Modules III

15-150

Lecture 19: November 14, 2024

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Carnegie Mellon University

Recap

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- Specification: **signature**.
- Implementation: **structure**.

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- Structures can **hide** auxiliary, implementation-specific components, not specified by signature.
- **Transparent ascription**: for undefined type specified in signature, **representation type** chosen by structure is **revealed**.
- **Opaque ascription**: for undefined type specified in signature, **representation type** chosen by structure is **hidden**.

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Representation invariants:

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Representation invariants:

- Hidden consistency condition enforced by structure.

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A closer look at representation invariants:

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We'll explain these ideas on an example, further illustrating:

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We'll explain these ideas on an example, further illustrating:

- A functional implementation of balanced trees.

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- Localize violation and characterize with weaker invariant.
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We'll explain these ideas on an example, further illustrating:

- A functional implementation of balanced trees.
- "Picture-guided programming" thanks to pattern matching.

Let's reconsider our dictionary

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```
signature DICT =
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  type key = string                     (* concrete  *)
  type 'a entry = key * 'a              (* concrete  *)
  type 'a dict                          (* abstract  *)
  val empty : 'a dict
  val lookup : 'a dict -> key -> 'a option
  val insert : 'a dict * 'a entry -> 'a dict
end
```

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→ Implement dictionary as a red black tree!

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Implement dictionary as a red black tree!

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Red Black Trees

