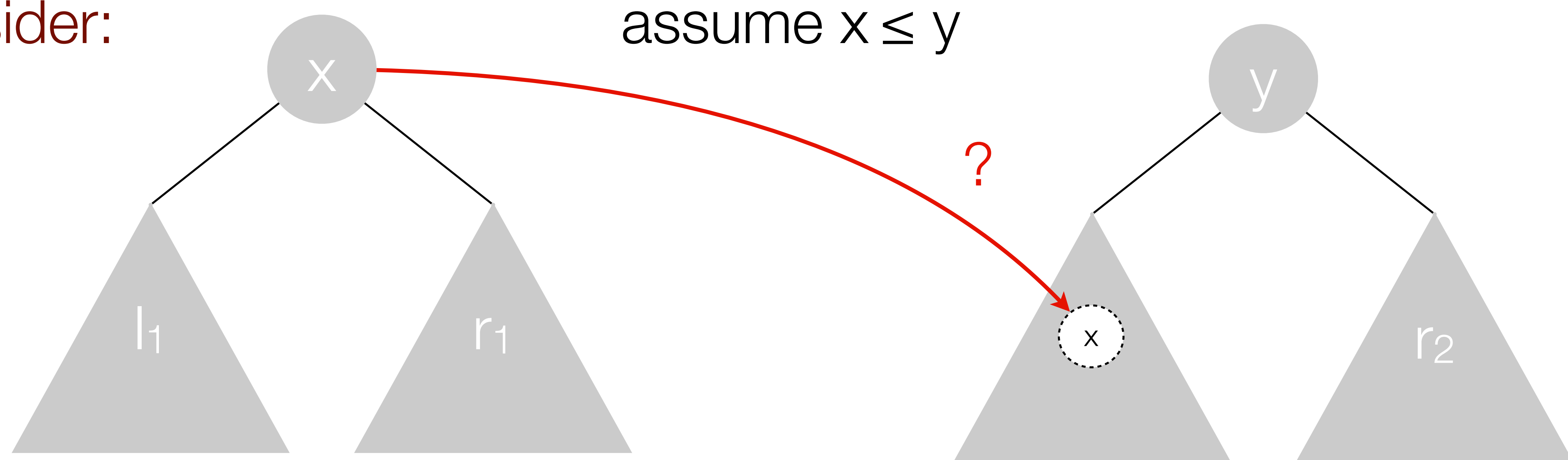

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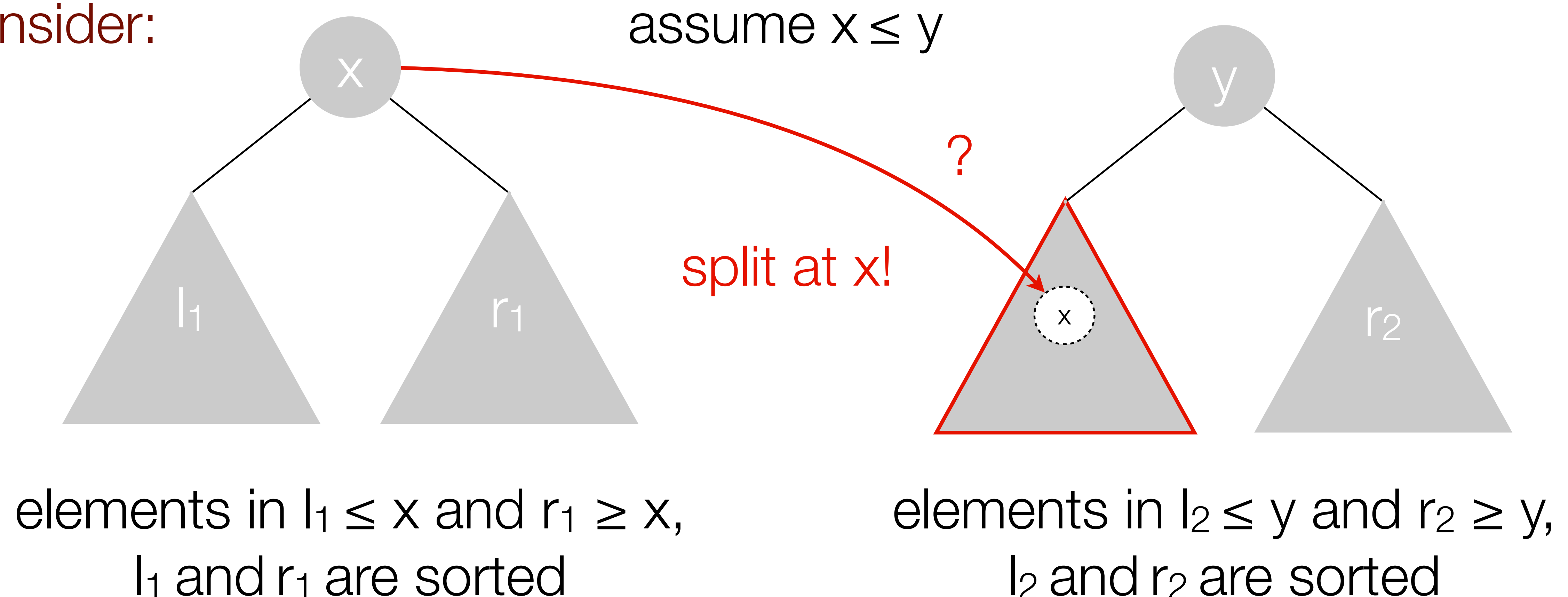