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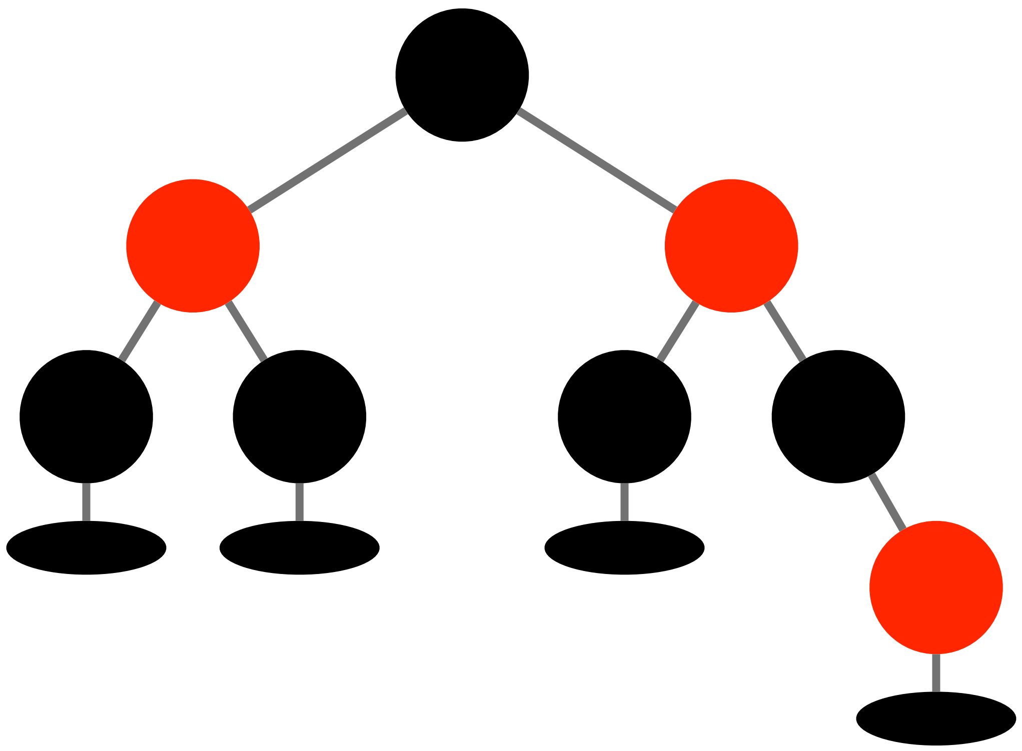

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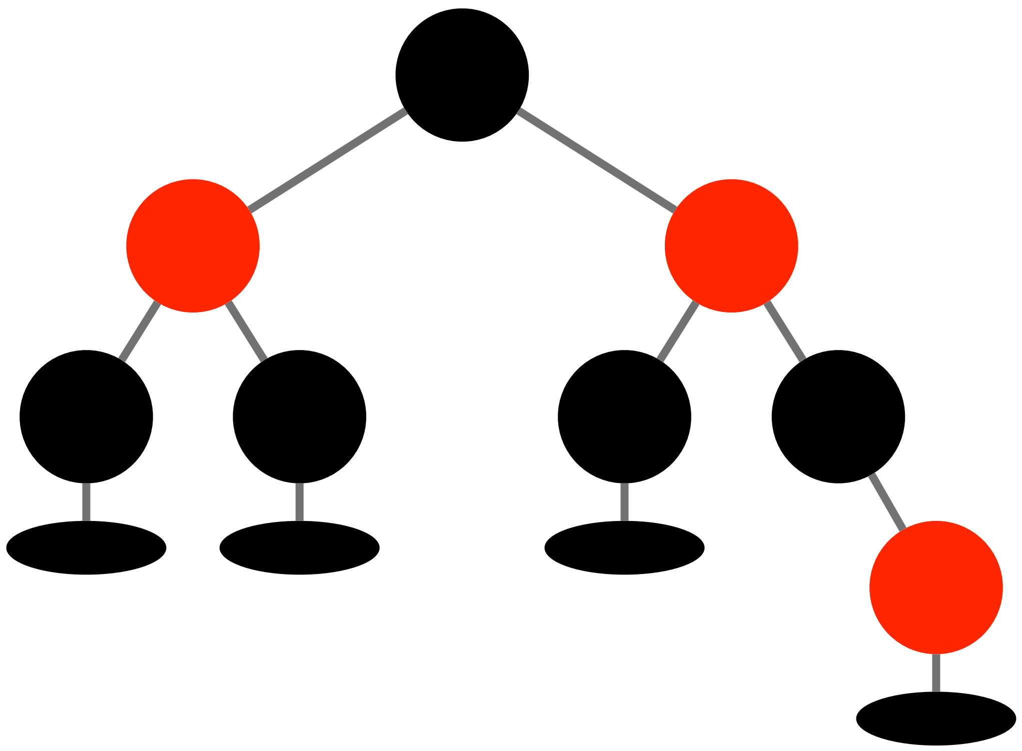

Red Black Trees

code shown
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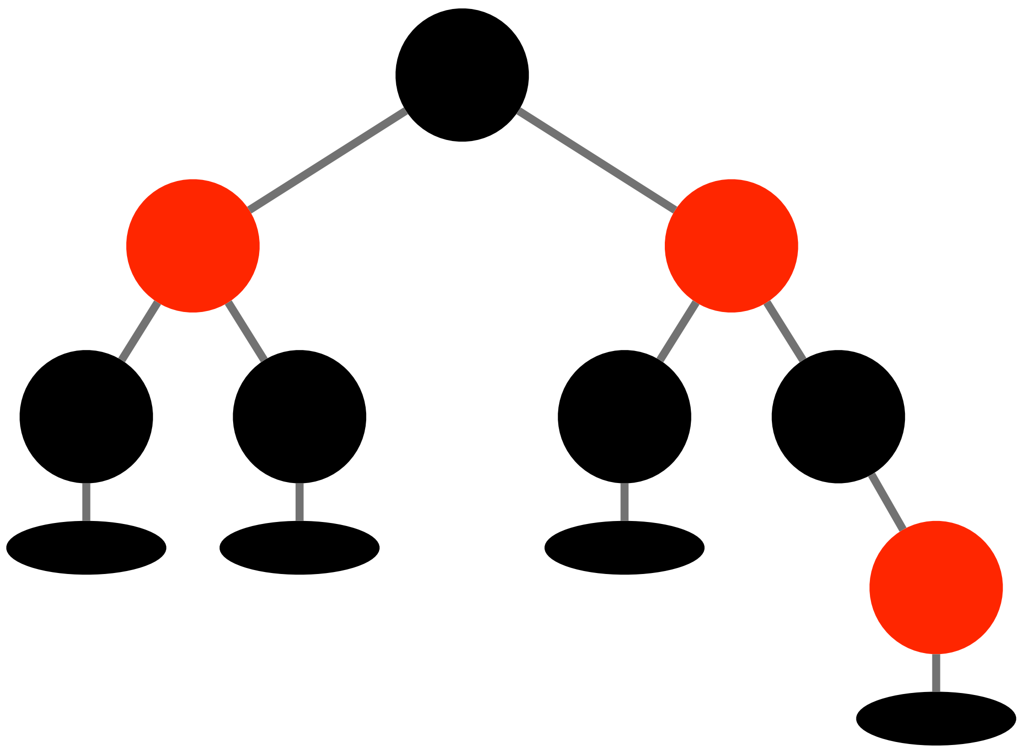

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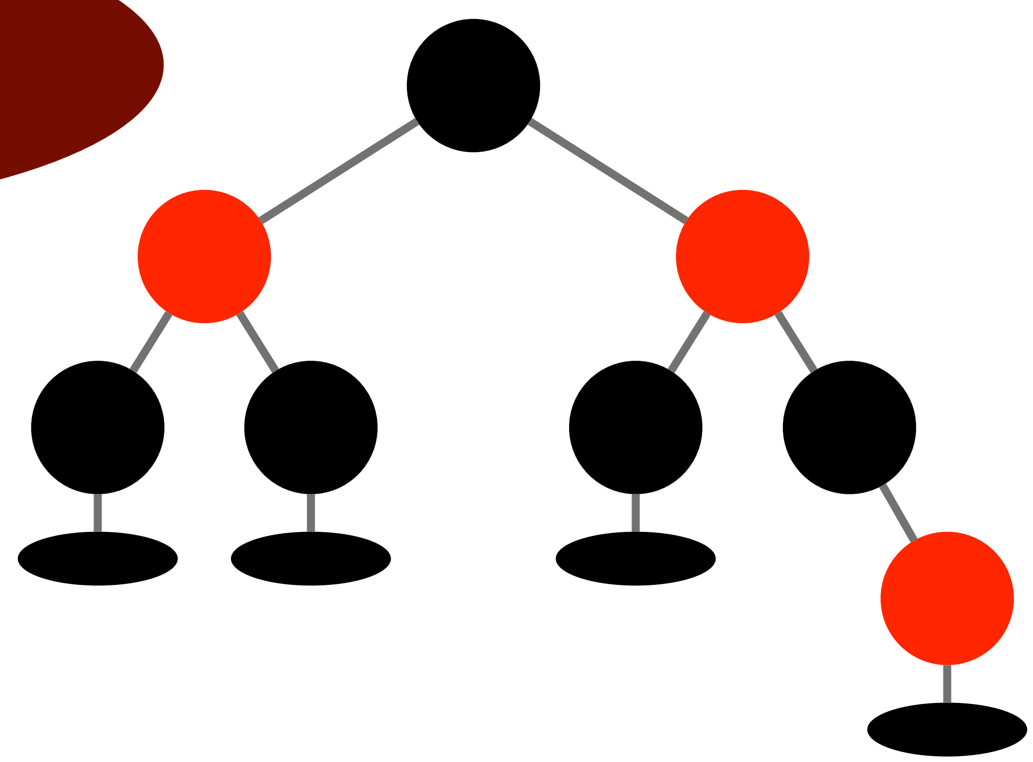
colors used for node coloring



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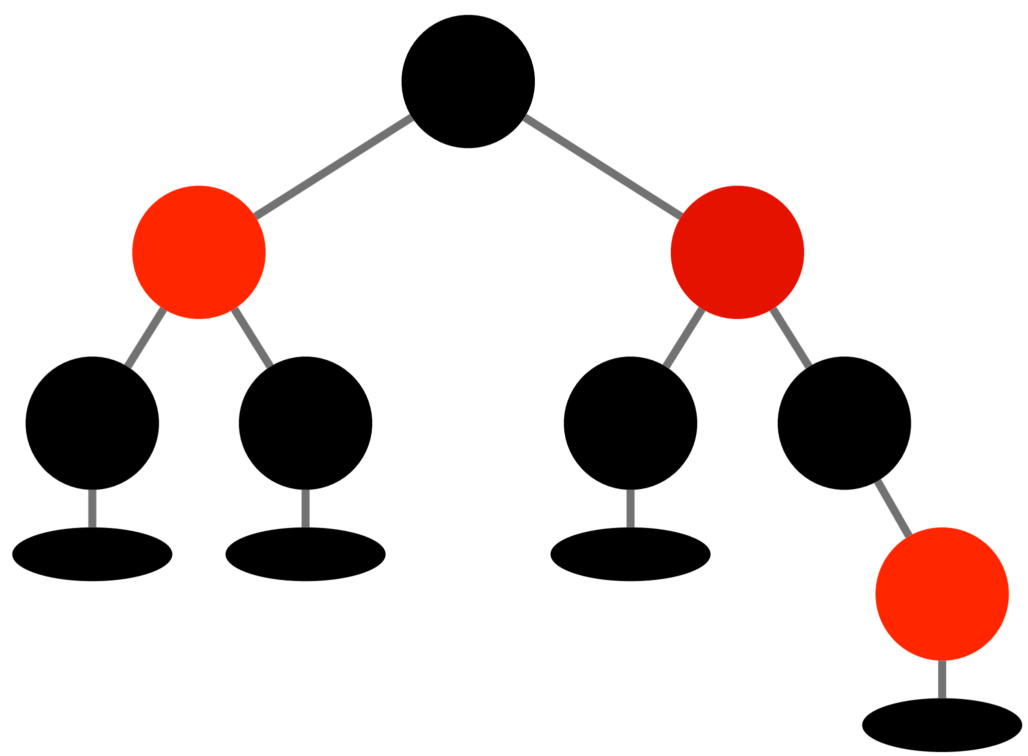

Red Black Trees

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datatype 'a dict =
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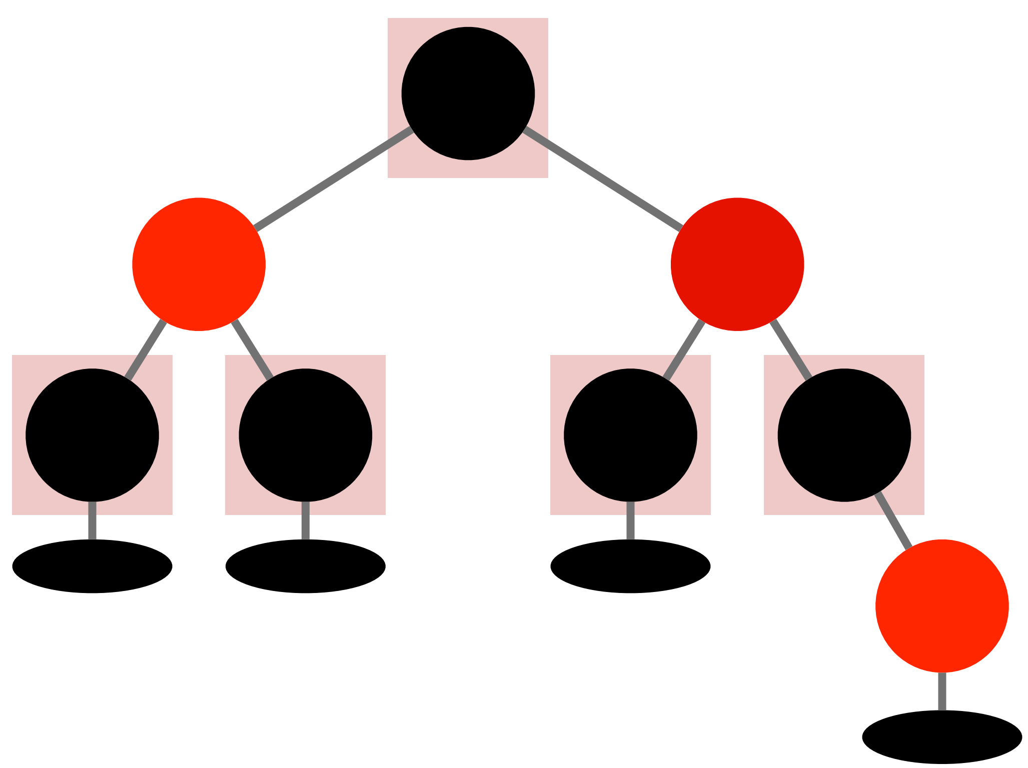
black nodes



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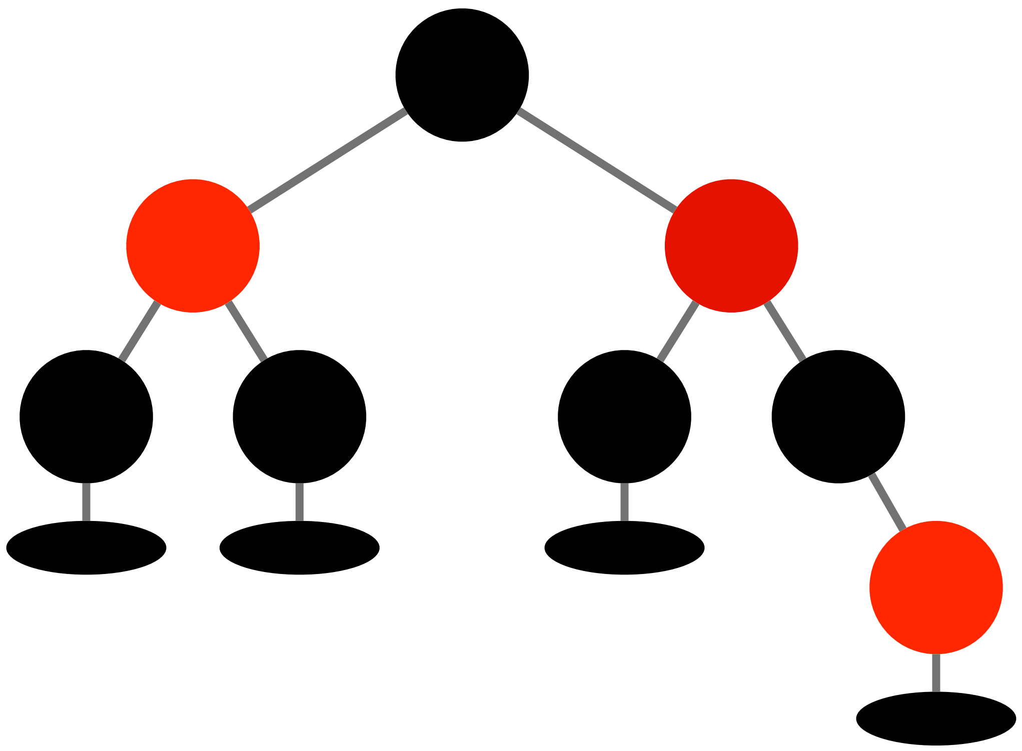

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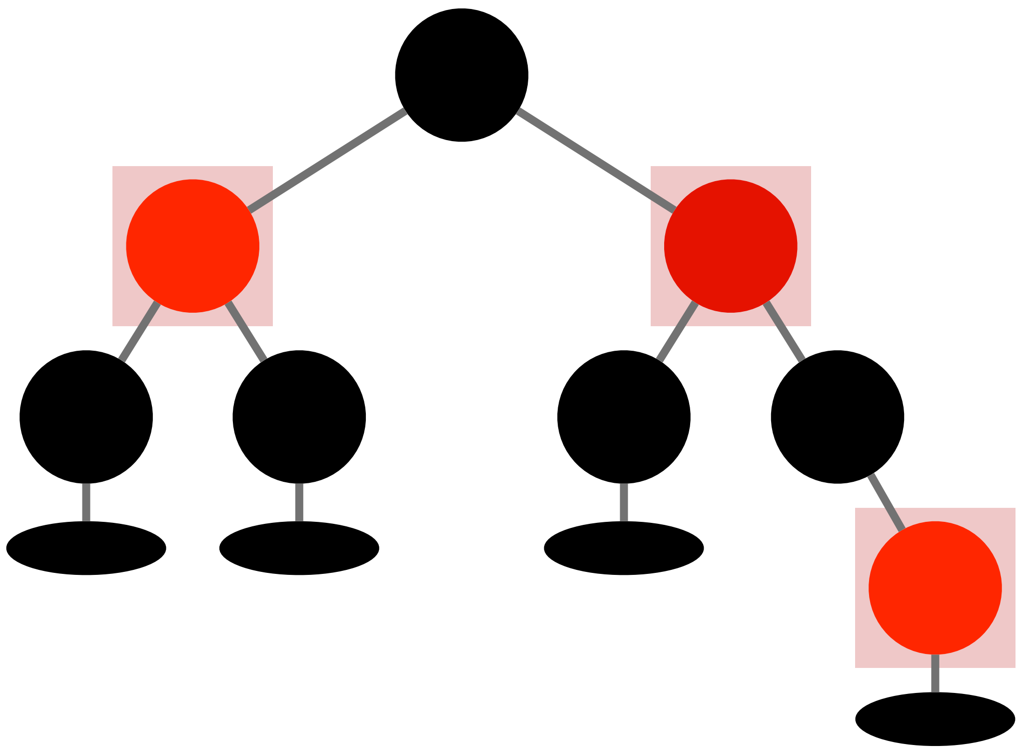

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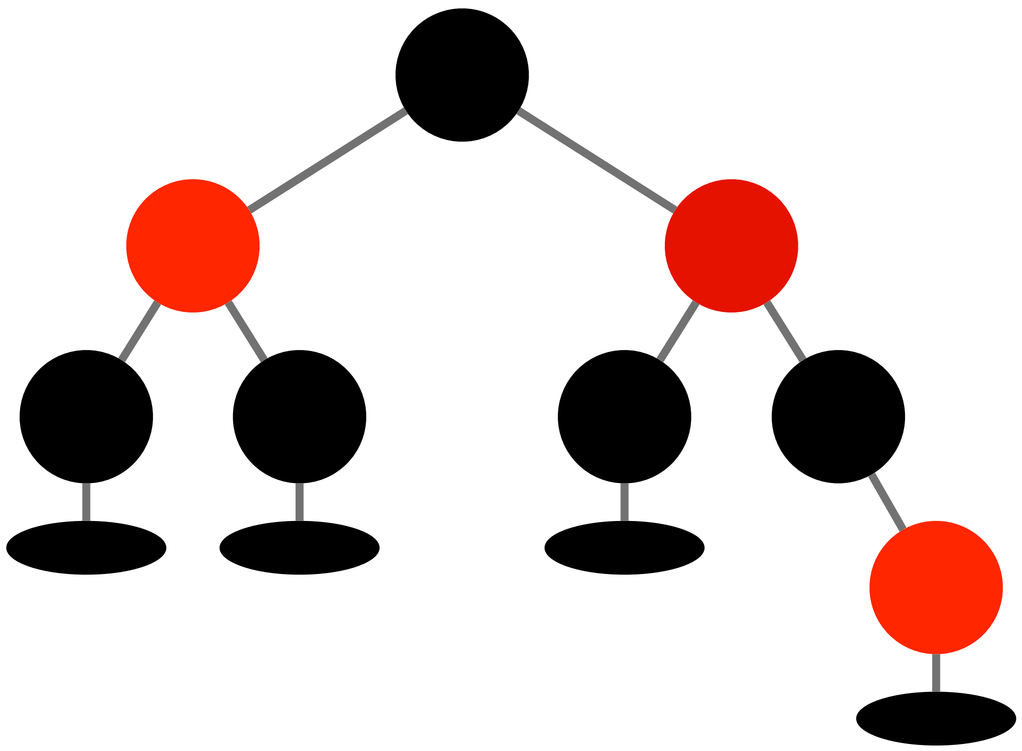

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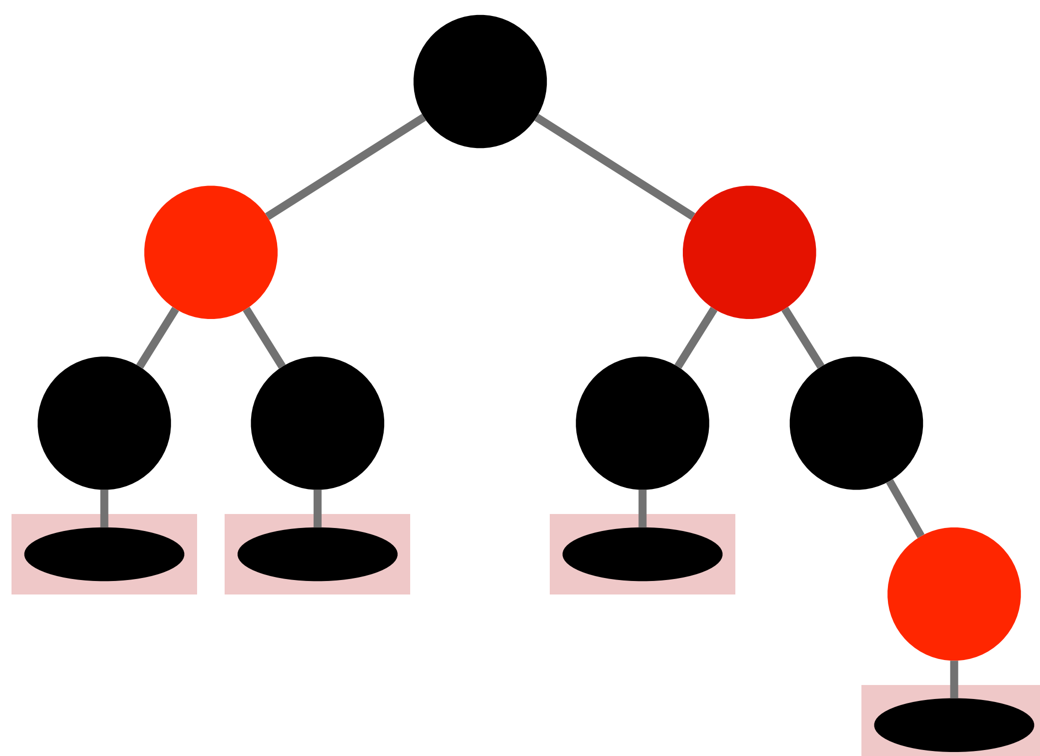

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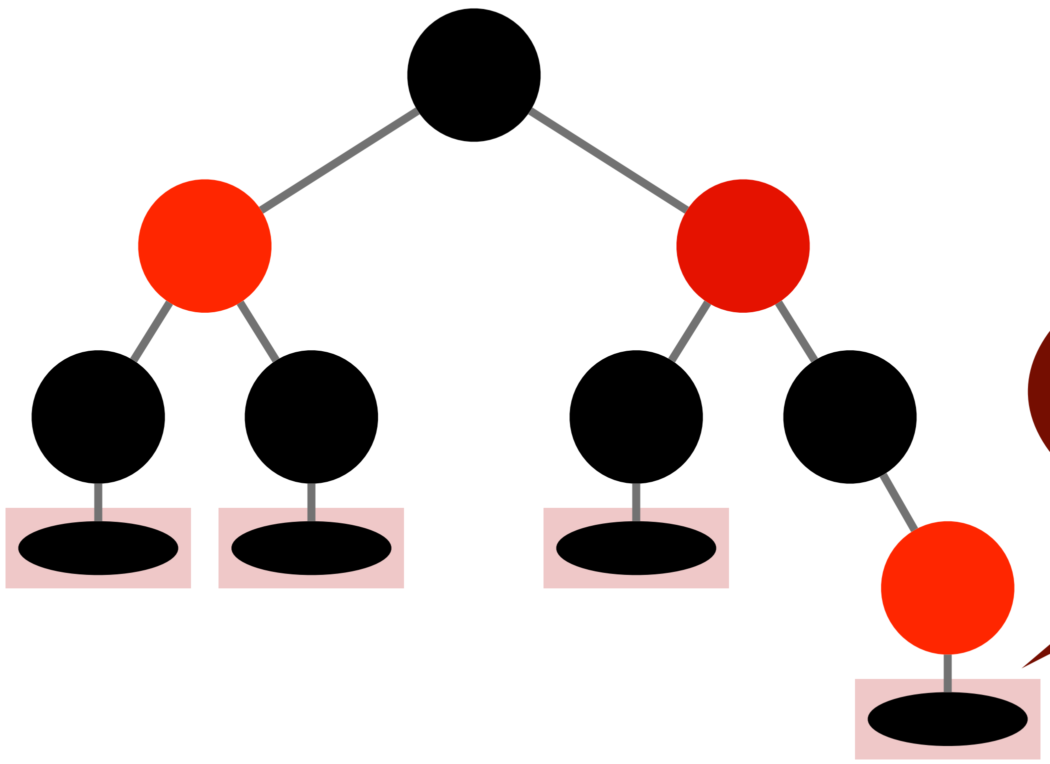