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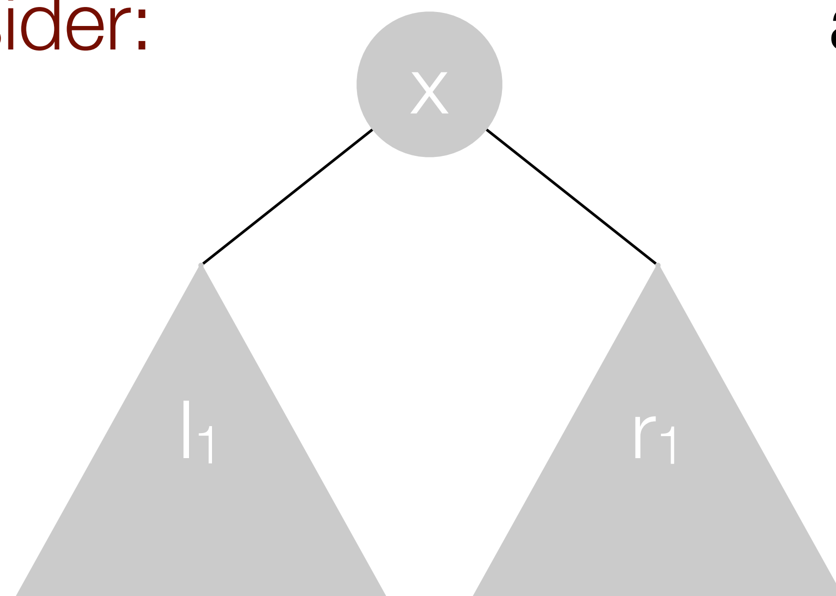
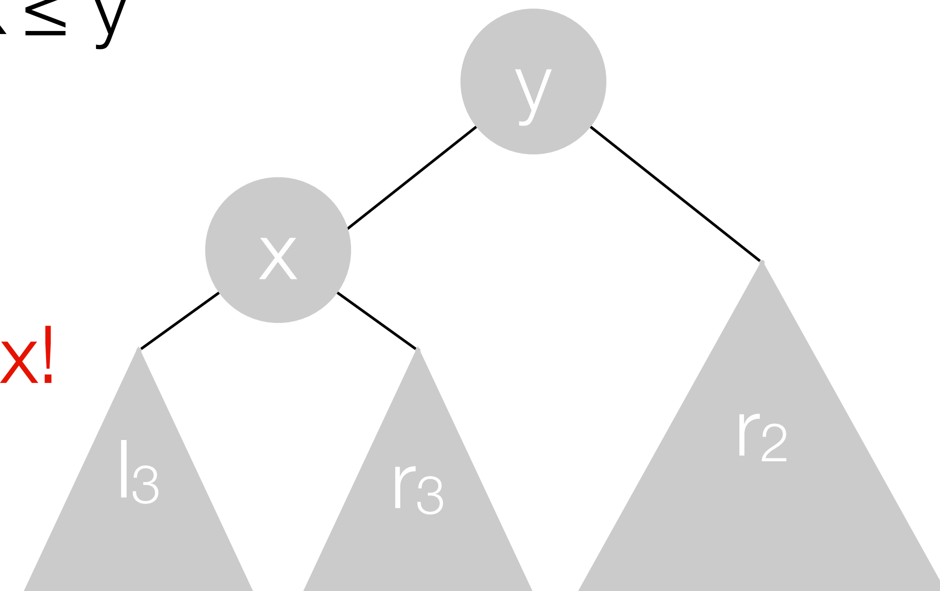


split at x!



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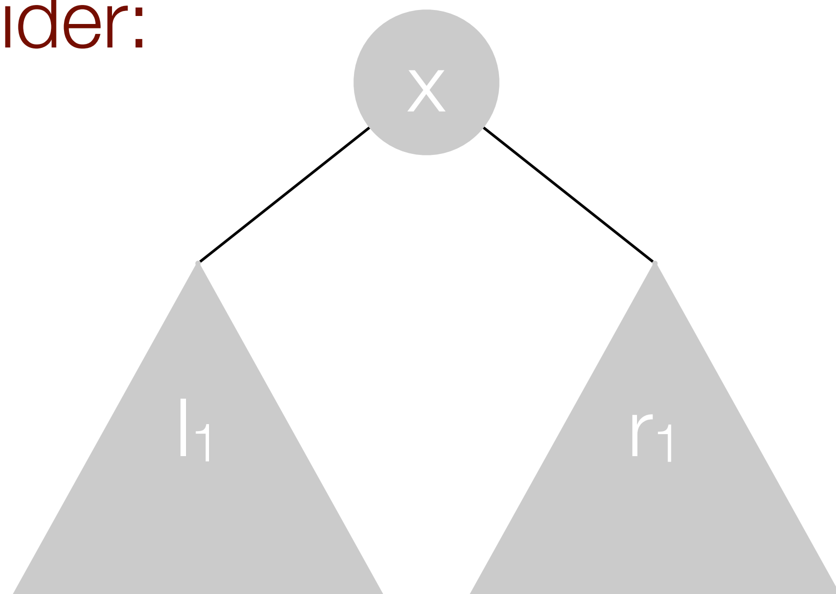

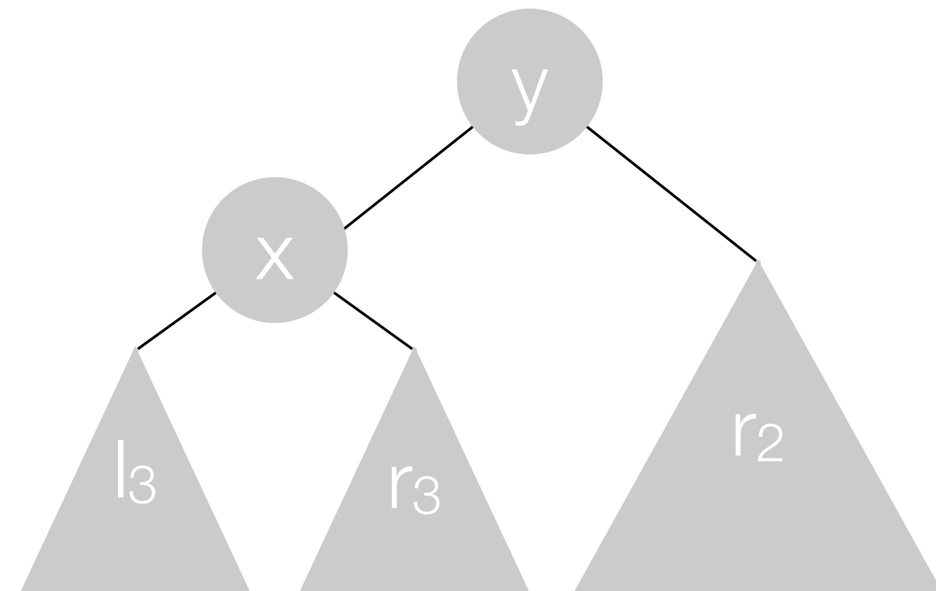

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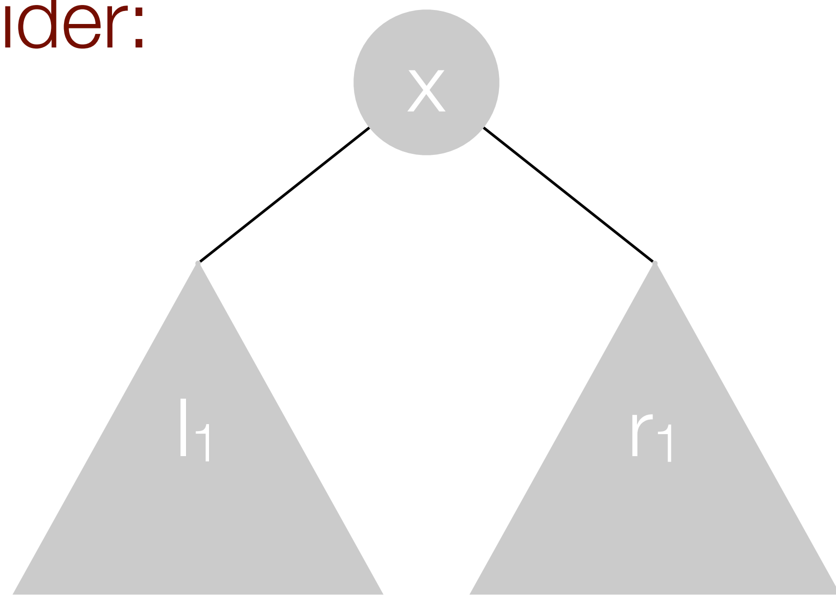

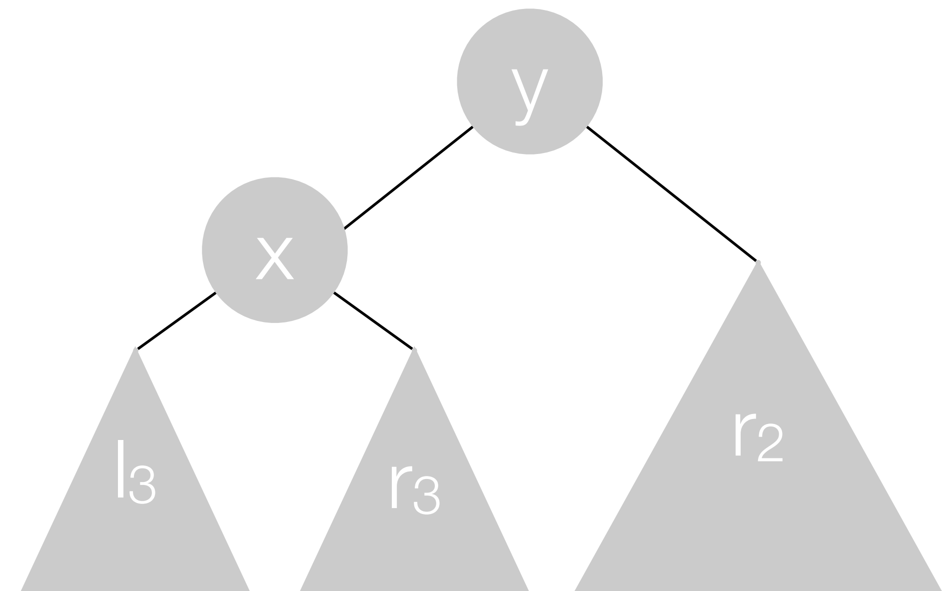

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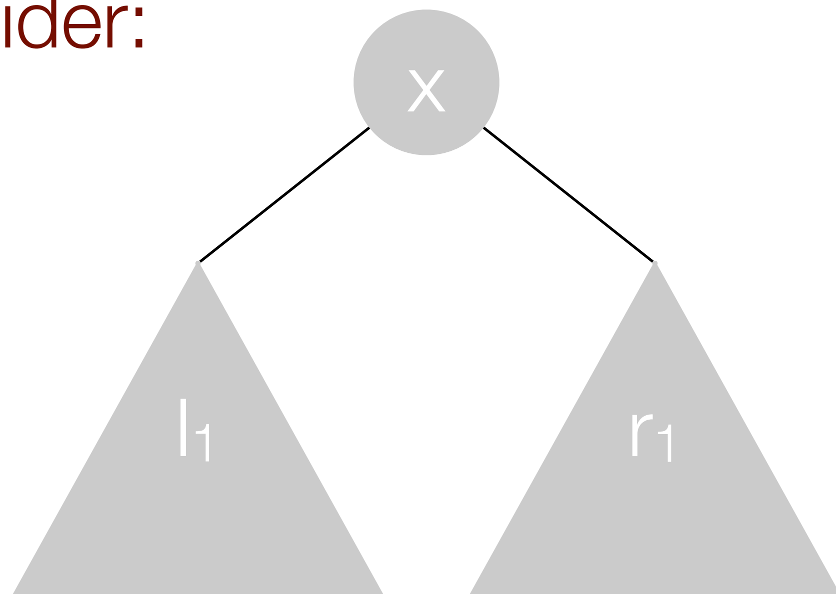