we'll suppress them, moving forward

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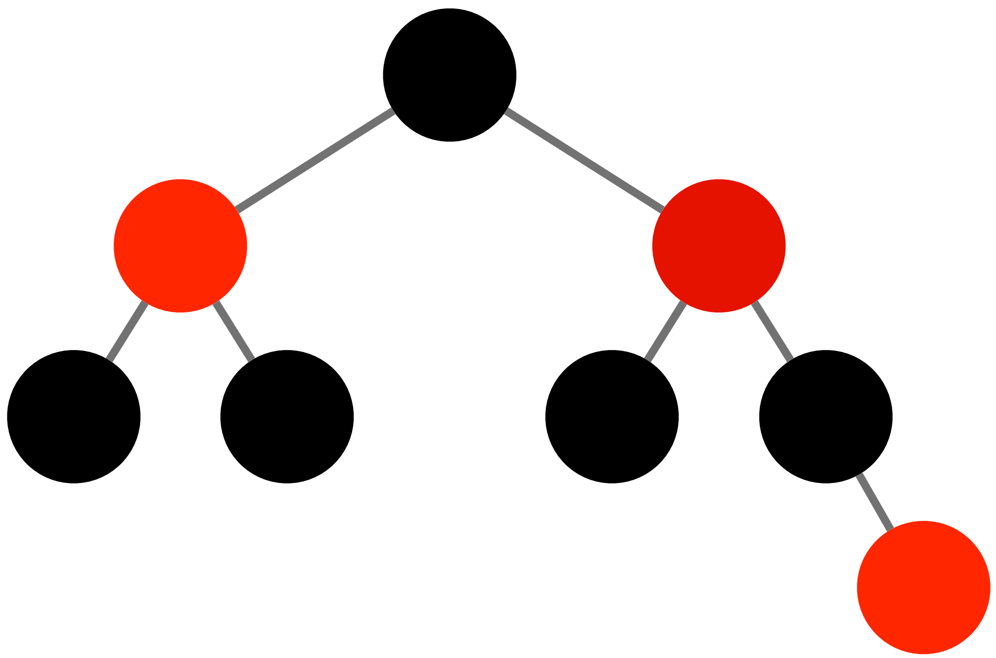
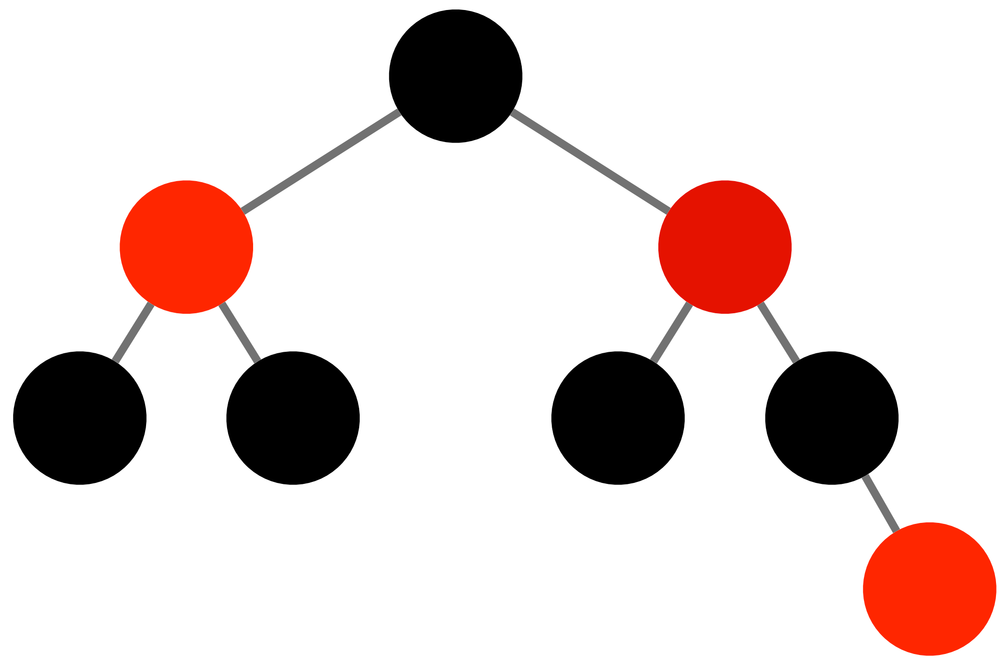


Red Black Tree (RBT) Invariant



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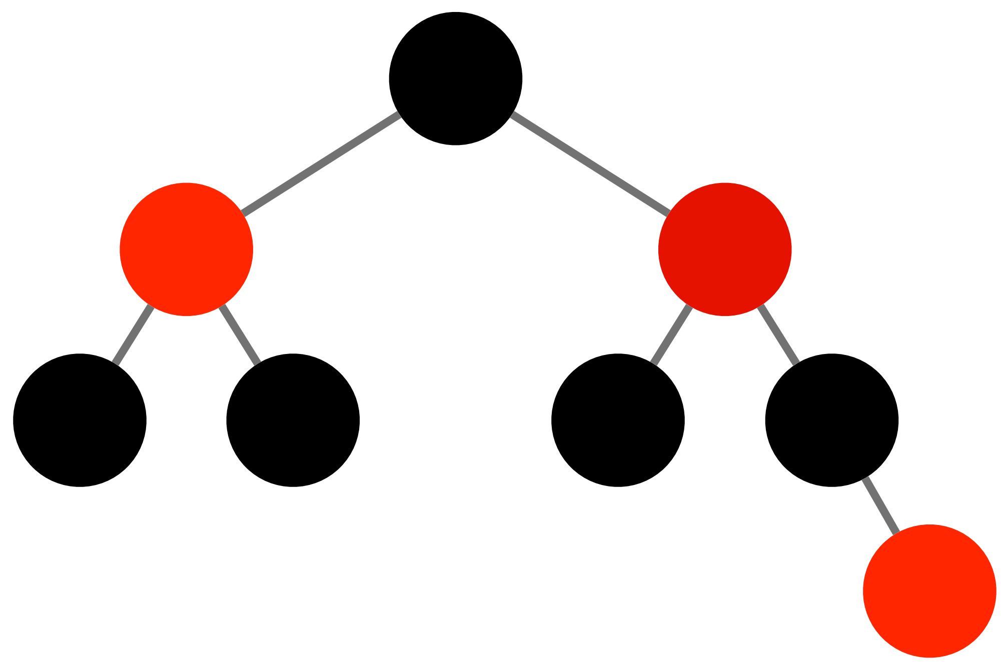

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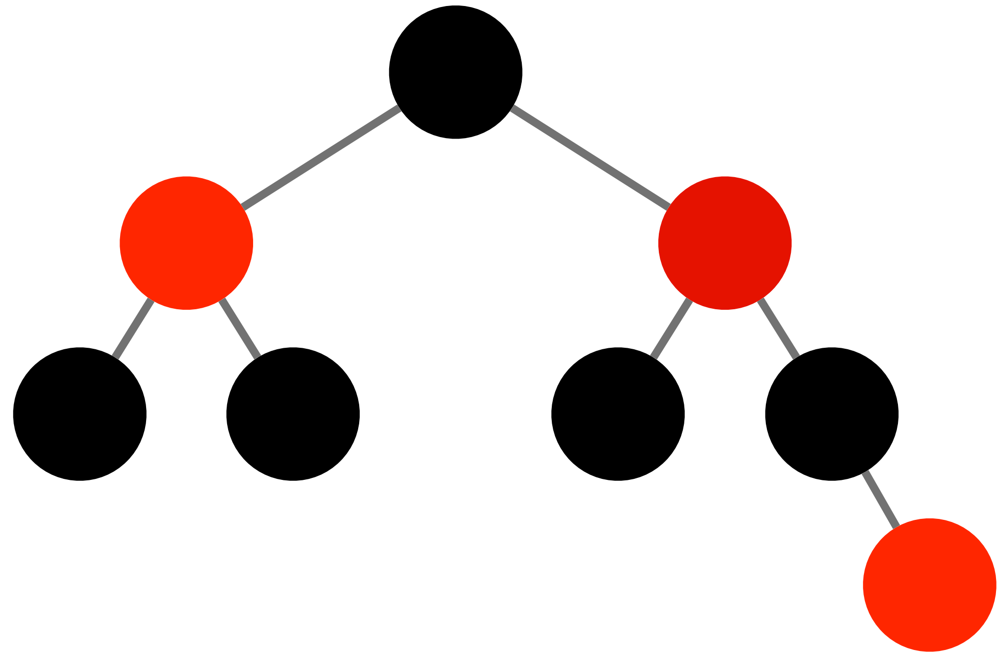

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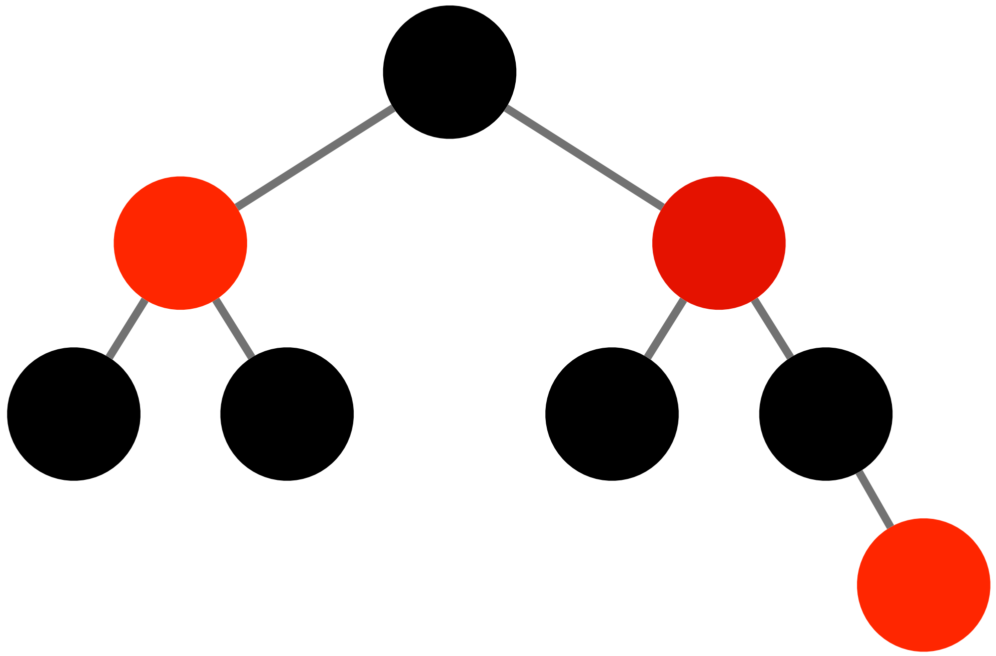

A Tree is **sorted** according to an entry's key.

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C **Black height**: for any node, the number of black nodes along any path from the node to a leaf (empty) is the same.

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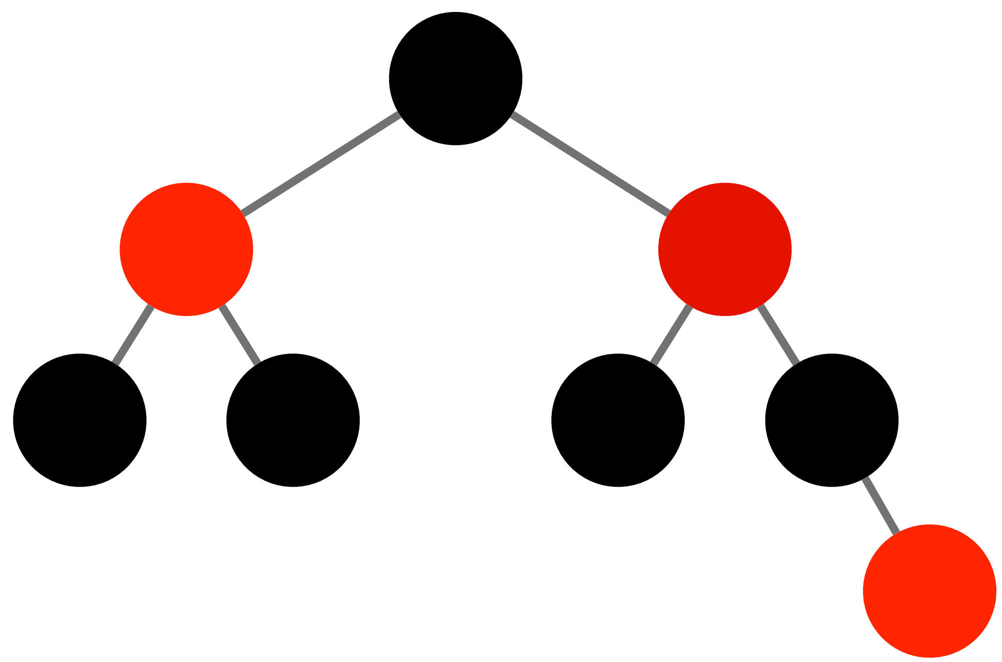

or, the number of black nodes from the root

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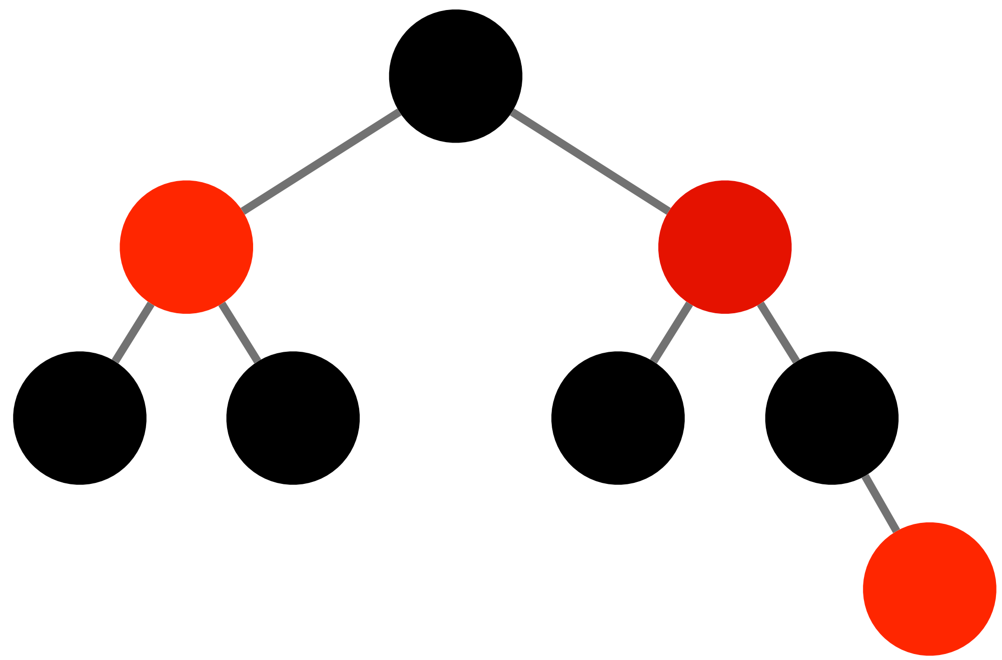

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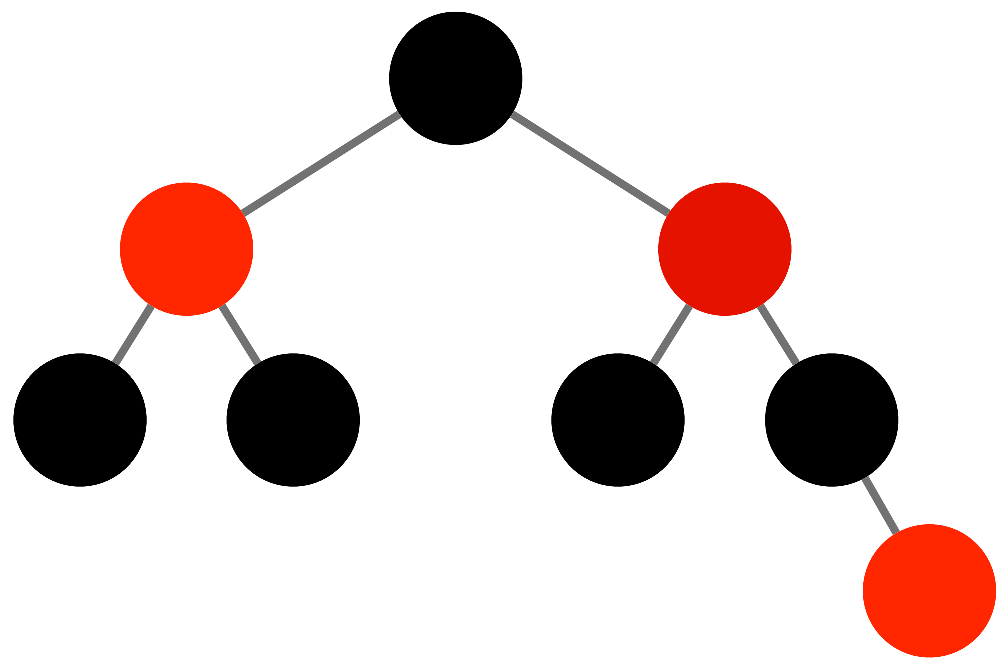

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This representation invariant ensures that tree is roughly balanced:

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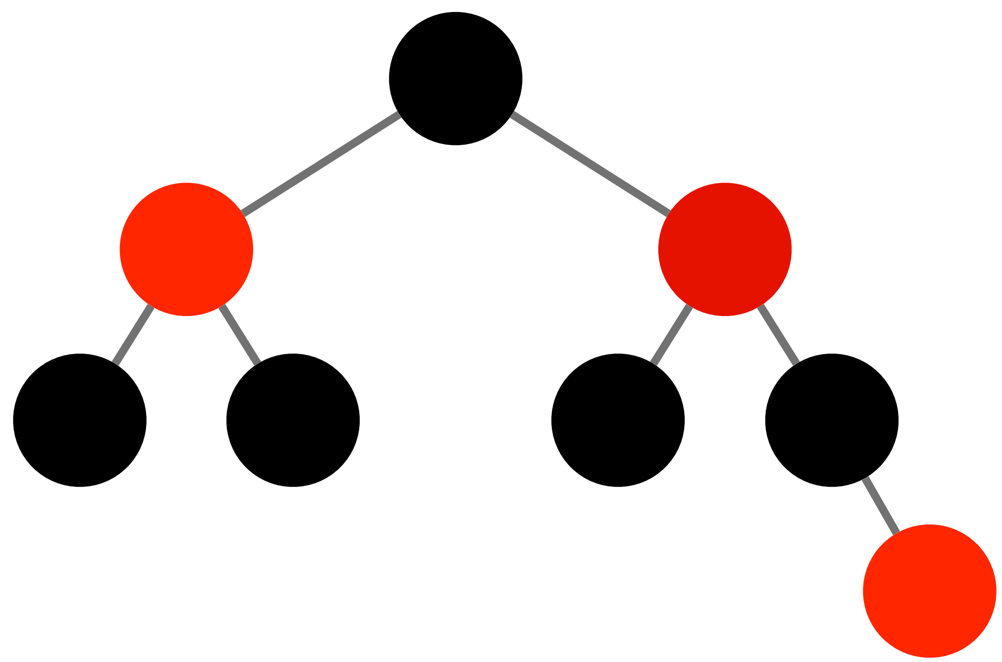

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$$\text{depth} \leq 2\log_2(|\text{nodes}| + 1)$$

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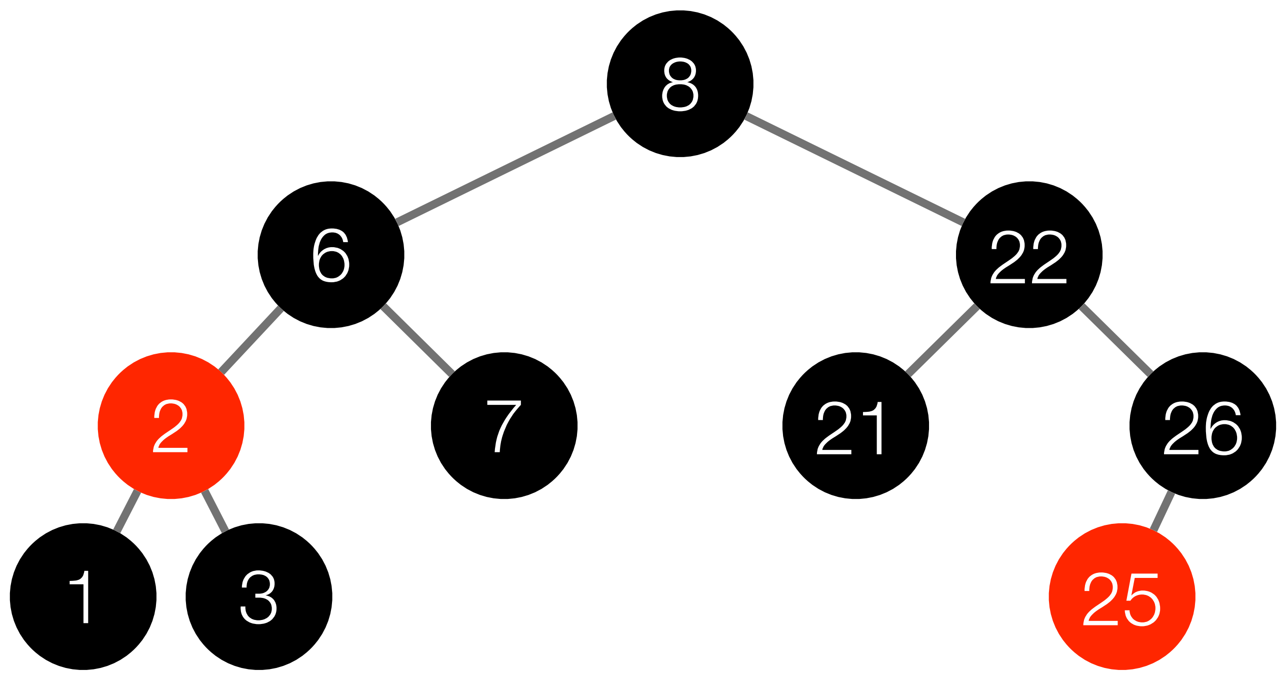

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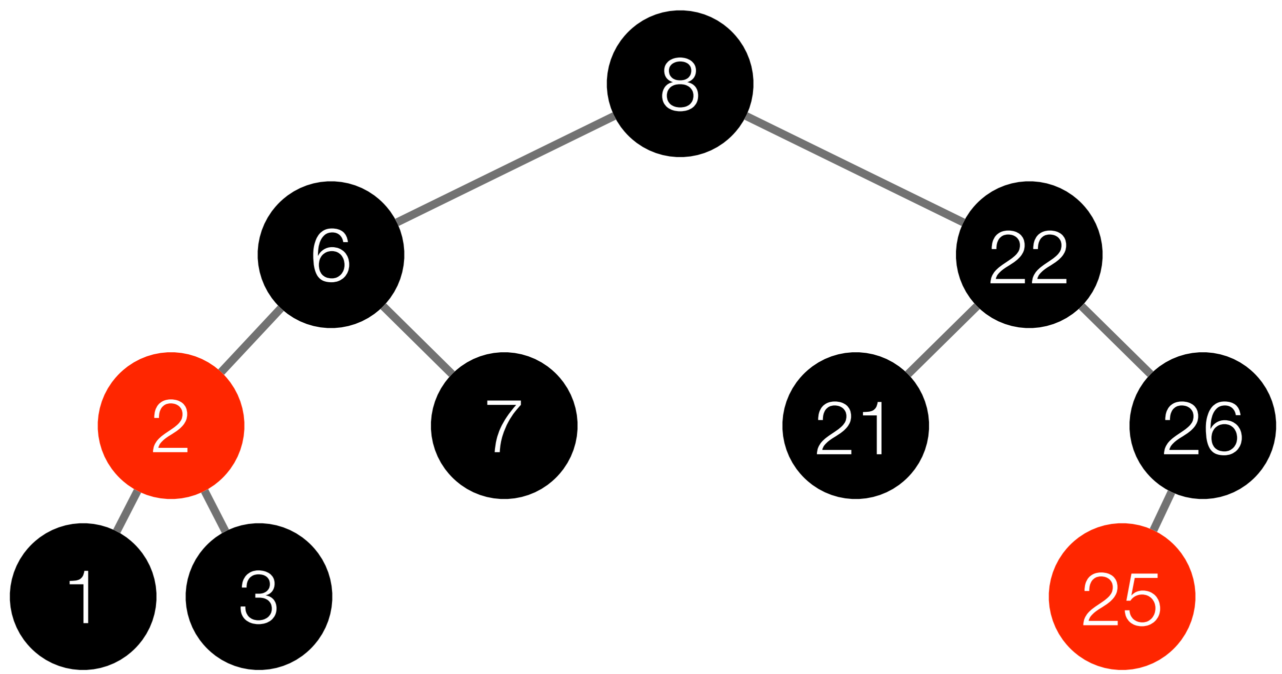

(For simplicity, we use integer keys and omit value part of an entry.)

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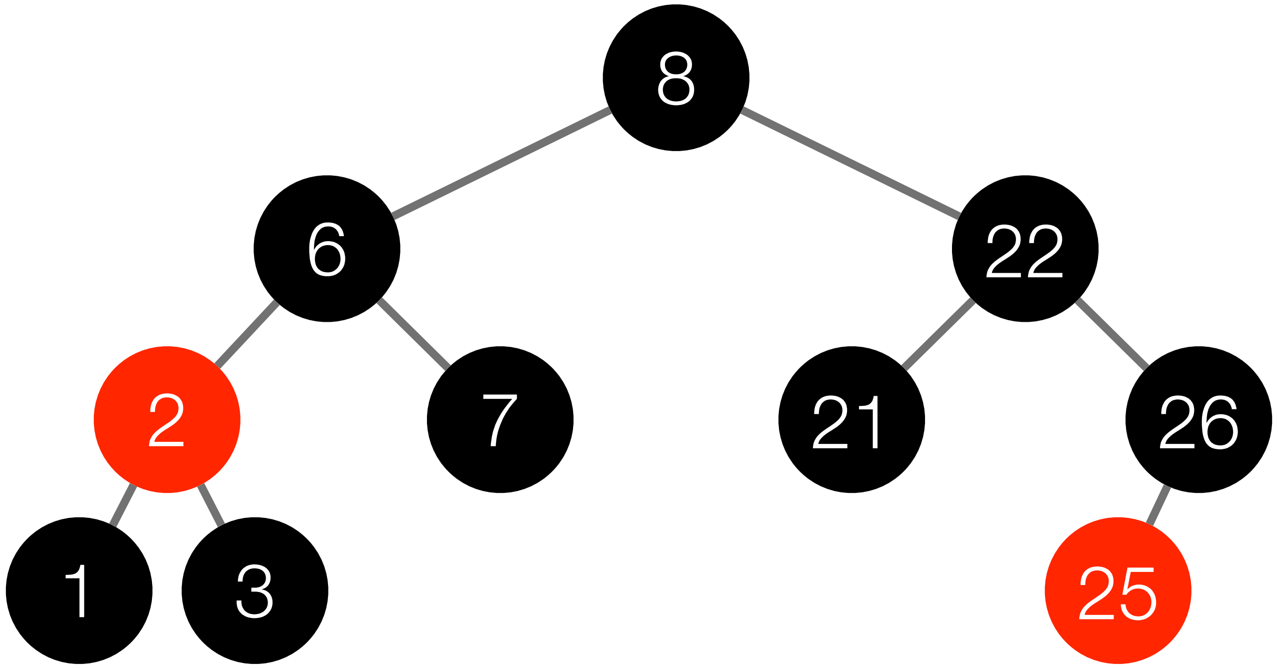

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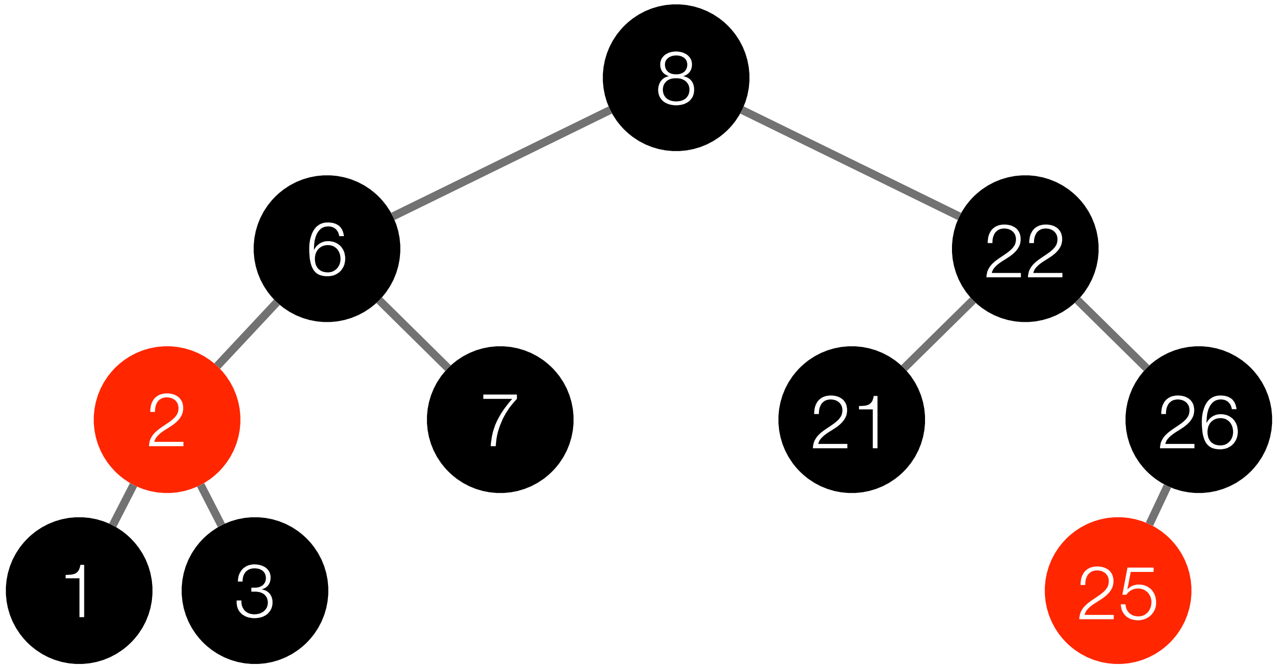

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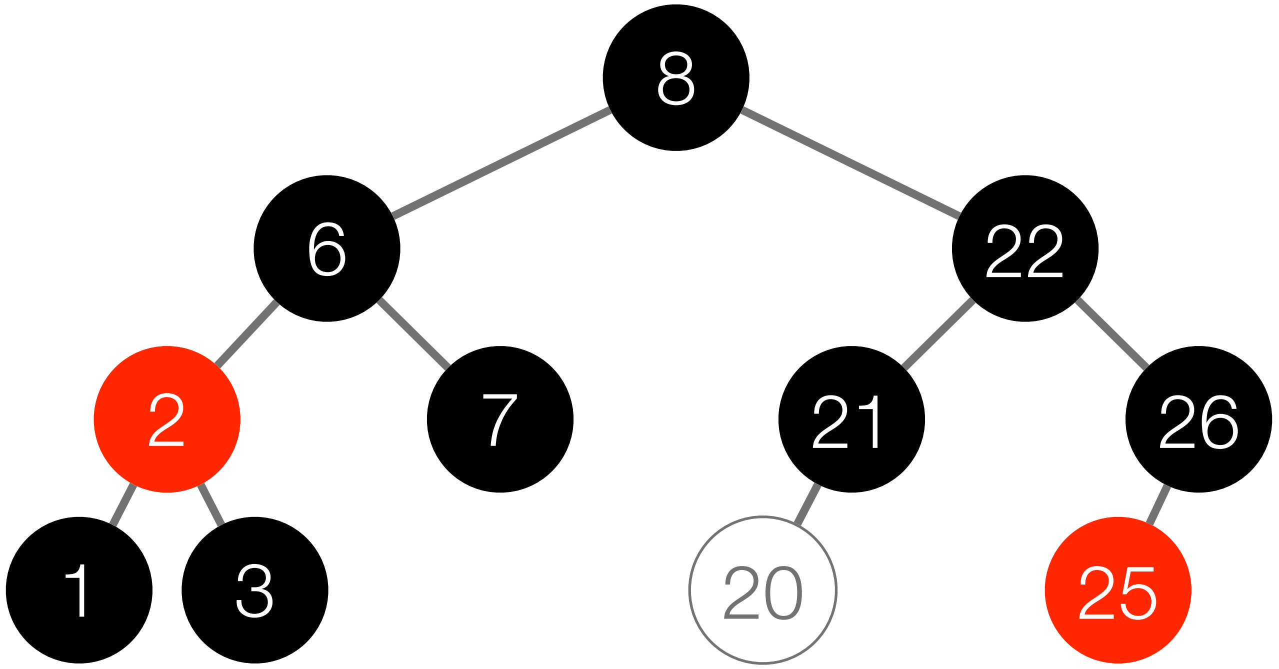

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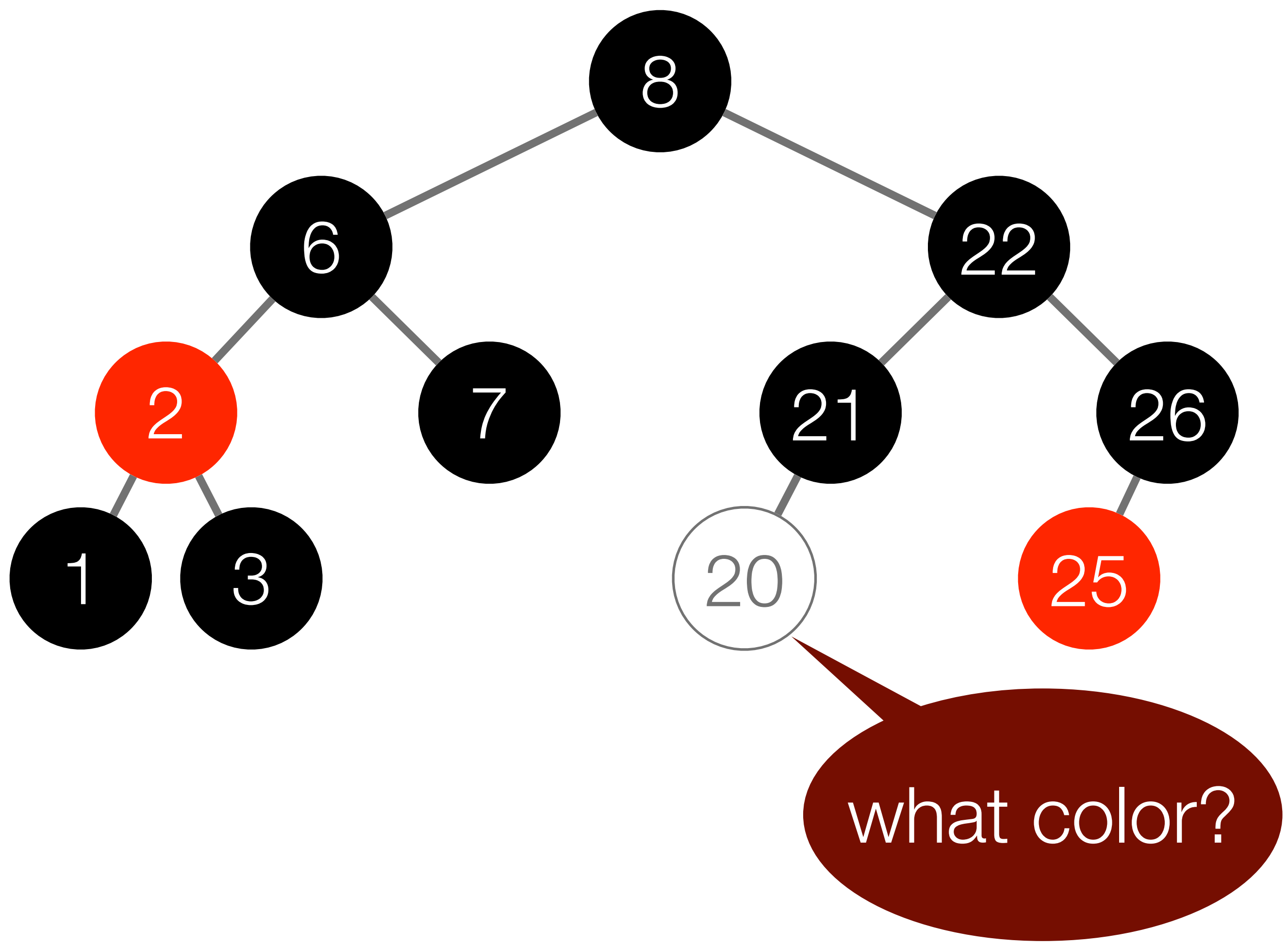

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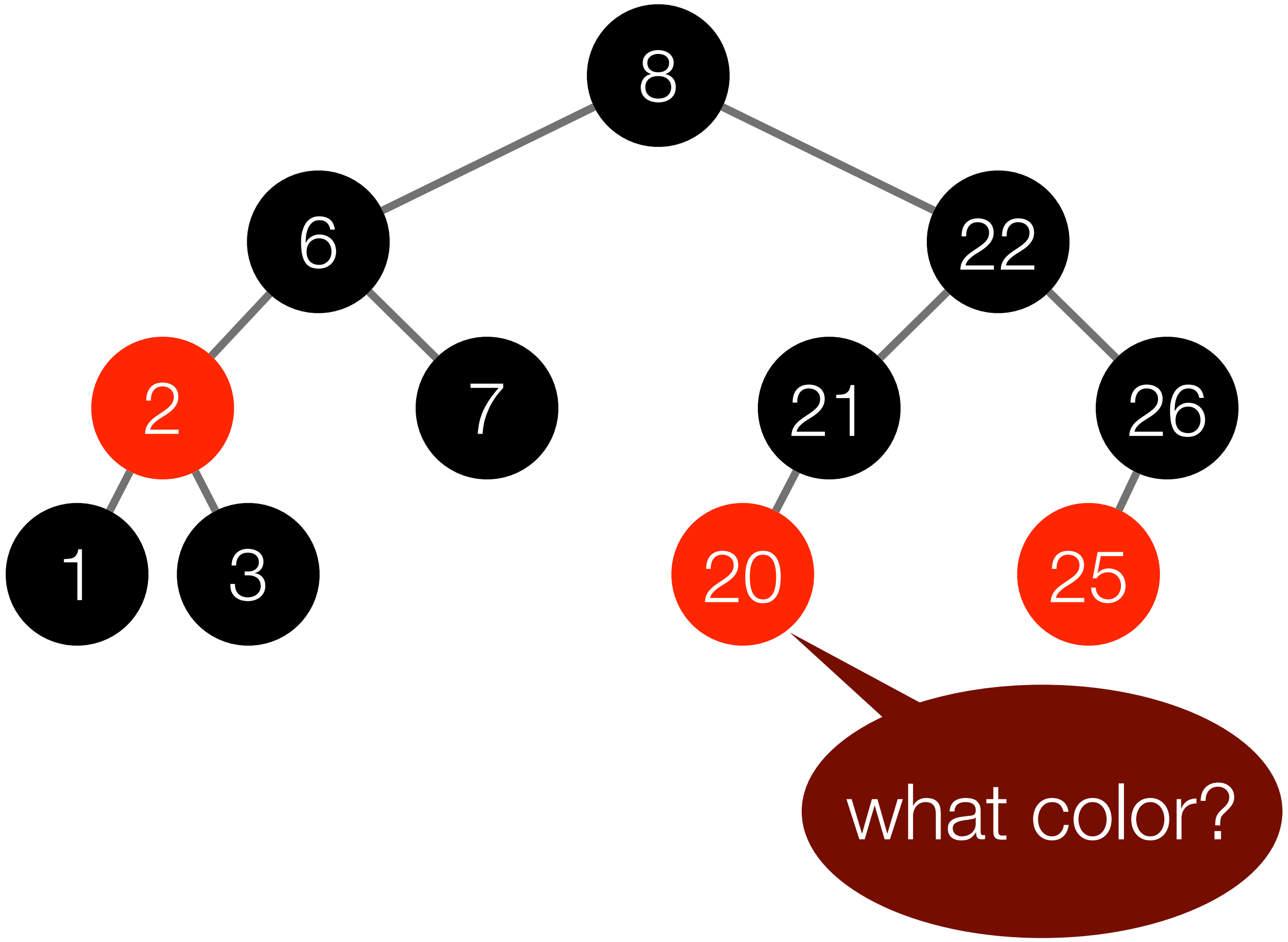

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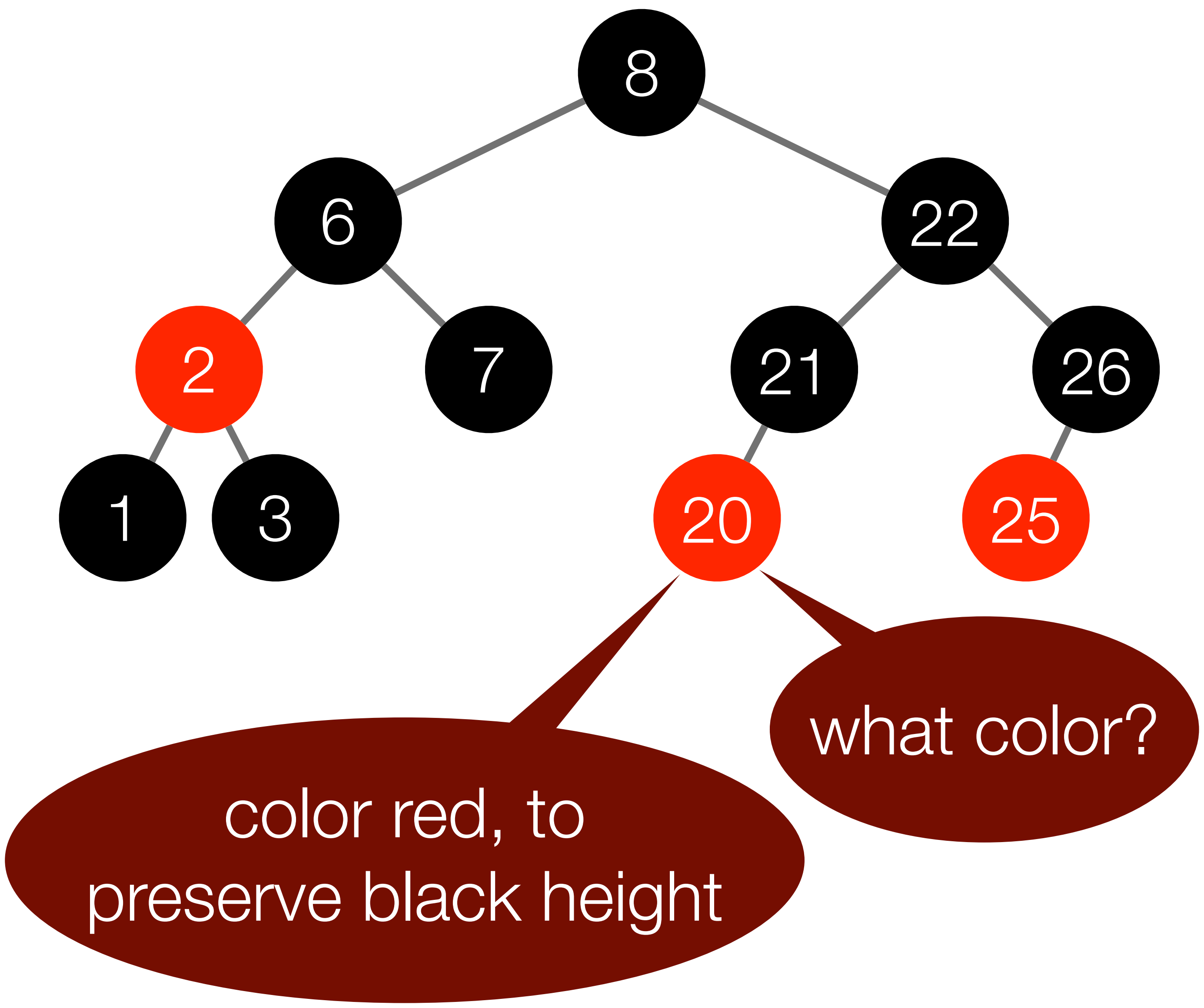

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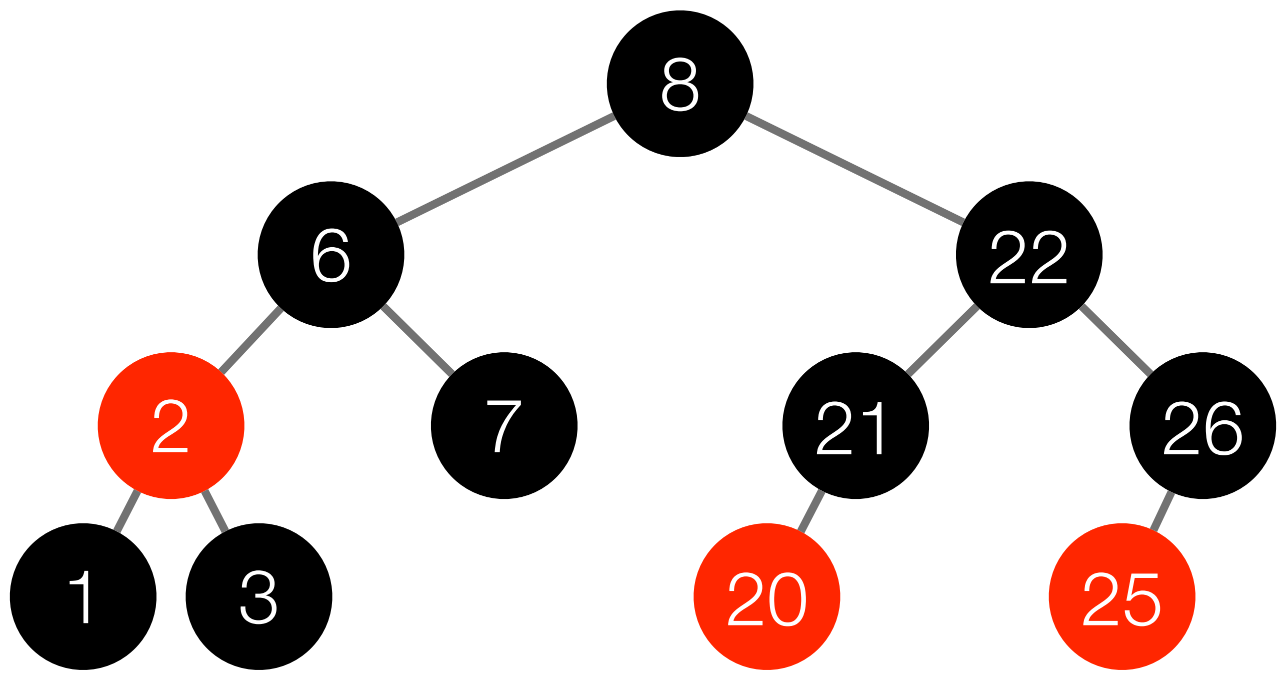