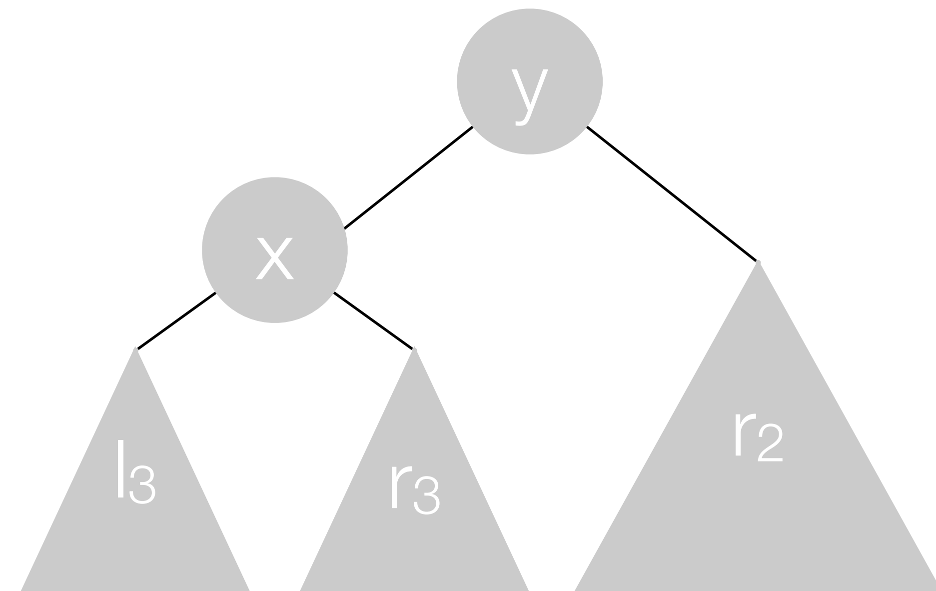

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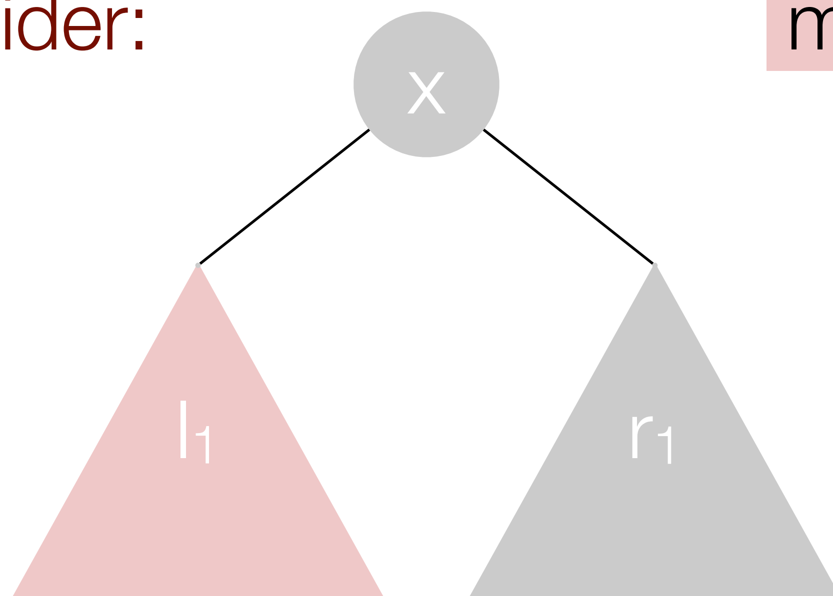
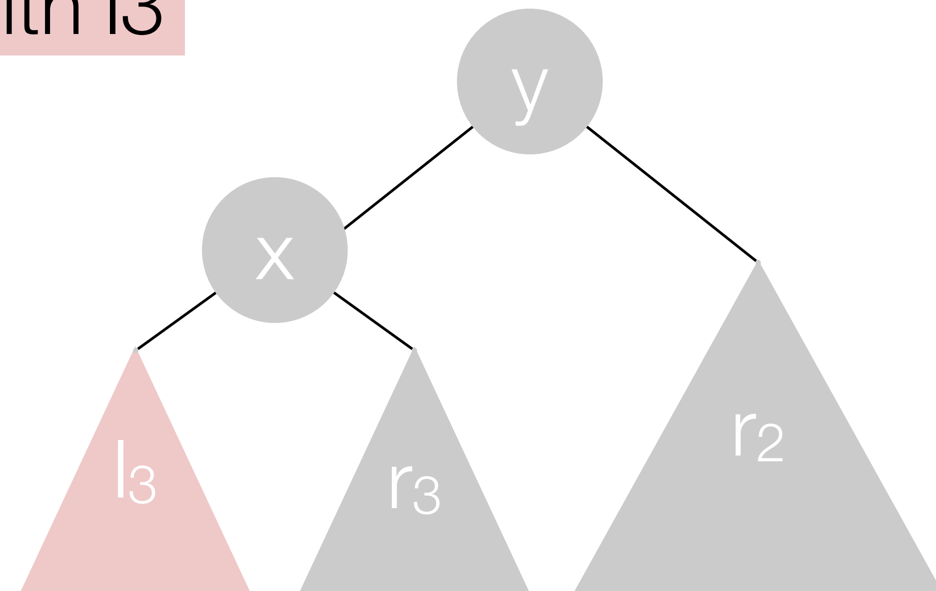
```

Consider:

merge l_1 with l3





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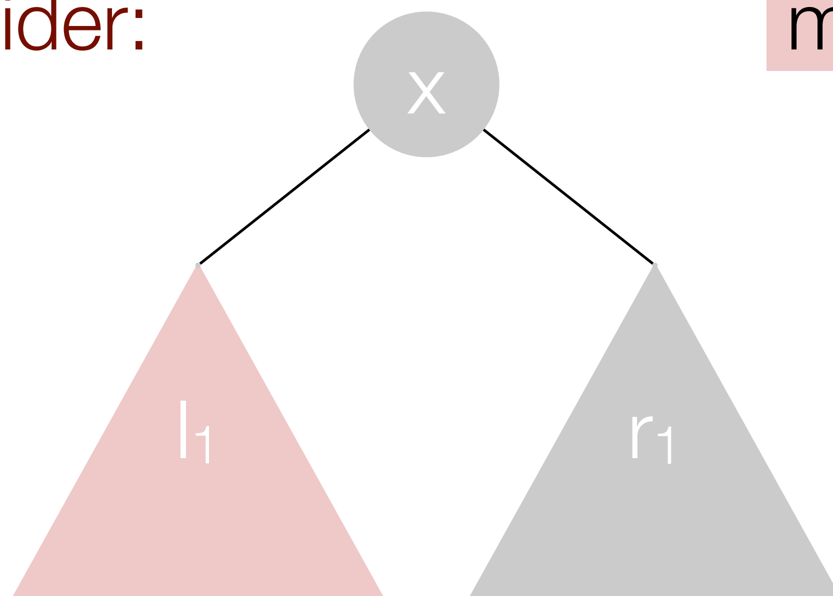

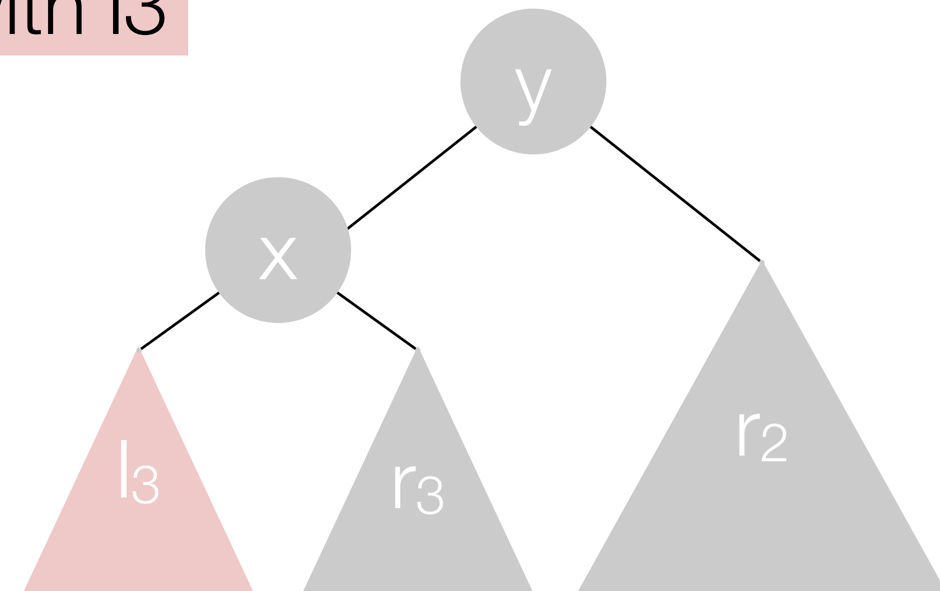

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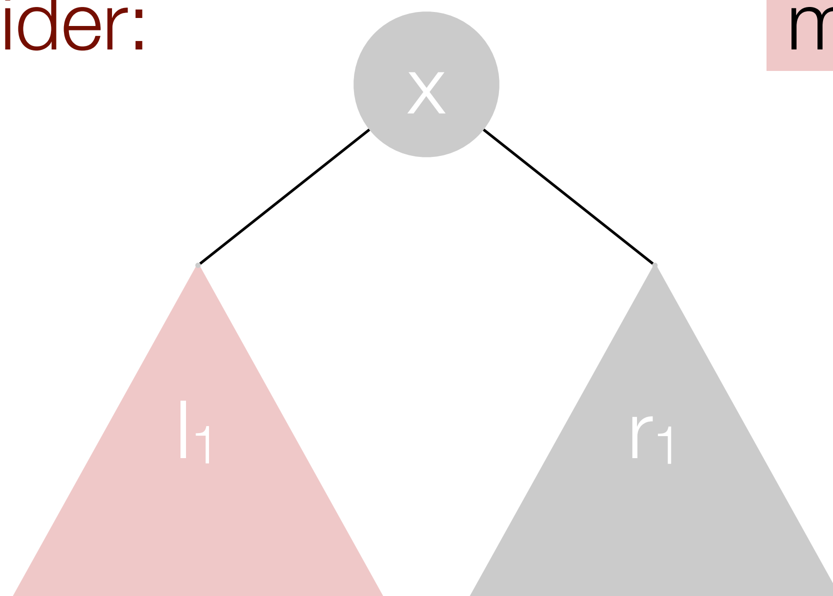

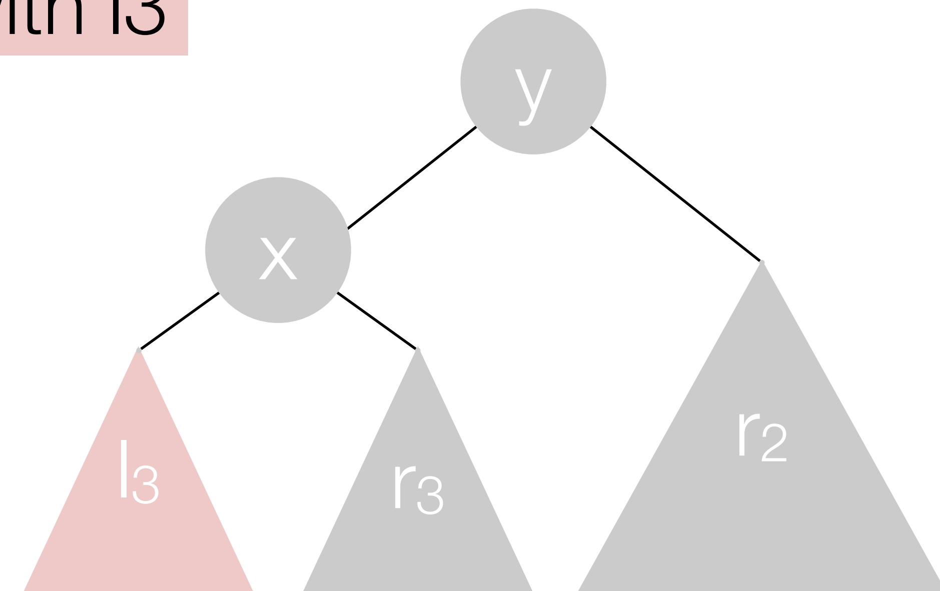

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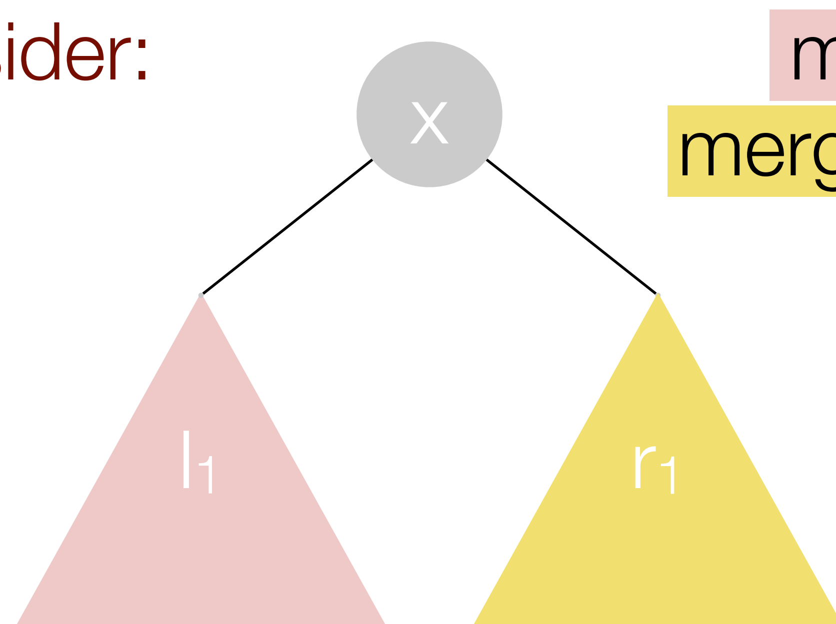
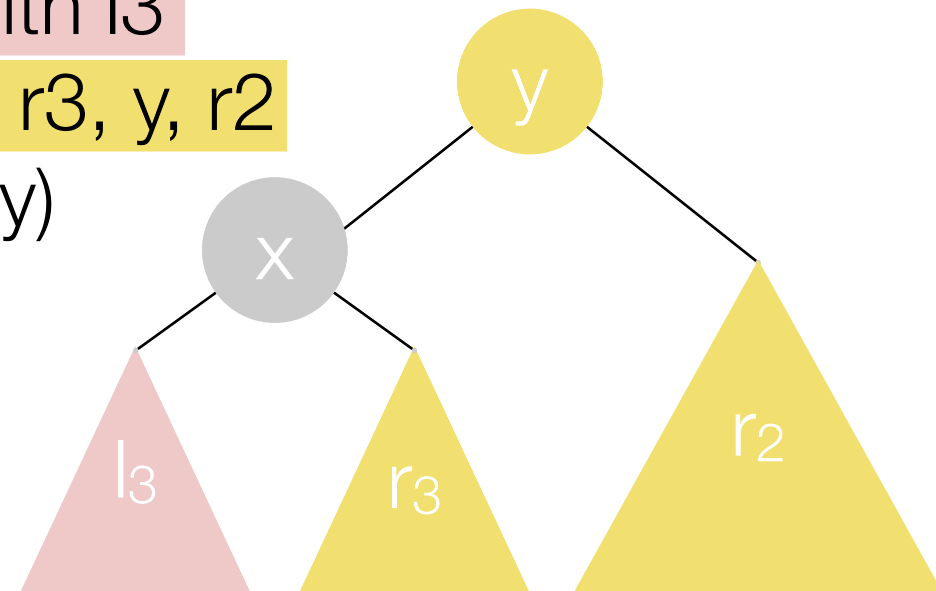


merge l_1 with l3

merge r1 with r3, y, r2