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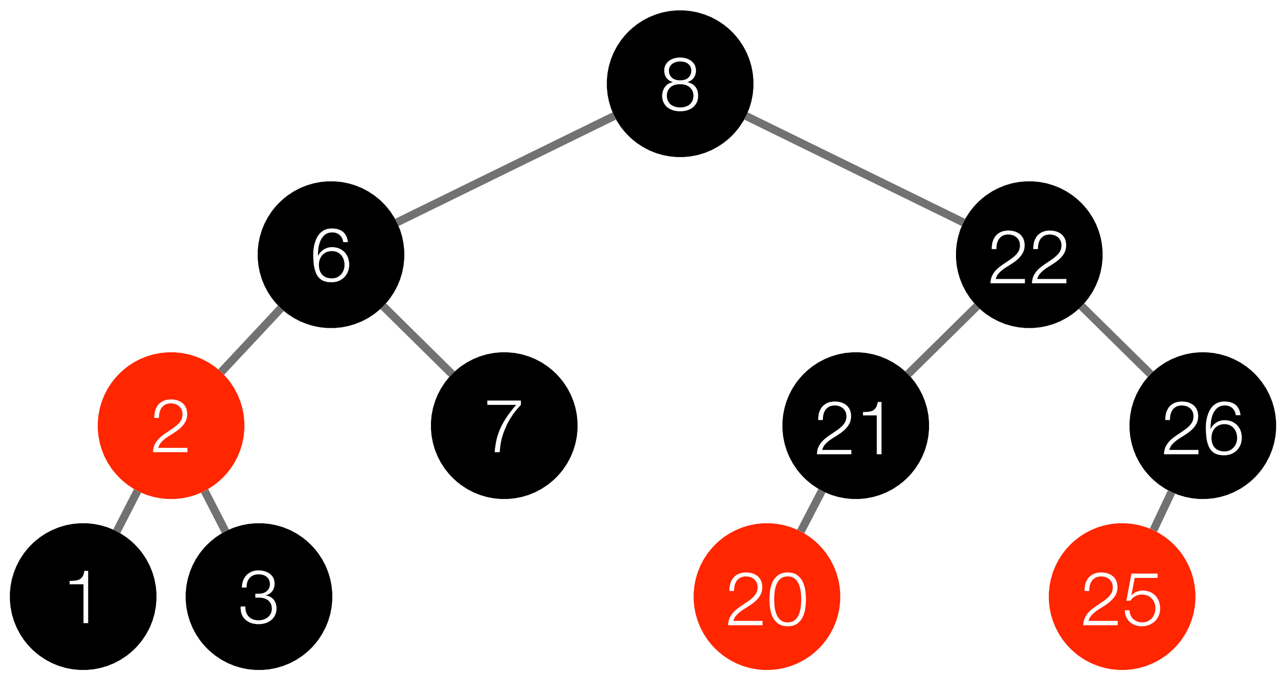
Now, let's insert 19:



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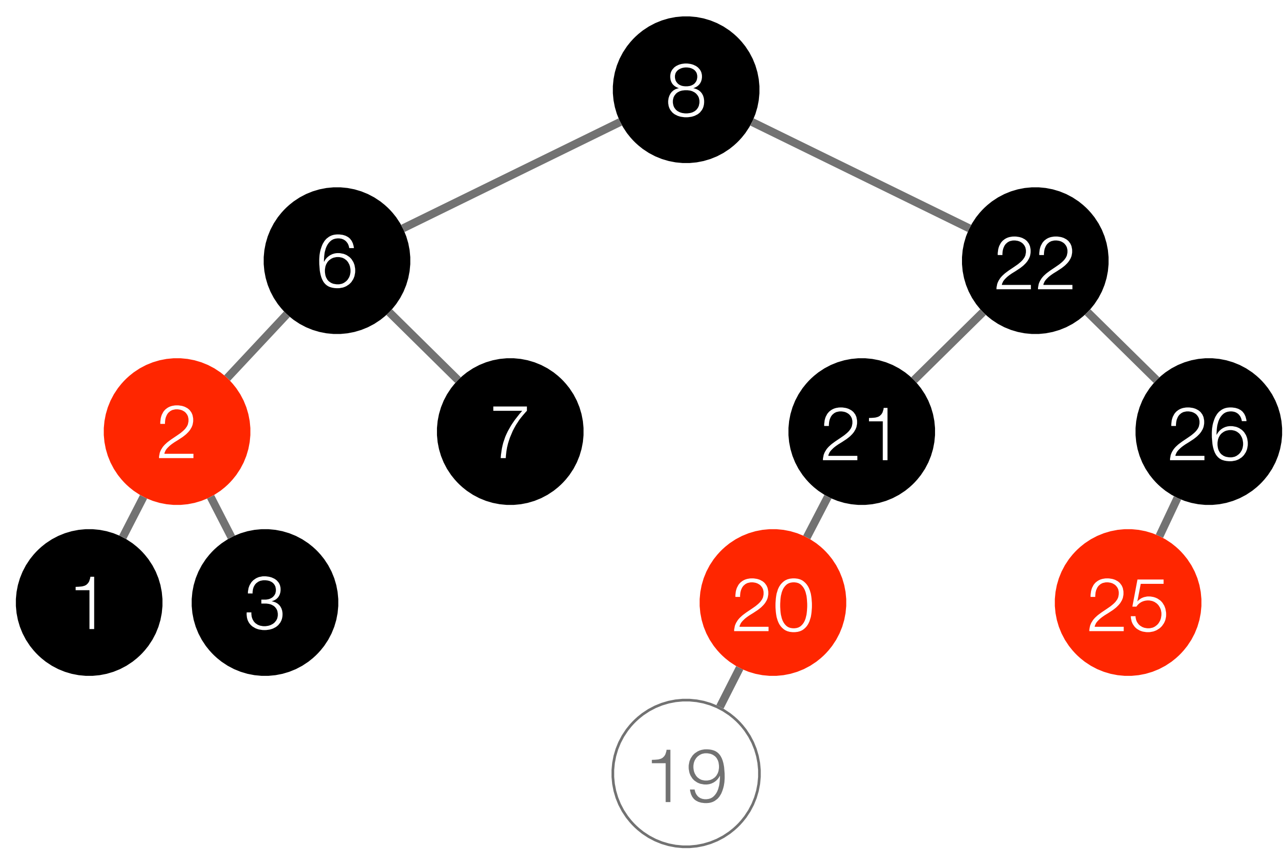

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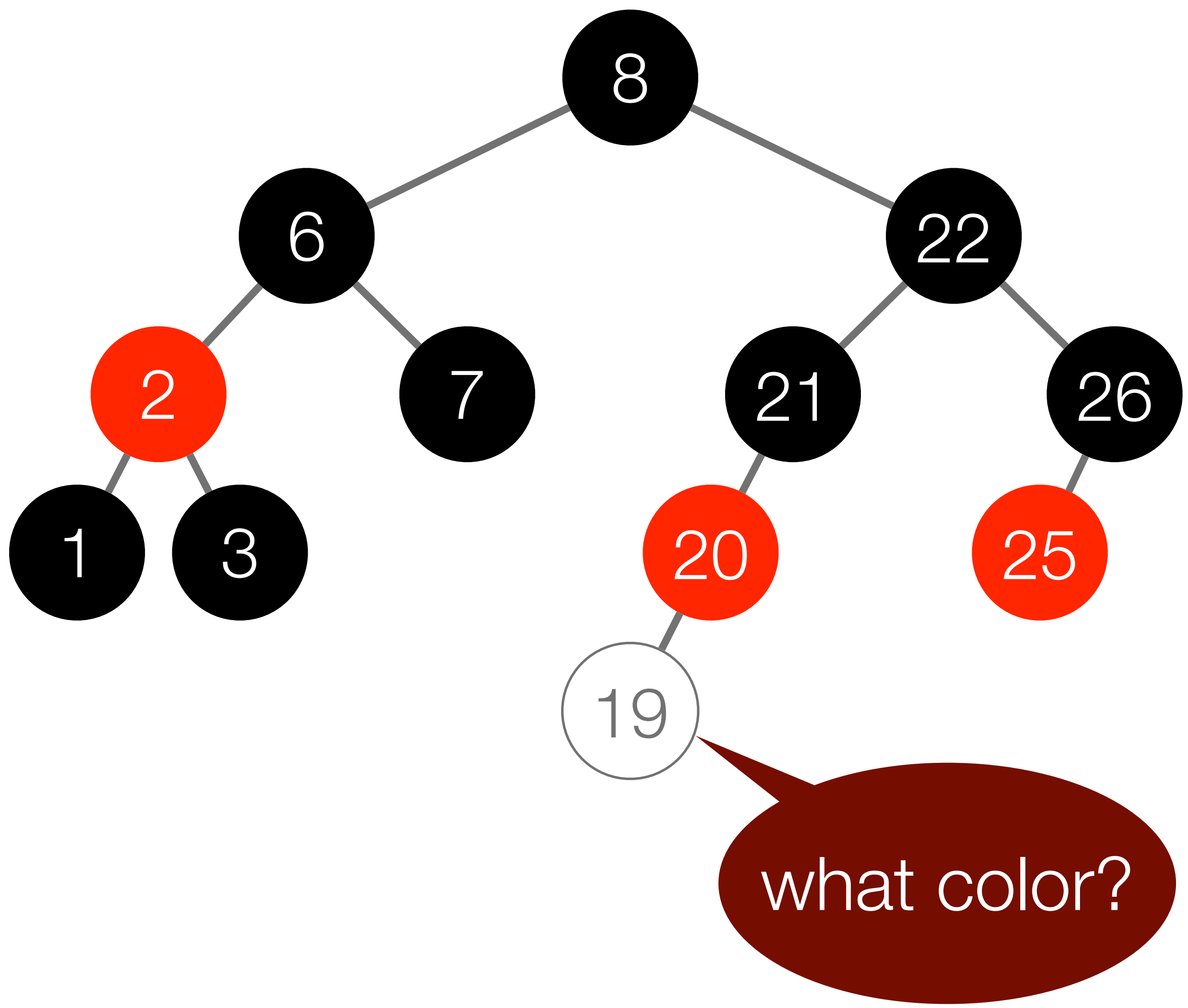

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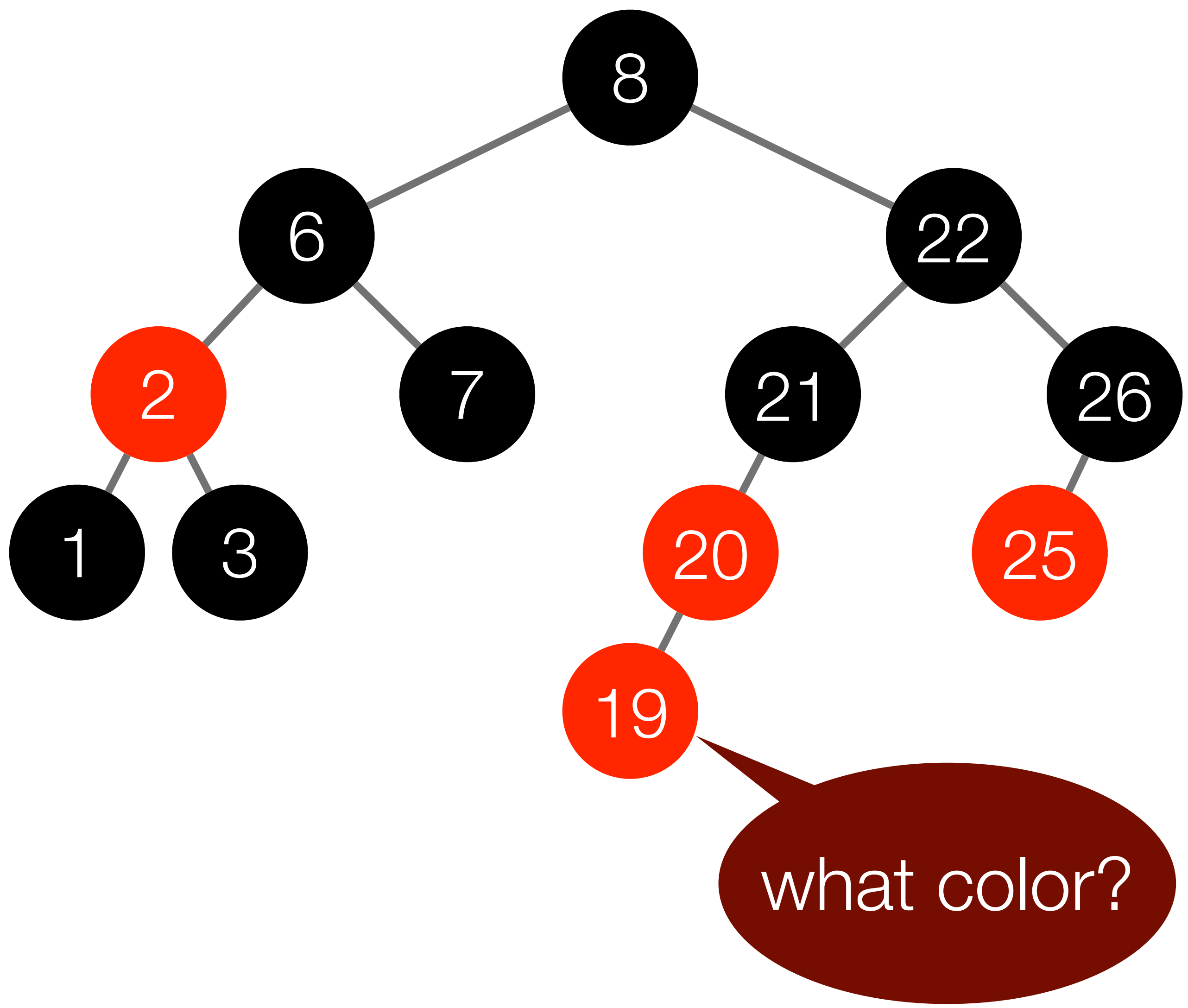

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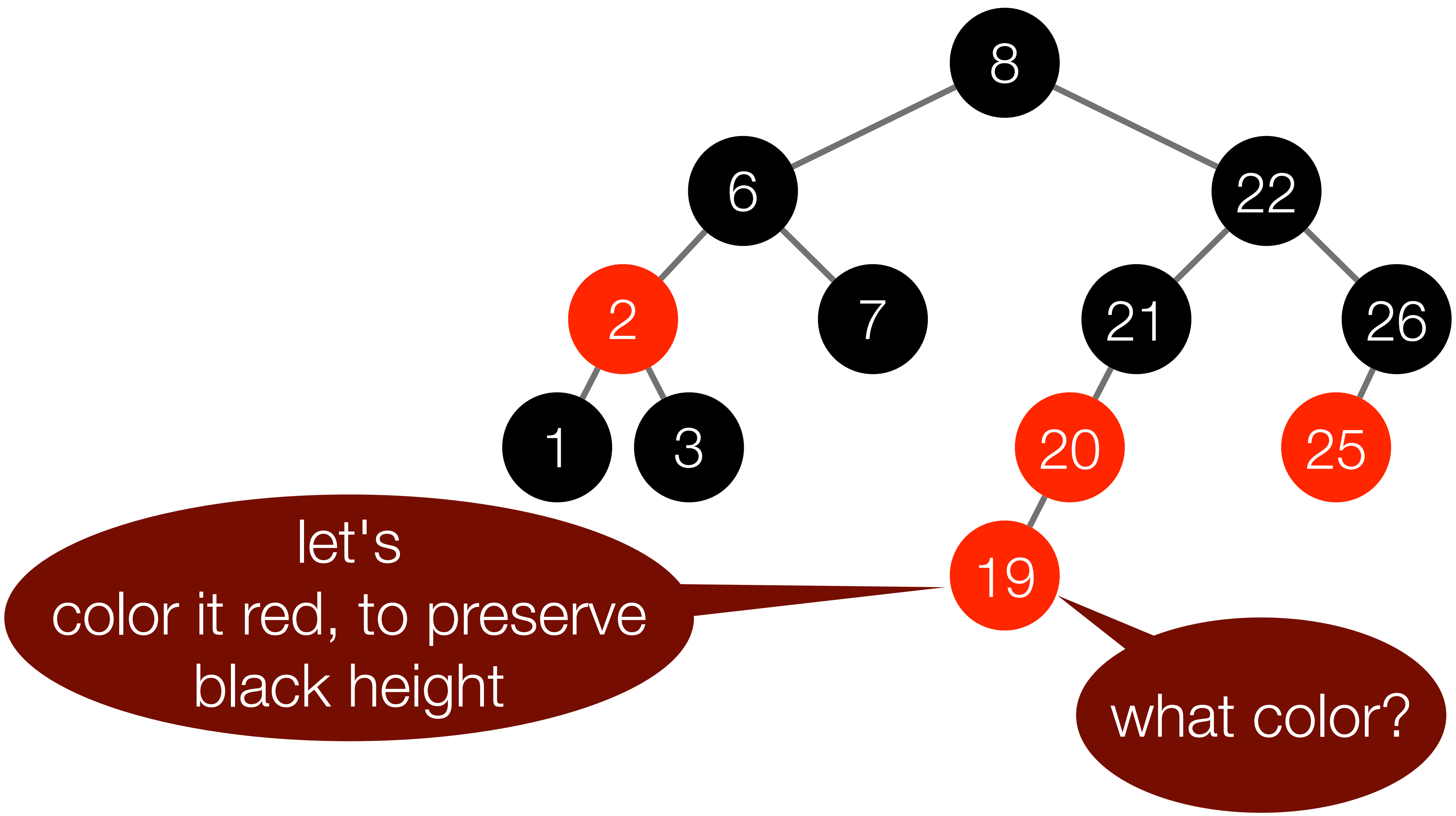

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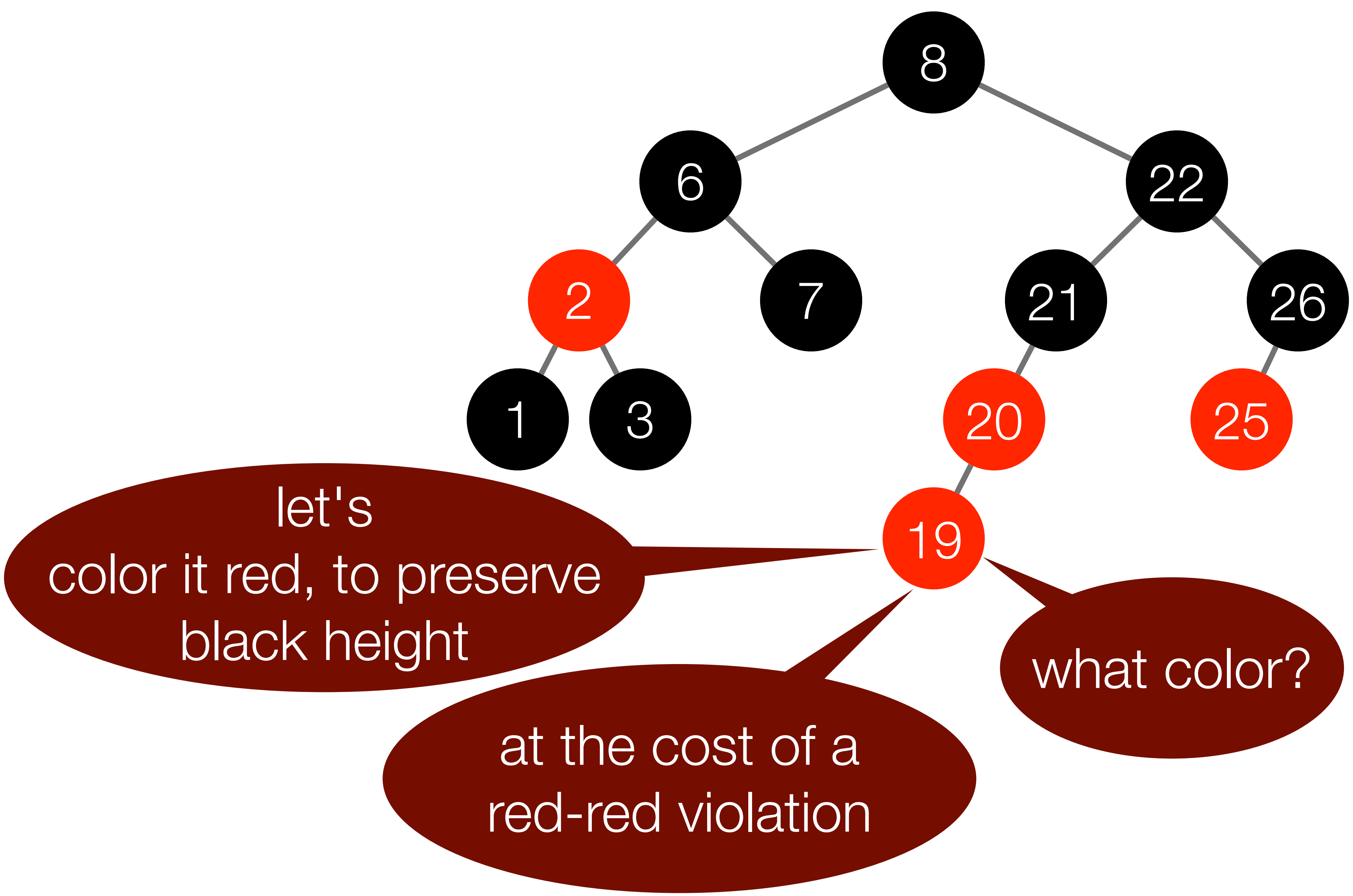

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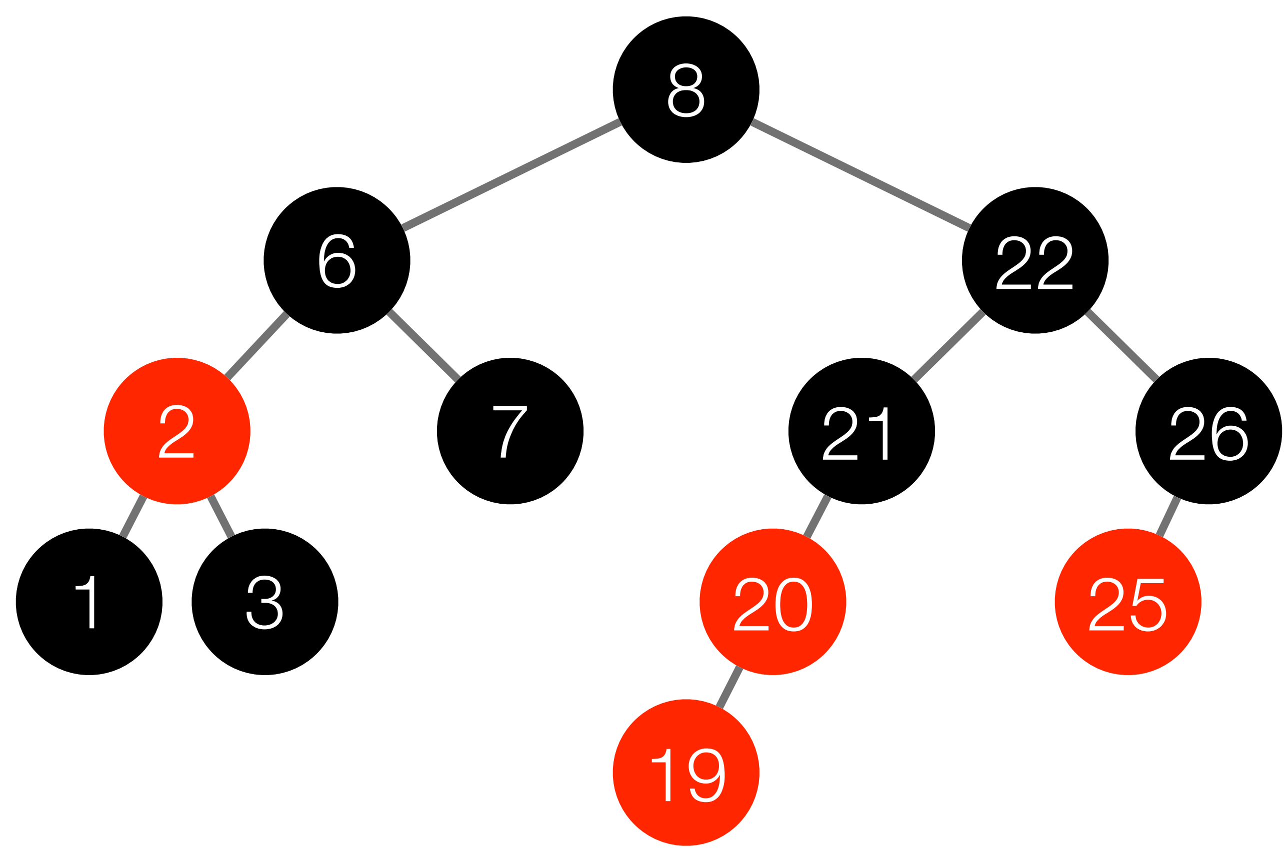

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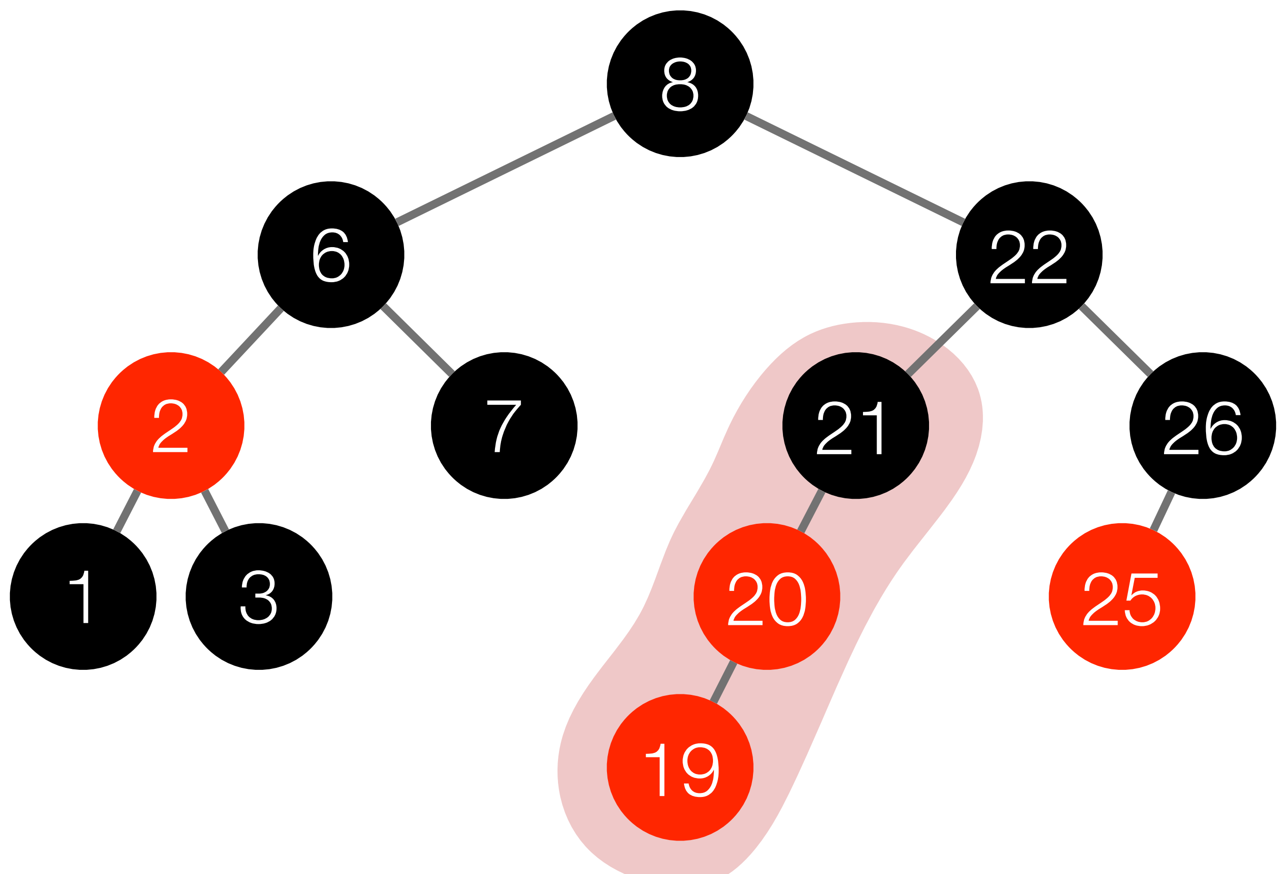

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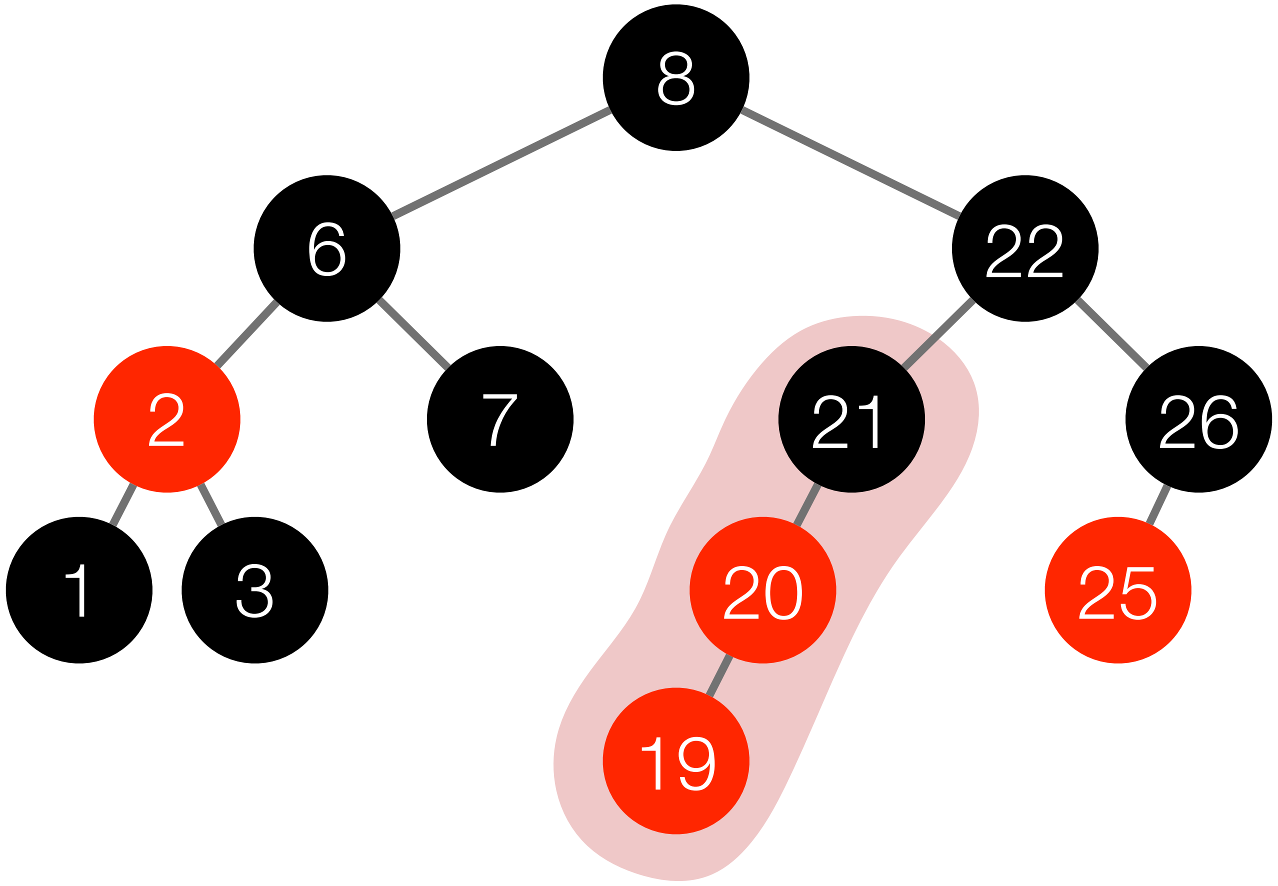

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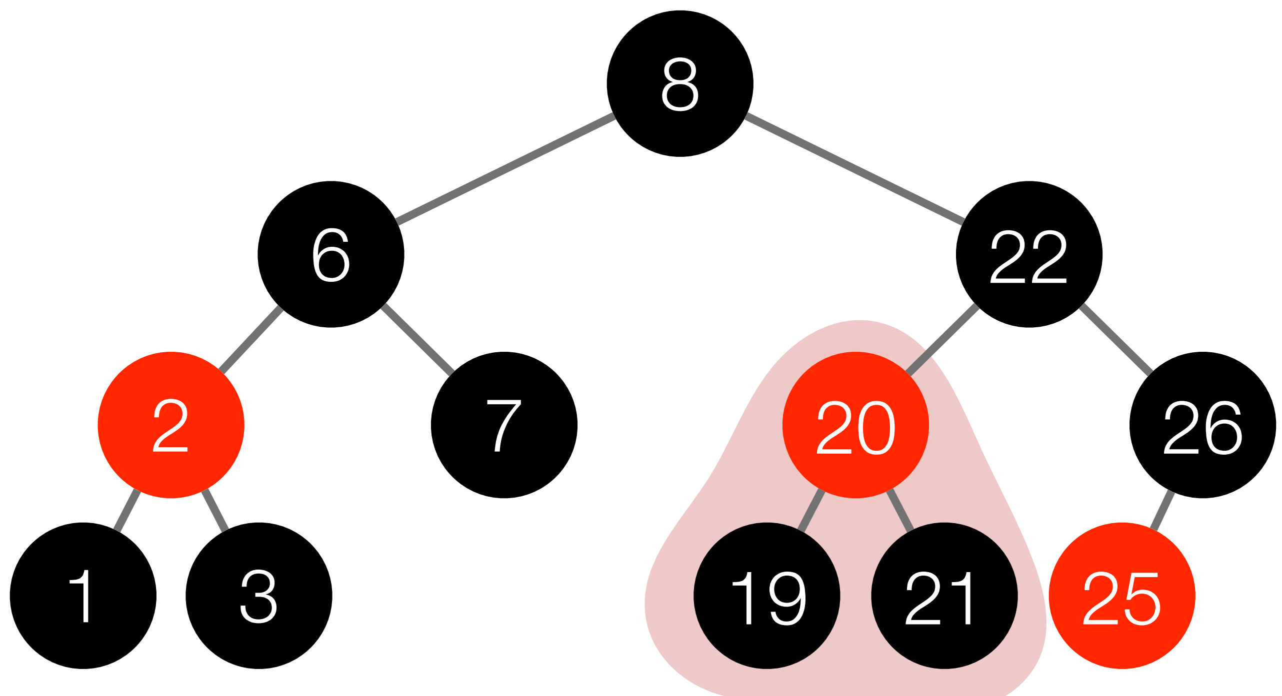

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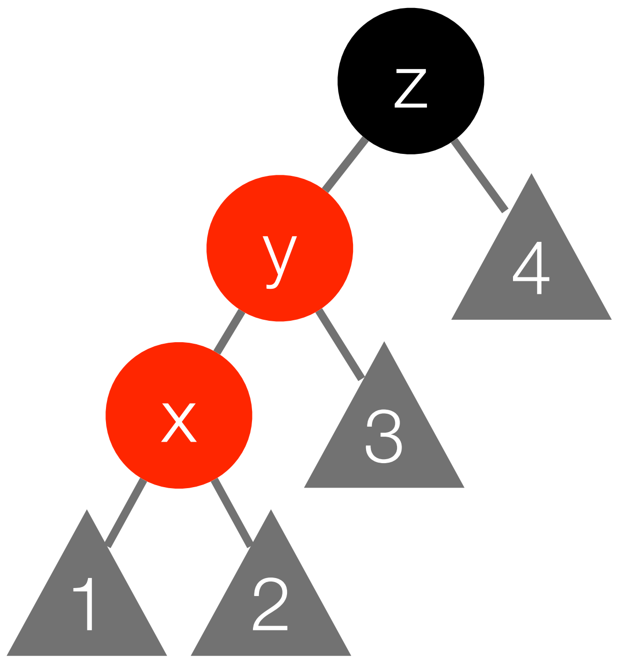

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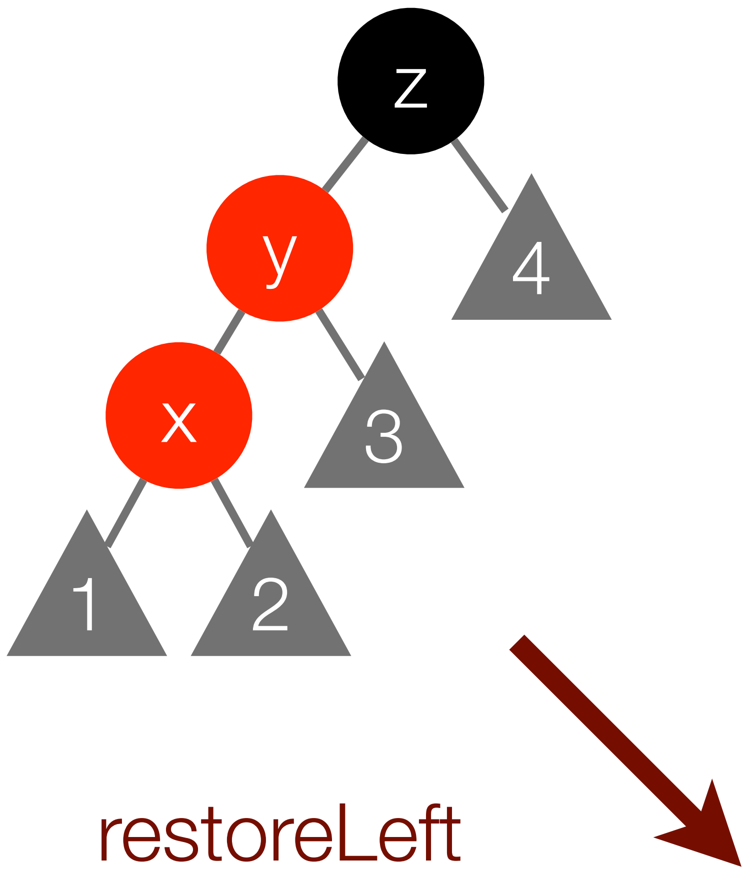
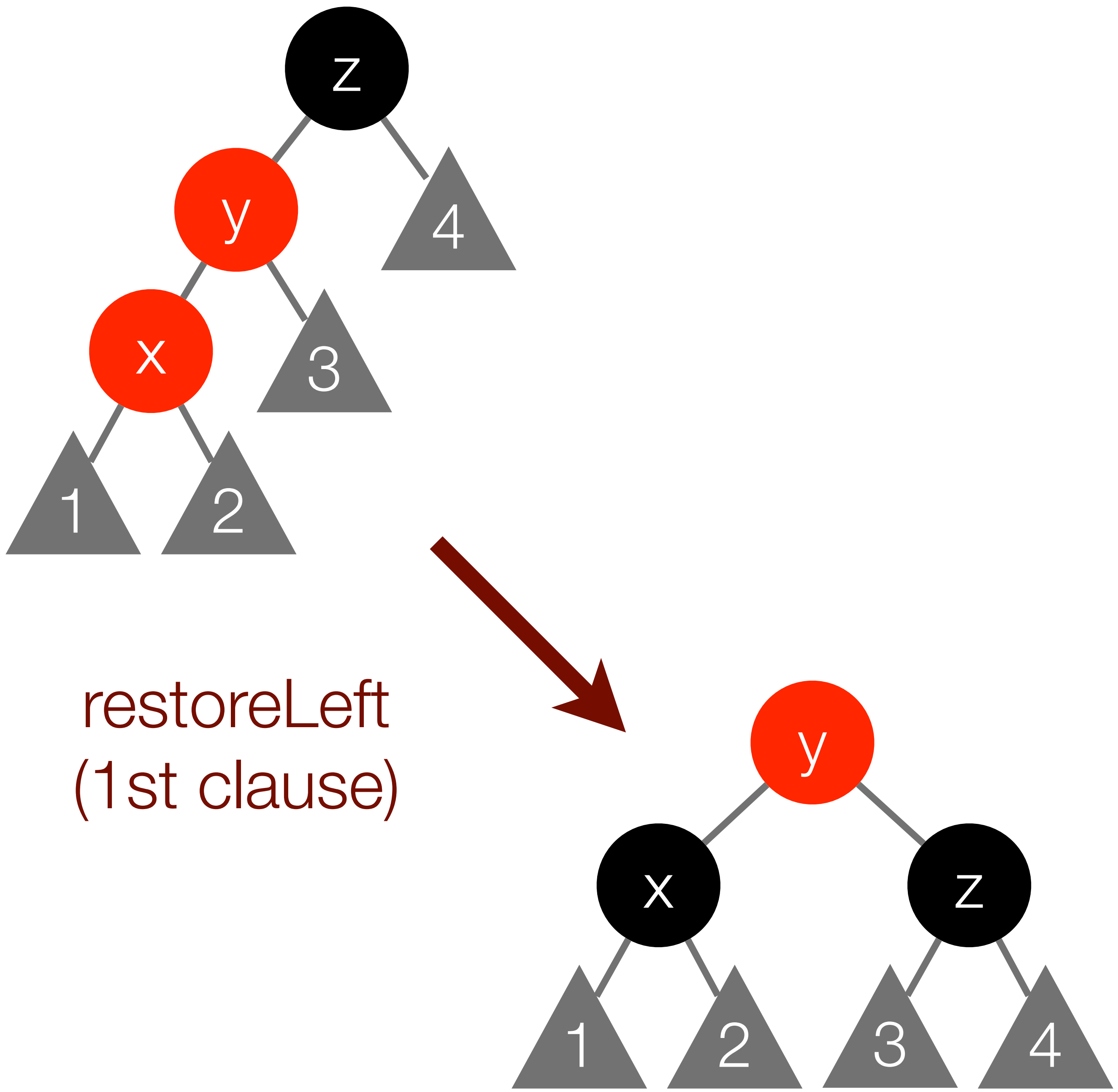