(for $x \leq y$)



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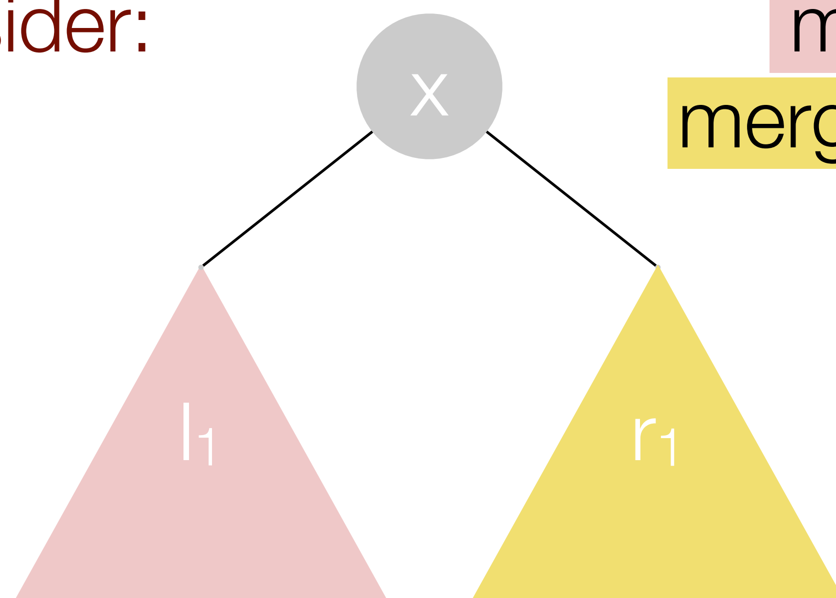

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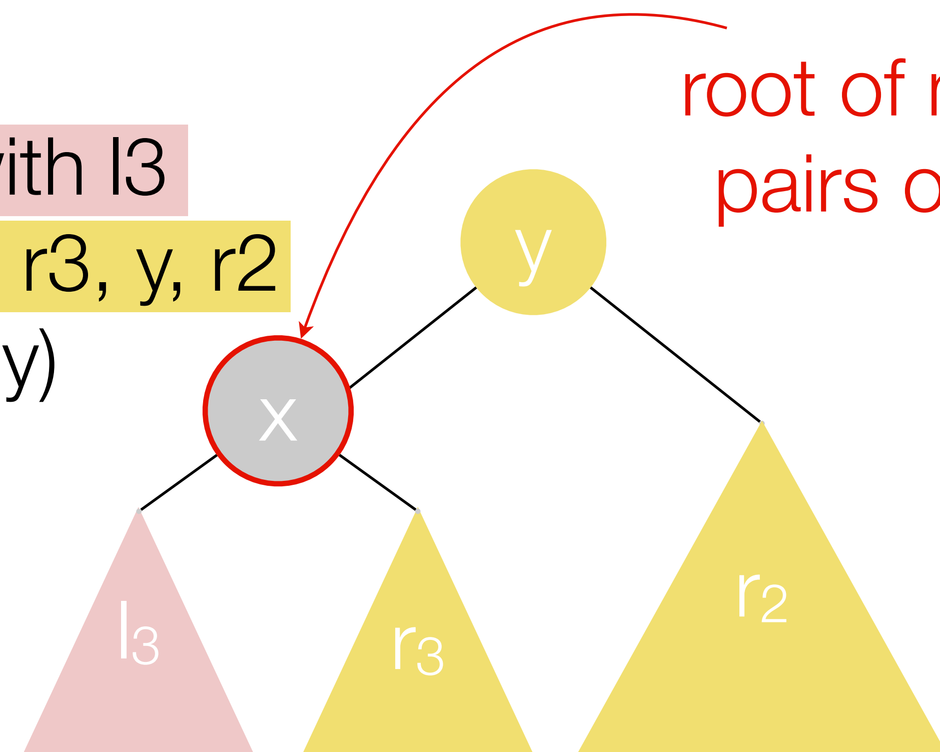
(for $x \leq y$)



root of merged pairs of trees

elements in $l_1 \leq x$ and $r_1 \geq x$,
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elements in l_3, $r_3 \leq y$ and $r_2 \geq y$,
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fun Merge (Empty : tree, t2 : tree) : tree =
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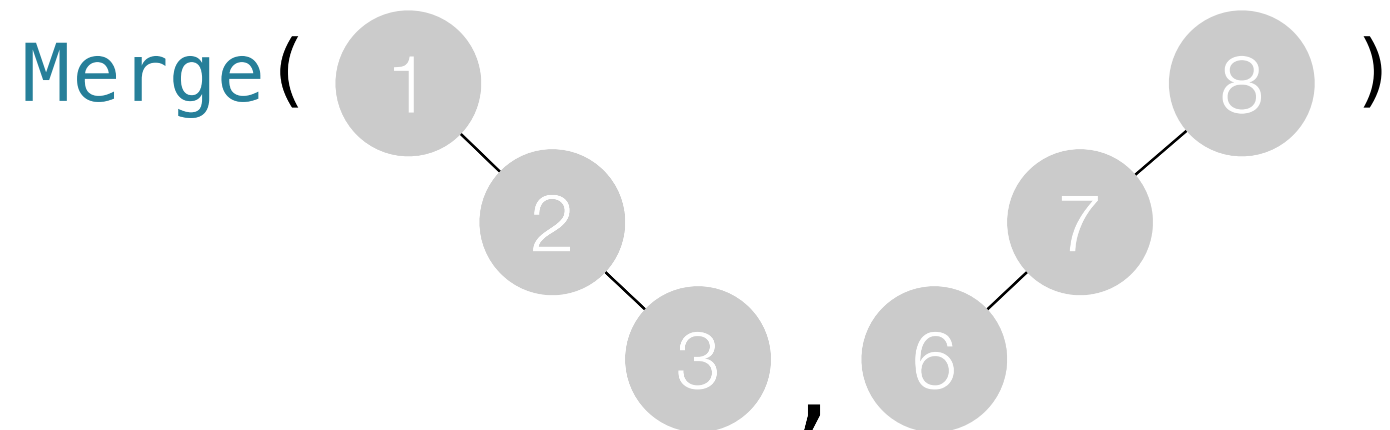

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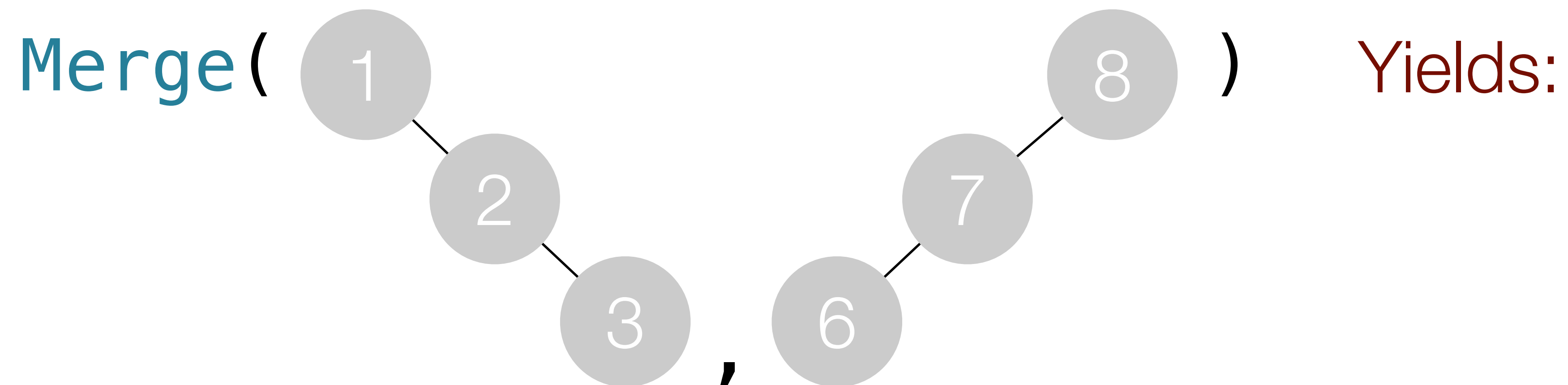

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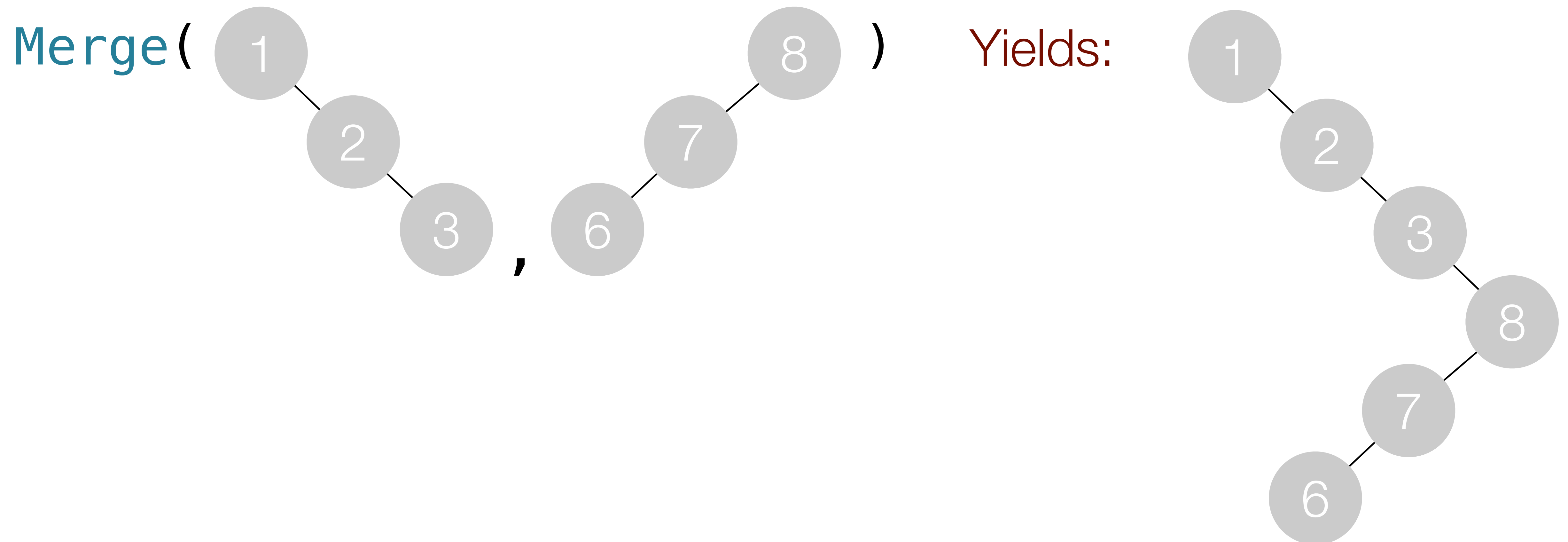

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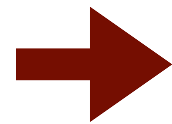

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      rebalance (Ins (x, Merge(Msort left, Msort right)))
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Let’s write the function split!

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```
(* SplitAt : int * tree -> tree * tree
   REQUIRES: t is sorted.
   ENSURES: SplitAt(x,t) returns a pair (t1,t2) of sorted
            trees such that:
            (a) t1 and t2 contain exactly the elements of t;