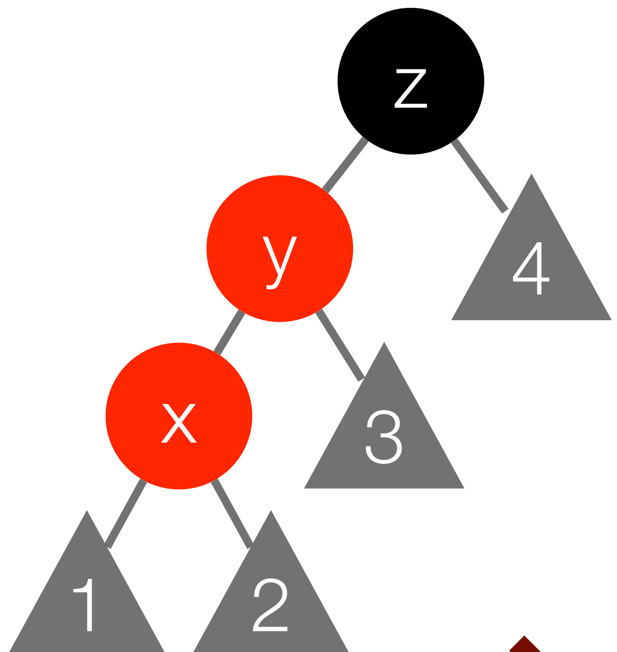
restoreLeft
(1st clause)

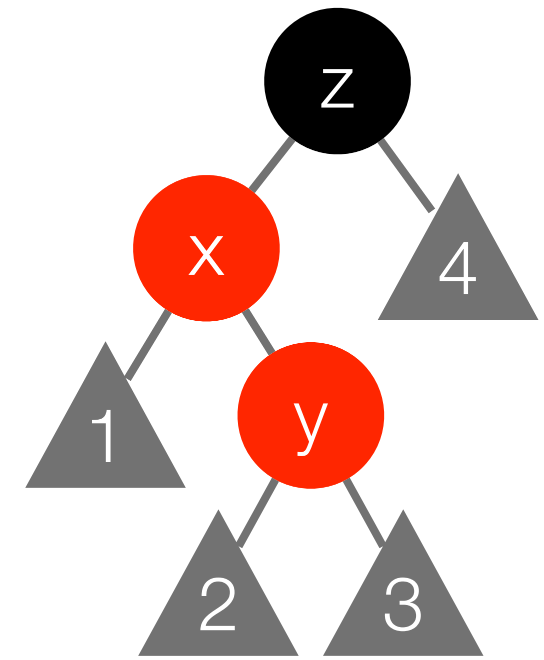
Two out of four possible rotations/re-coloring



restoreLeft
(1st clause)

Two out of four possible rotations/re-coloring





restoreLeft
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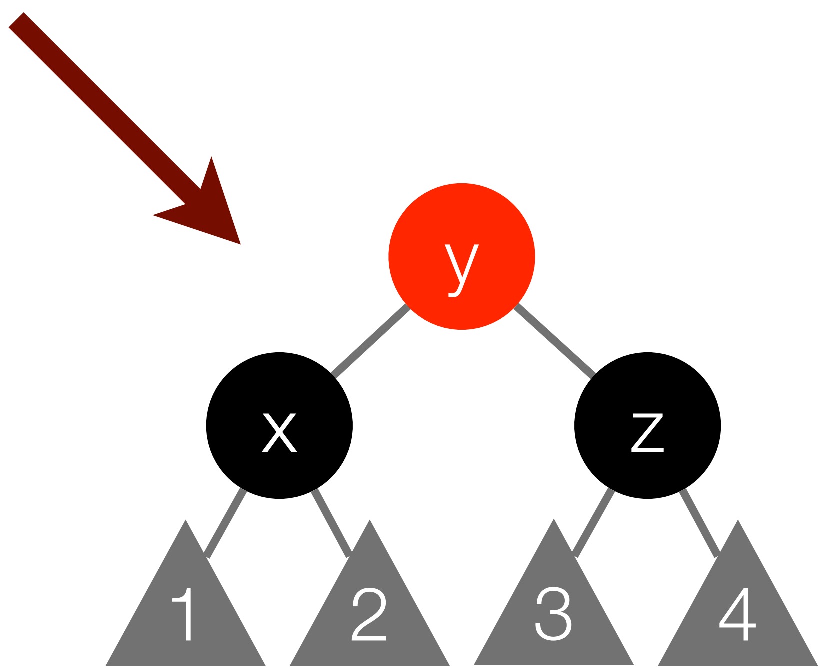
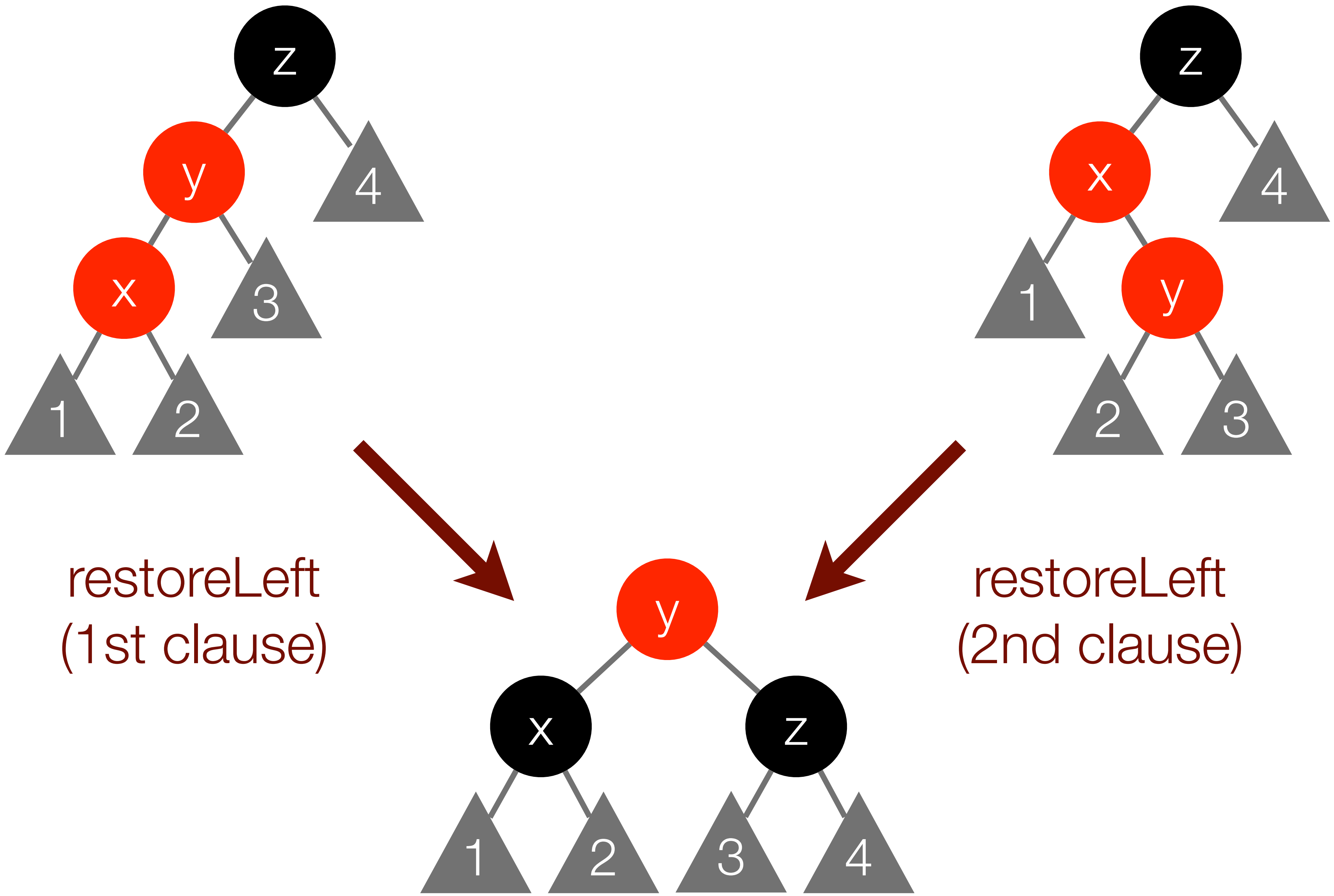


Two out of four possible rotations/re-coloring



The other two possible rotations/re-coloring

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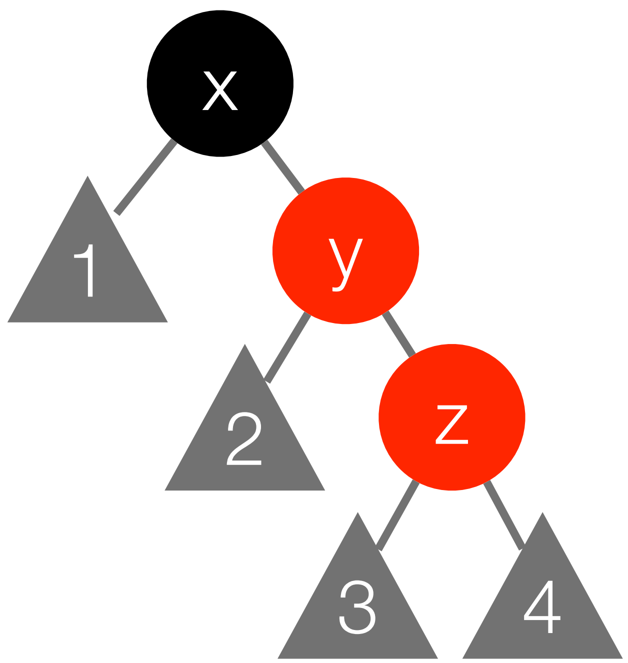
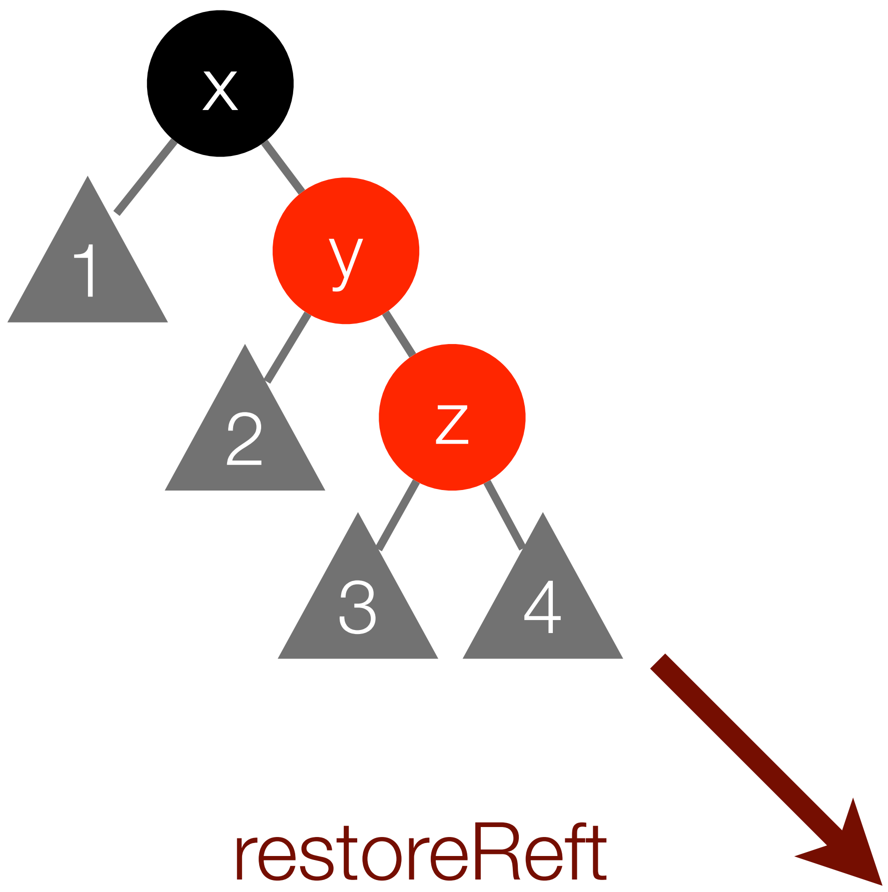
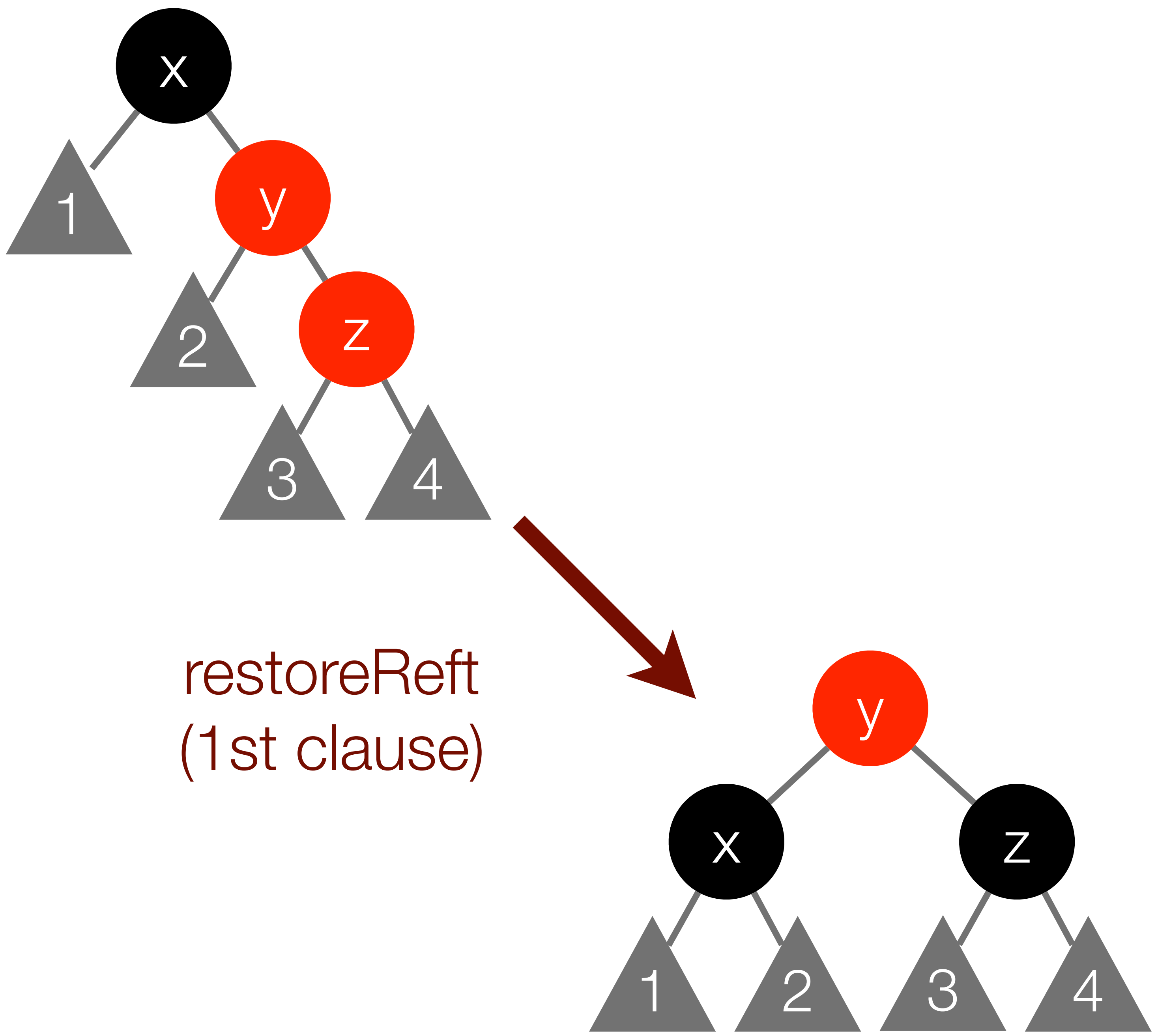


The other two possible rotations/re-coloring



restoreReft
(1st clause)

The other two possible rotations/re-coloring



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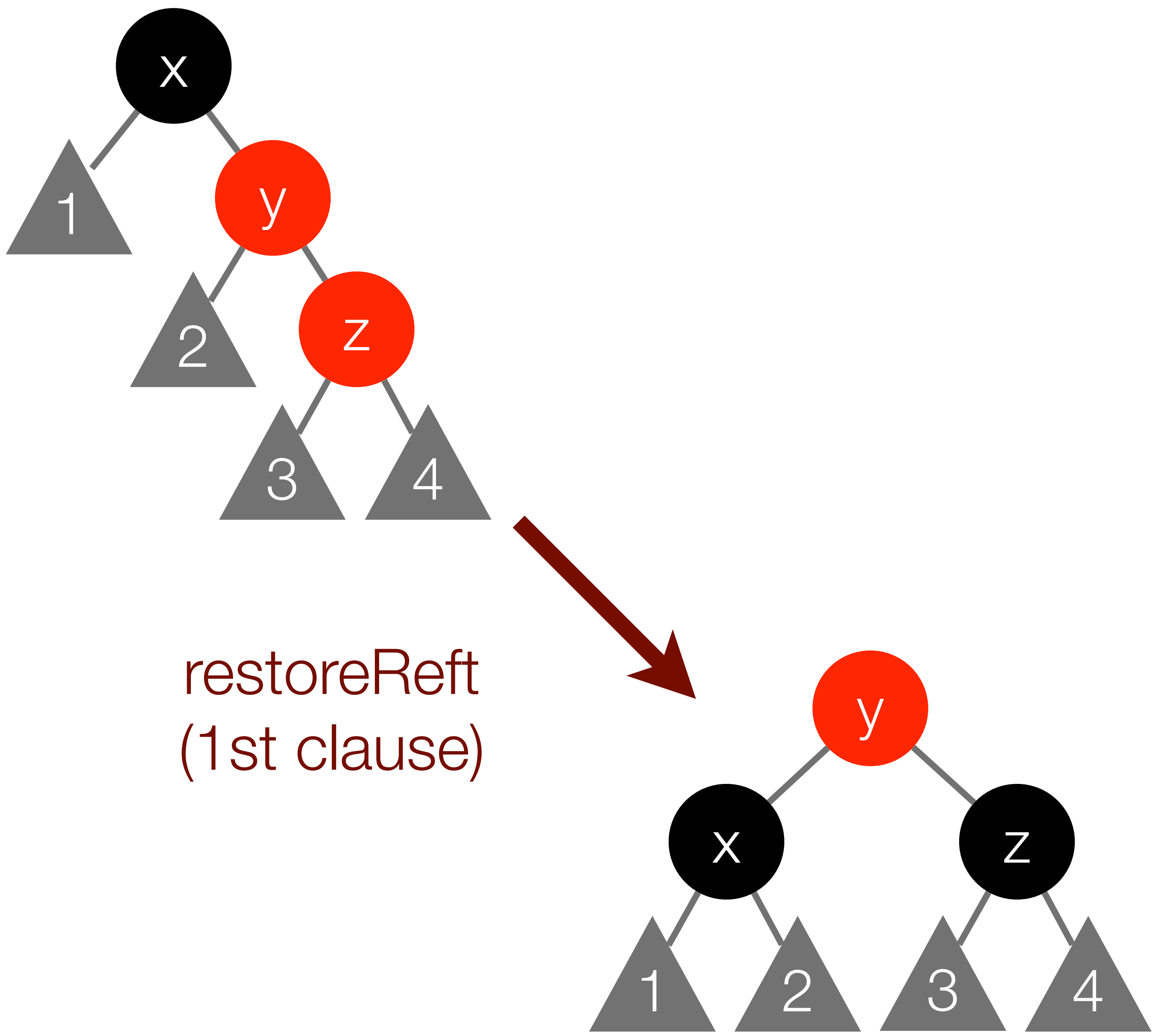

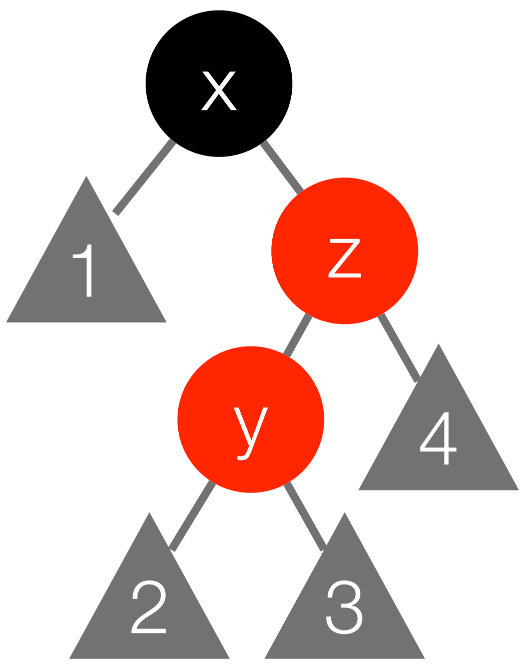

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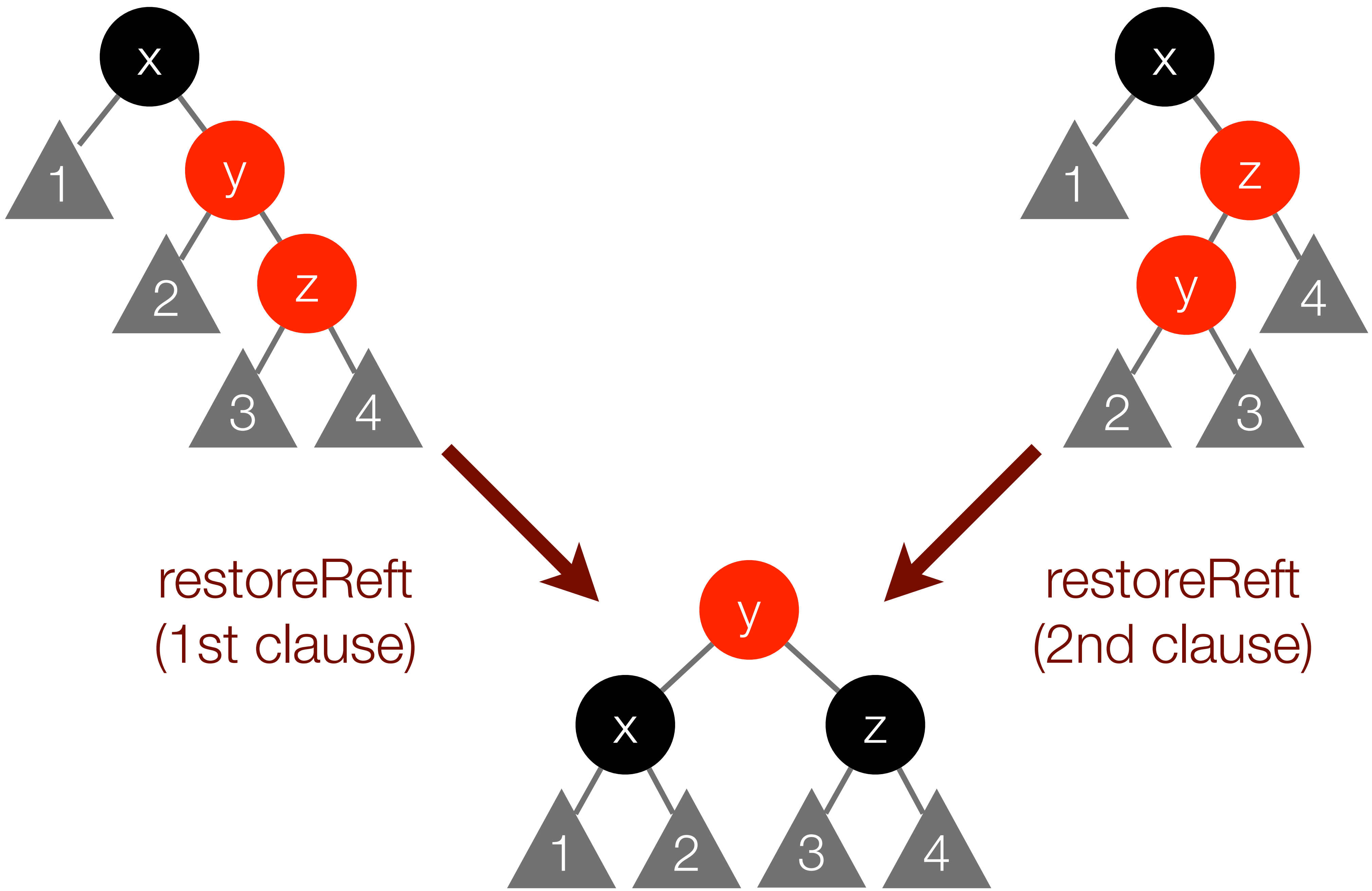

Let's look at another example

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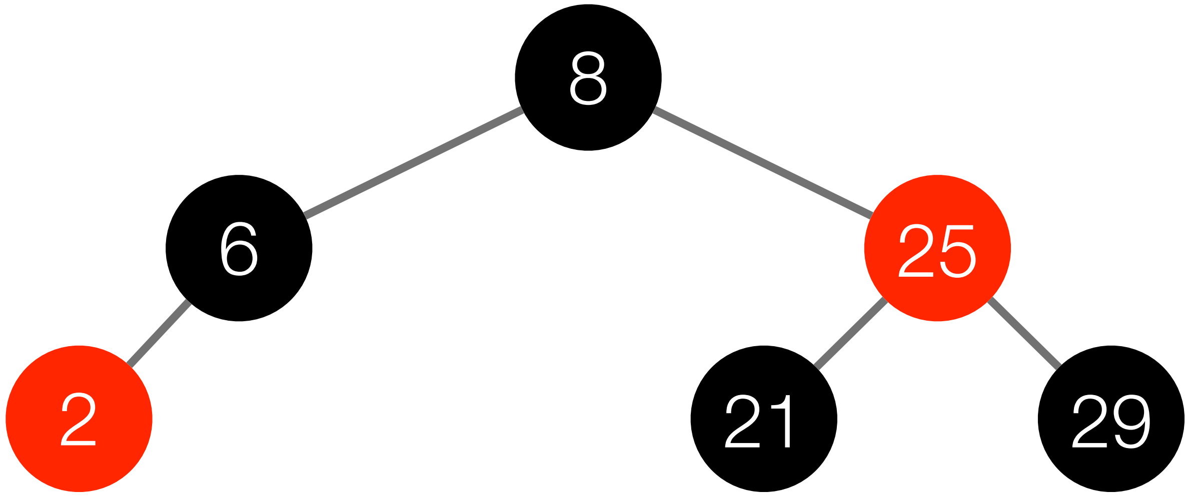