            (b) the elements of t1 are LESS or EQUAL to x;
            (c) the elements of t2 are GREATER or EQUAL to x.
*)
```

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```
fun SplitAt(x: int, Empty: tree): tree * tree =
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fun SplitAt(x: int, Empty: tree): tree * tree = (Empty, Empty)
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fun SplitAt(x: int, Empty: tree): tree * tree = (Empty, Empty)
  | SplitAt(x, Node(left, y, right)) =
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fun SplitAt(x: int, Empty: tree): tree * tree = (Empty, Empty)
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We have: $S_{\texttt{Ins}}(d)$ is $O(d)$.
$S_{\texttt{SplitAt}}(d)$ is $O(d)$.
$S_{\texttt{Merge}}(d)$ is $O(d_1 d_2)$.

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If we rebalance as a final step in Msort, then:

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Consequently: $S_{Msort}(d)$ is $O(d^3)$, ie $O((\log n)^3)$.

Sorting overview

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	list isort	list msort	tree msort
Work:	$O(n^2)$	$O(n \log n)$	$O(n \log n)$
Span:	$O(n^2)$	$O(n)$	$O((\log n)^3)$

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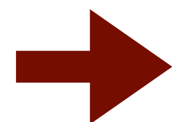

In 15-210, span of tree msort can be reduced to $O((\log n)^3)$!