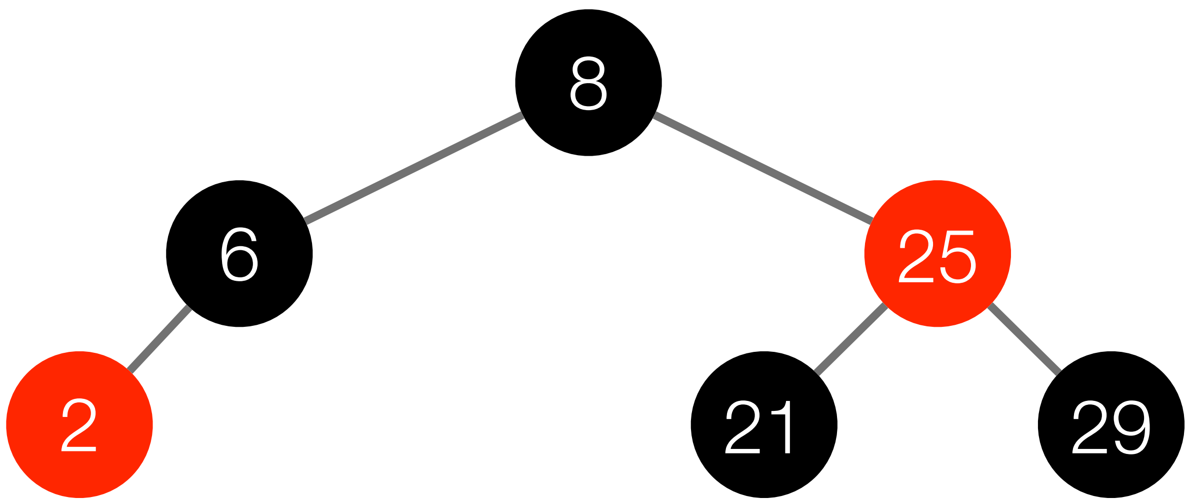


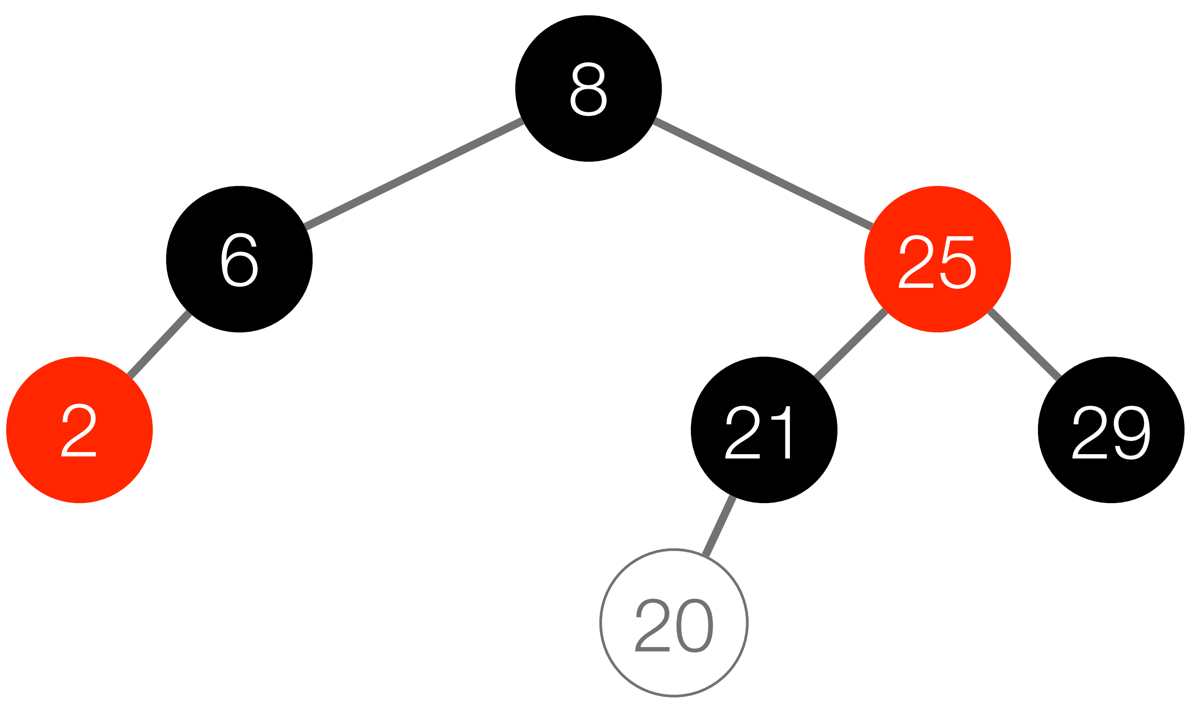
Let's look at another example

Let's insert 20:



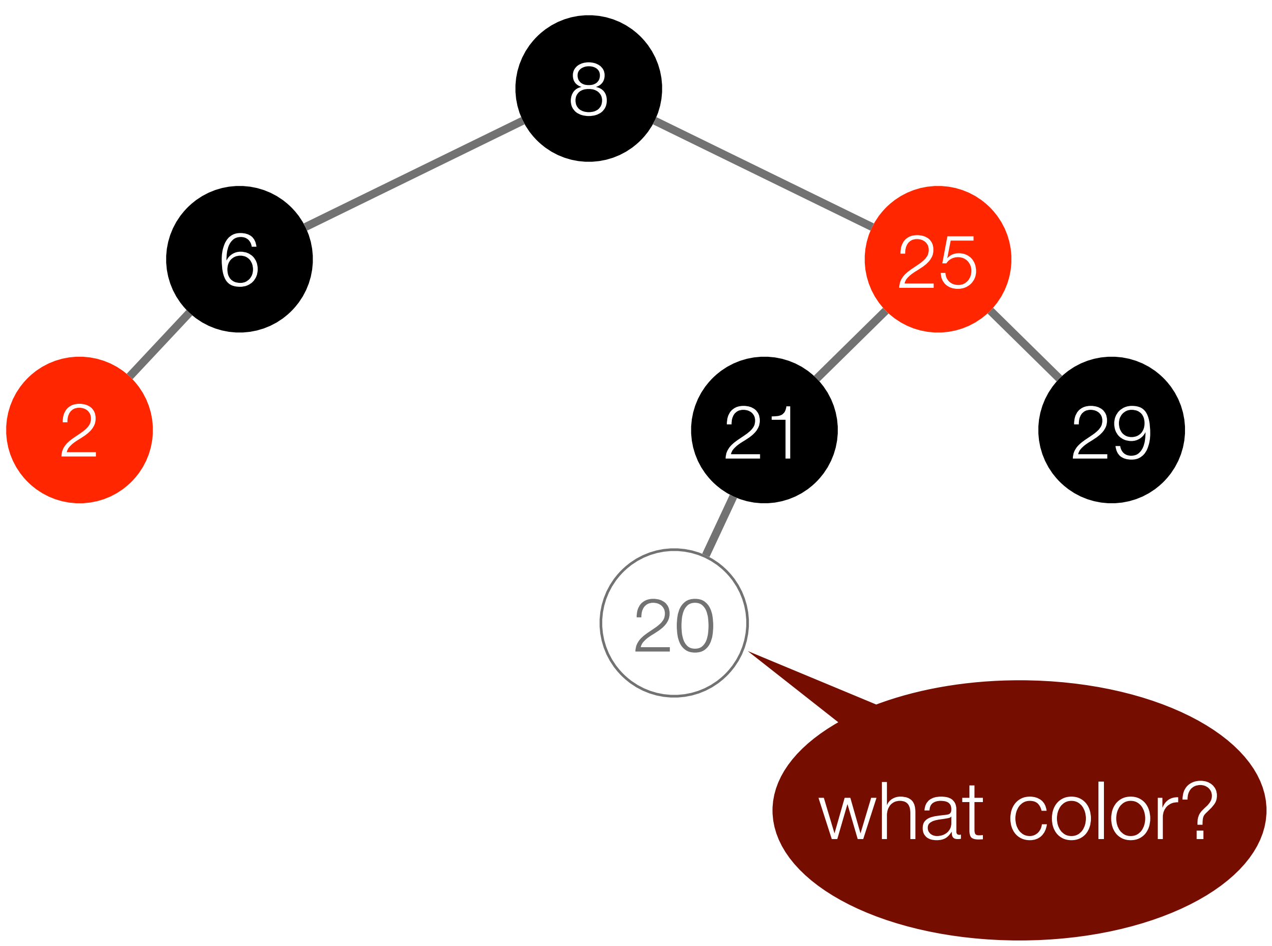
Let's look at another example

Let's insert 20:



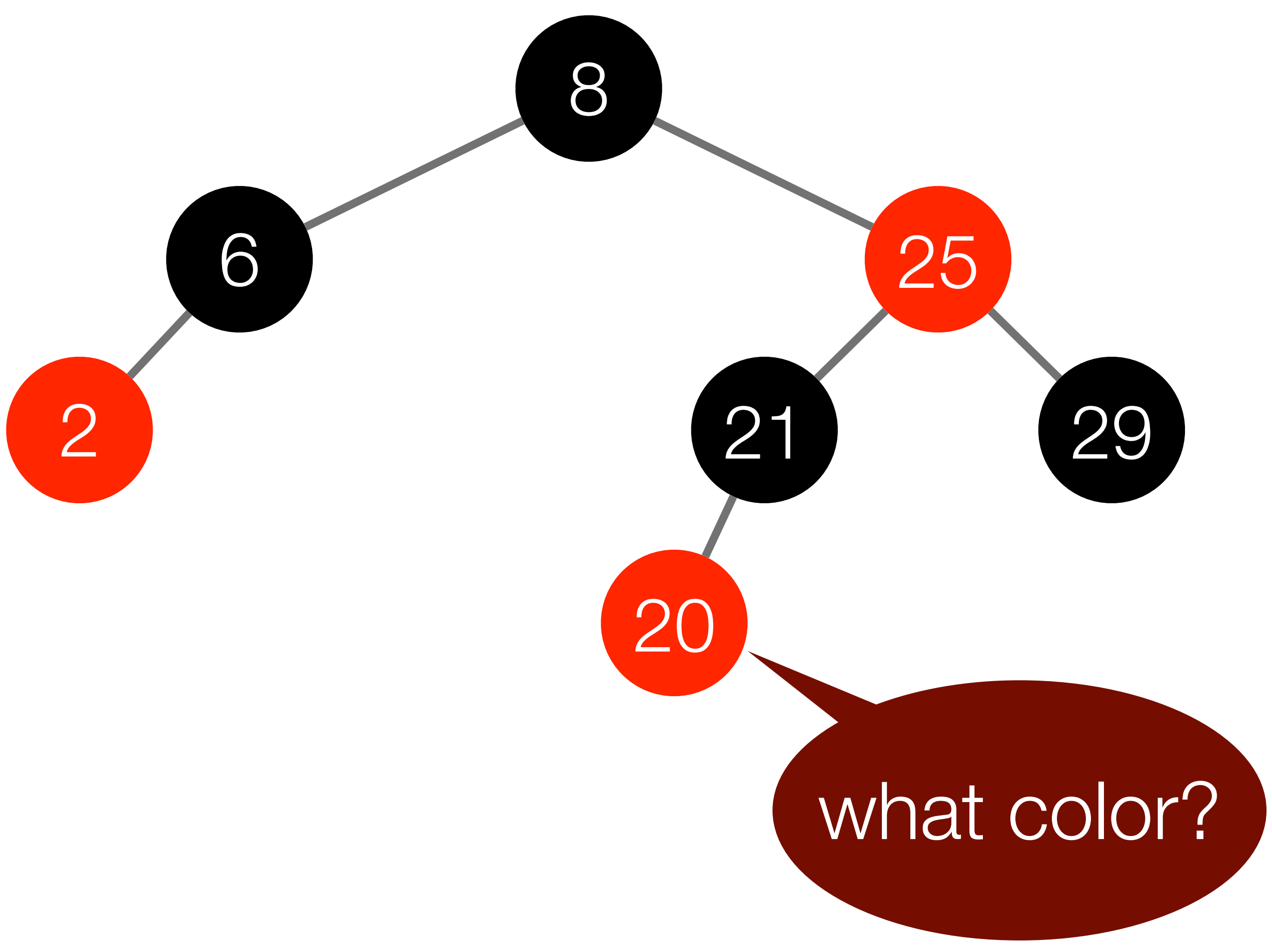
Let's look at another example

Let's insert 20:



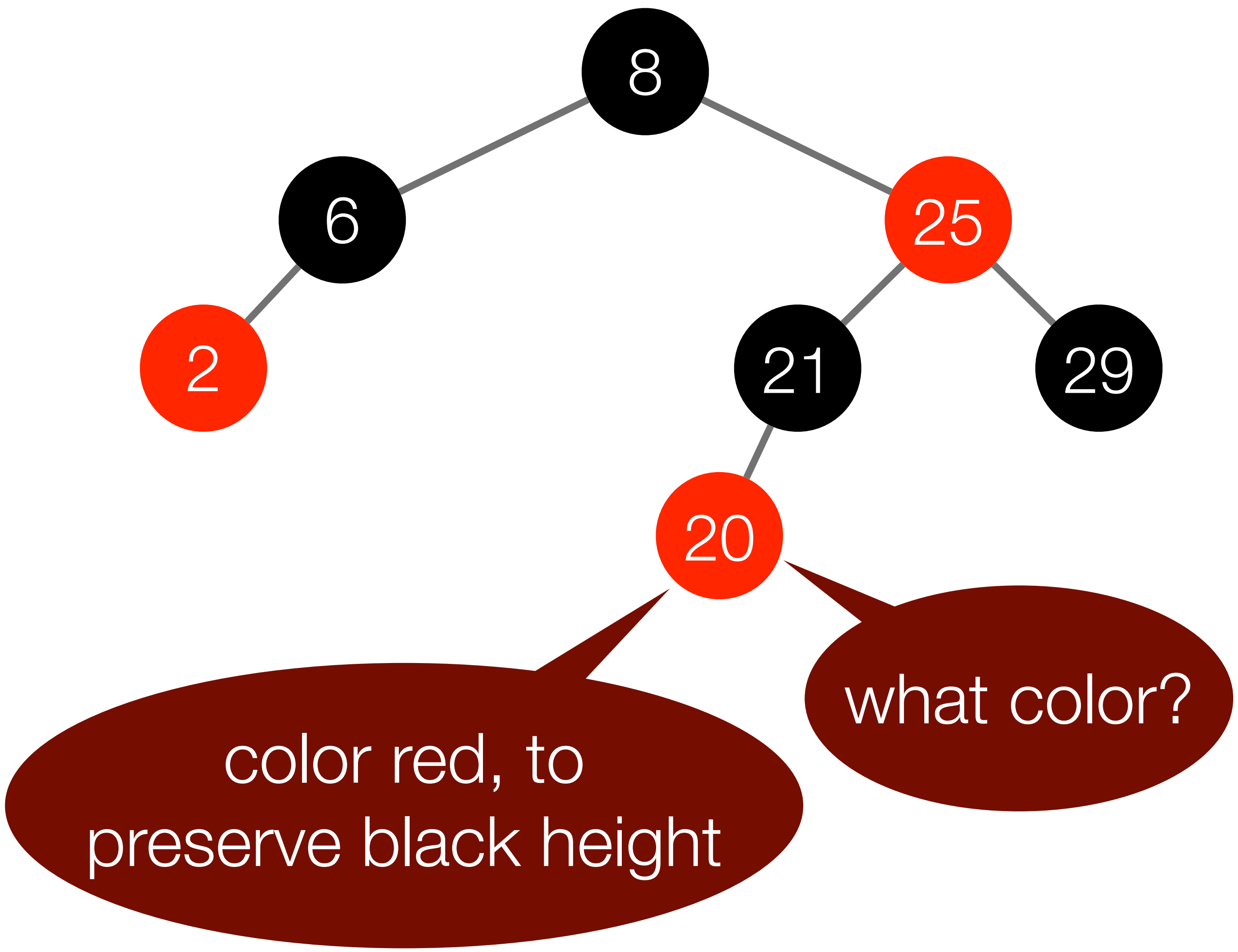
Let's look at another example

Let's insert 20:



Let's look at another example

Let's insert 20:



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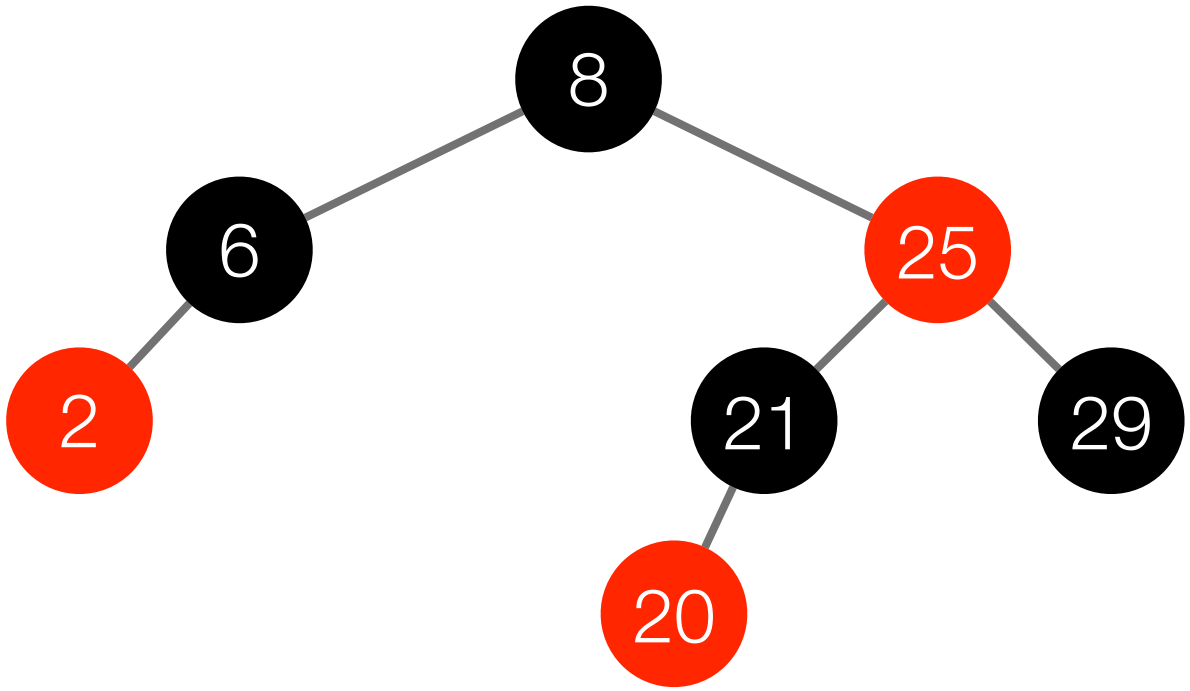
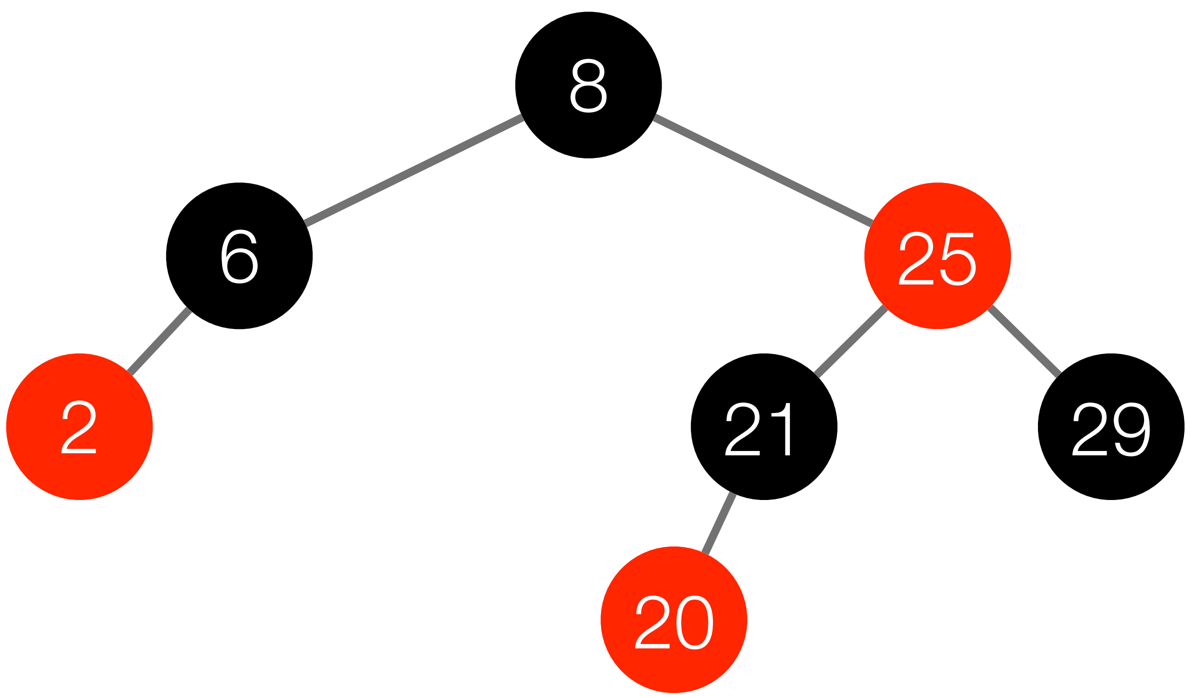


Let's look at another example

Now, let's insert 19:



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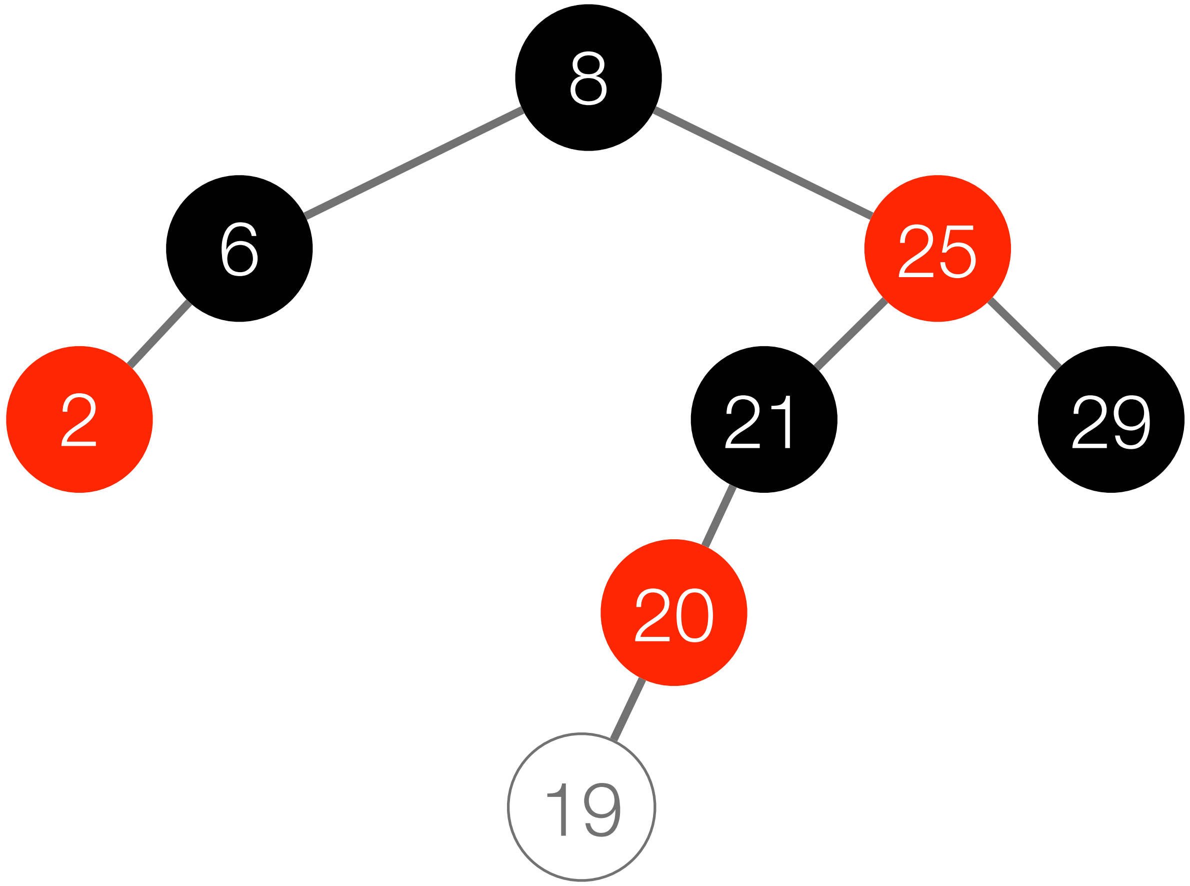

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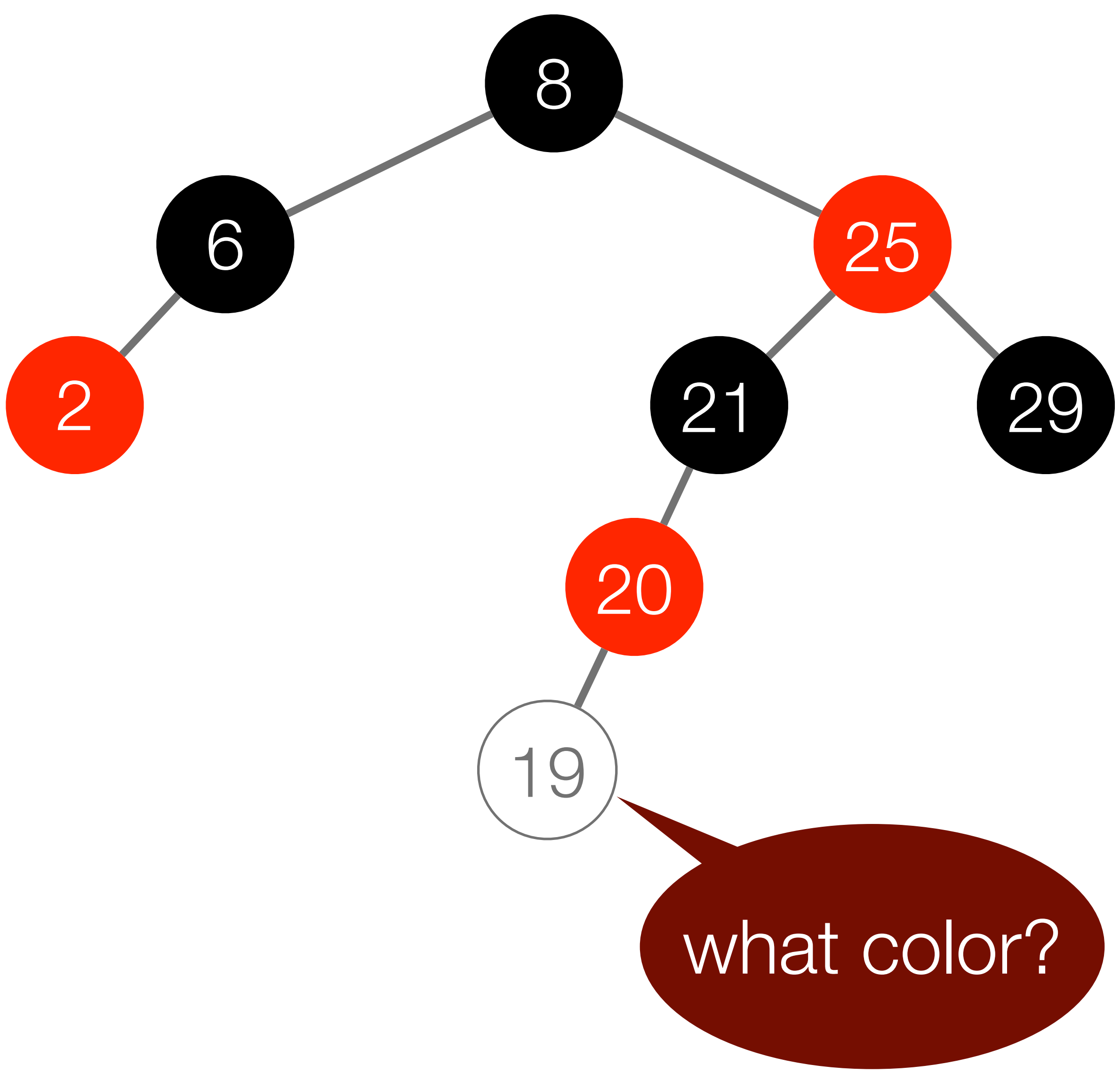

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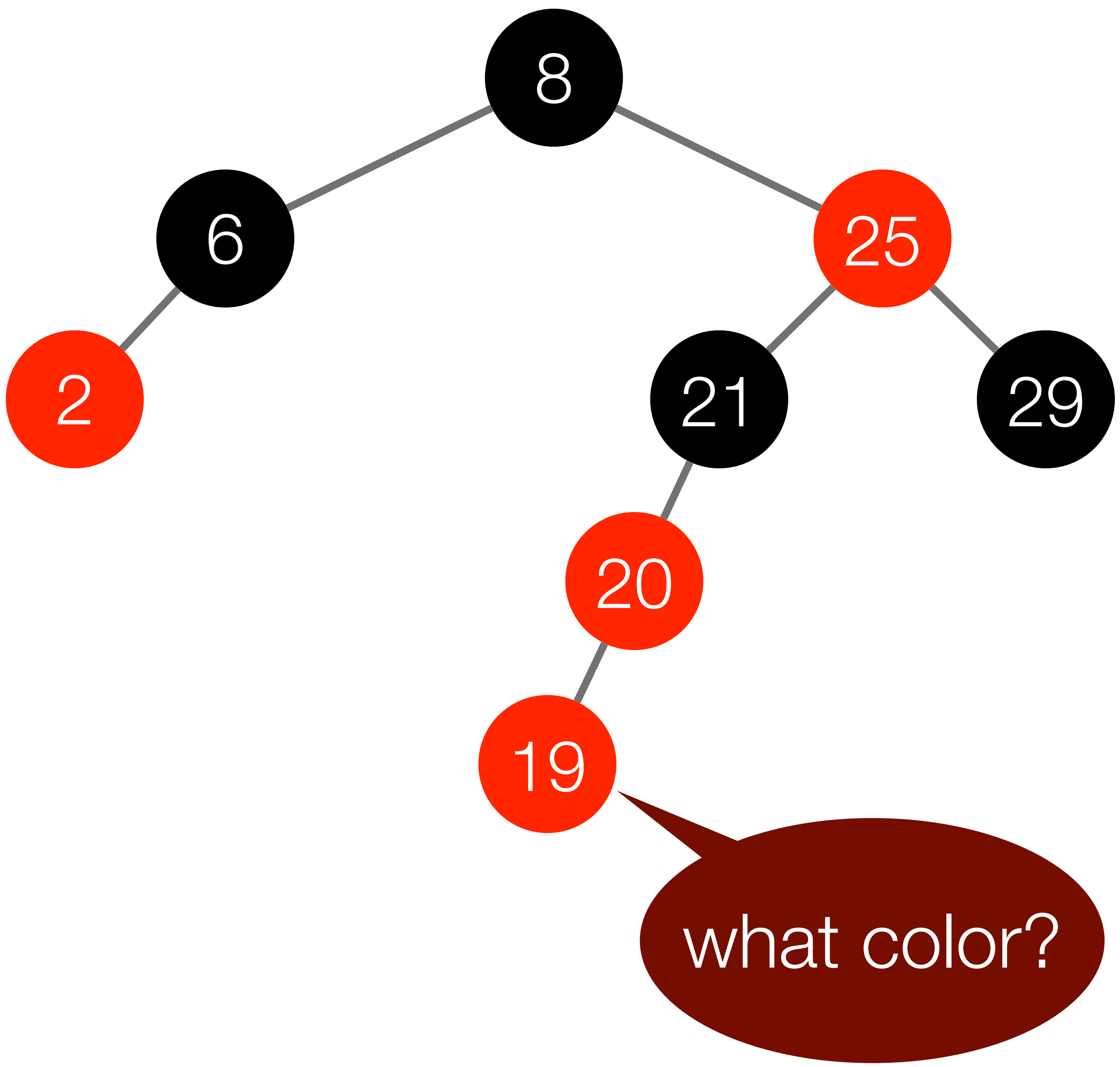

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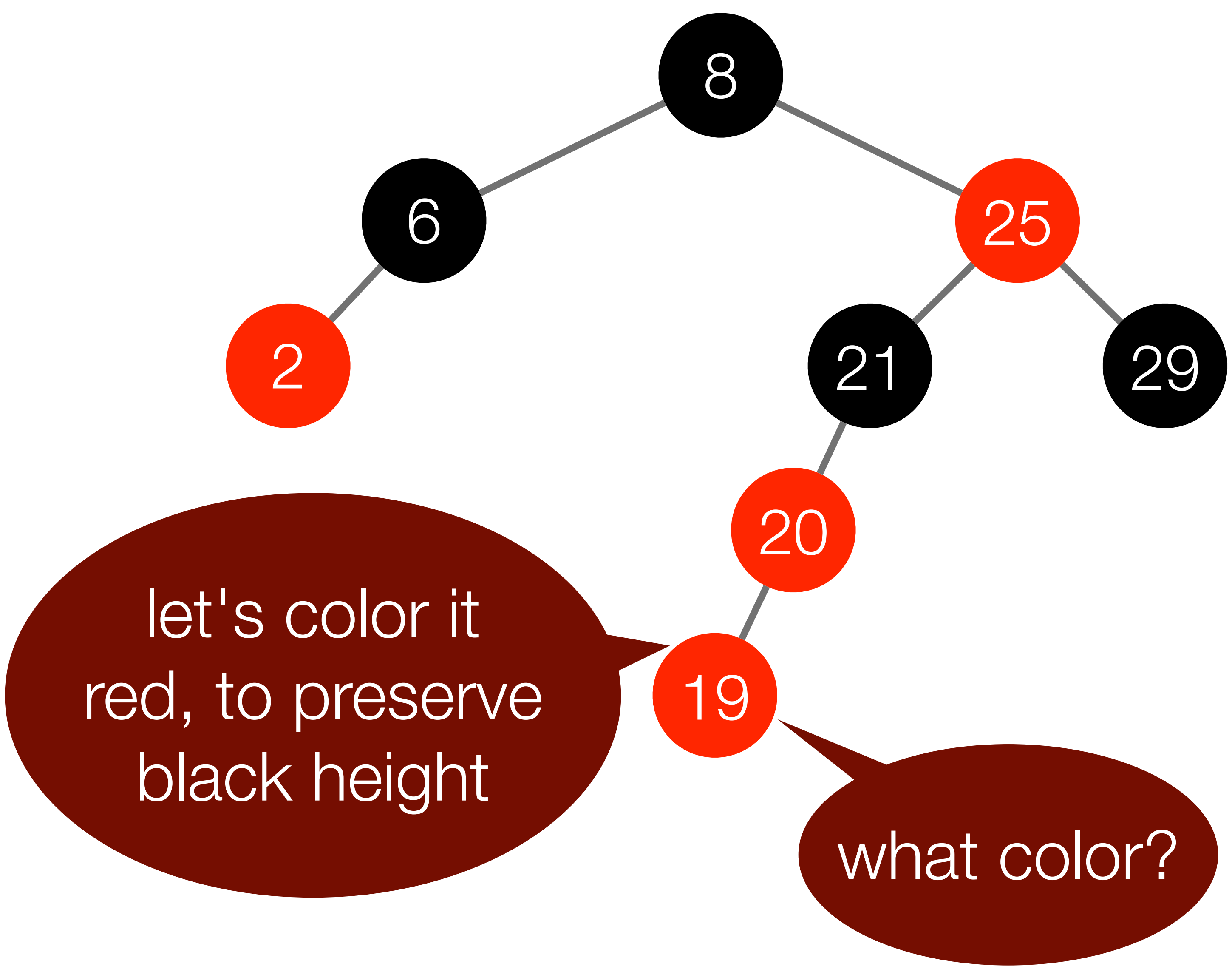

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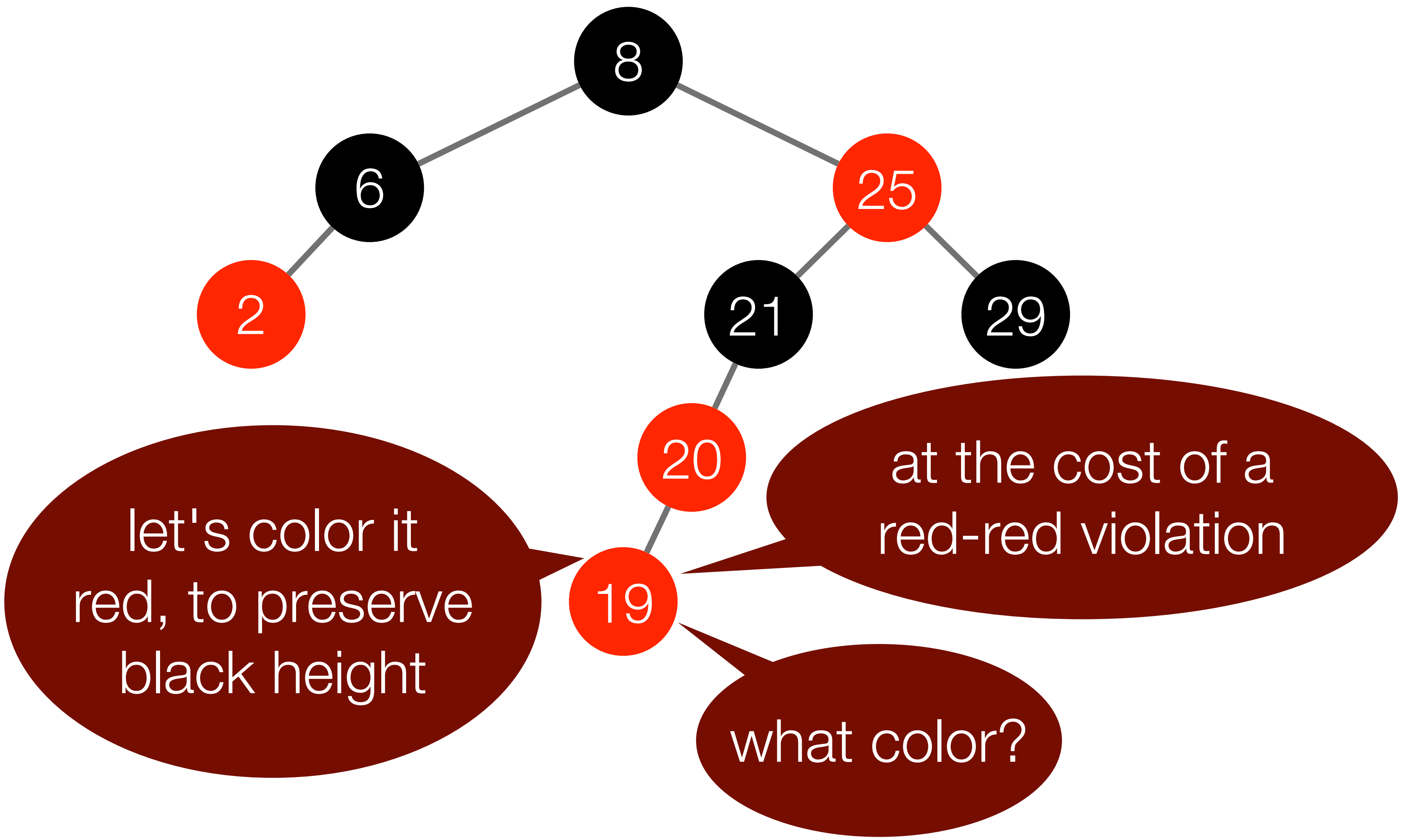

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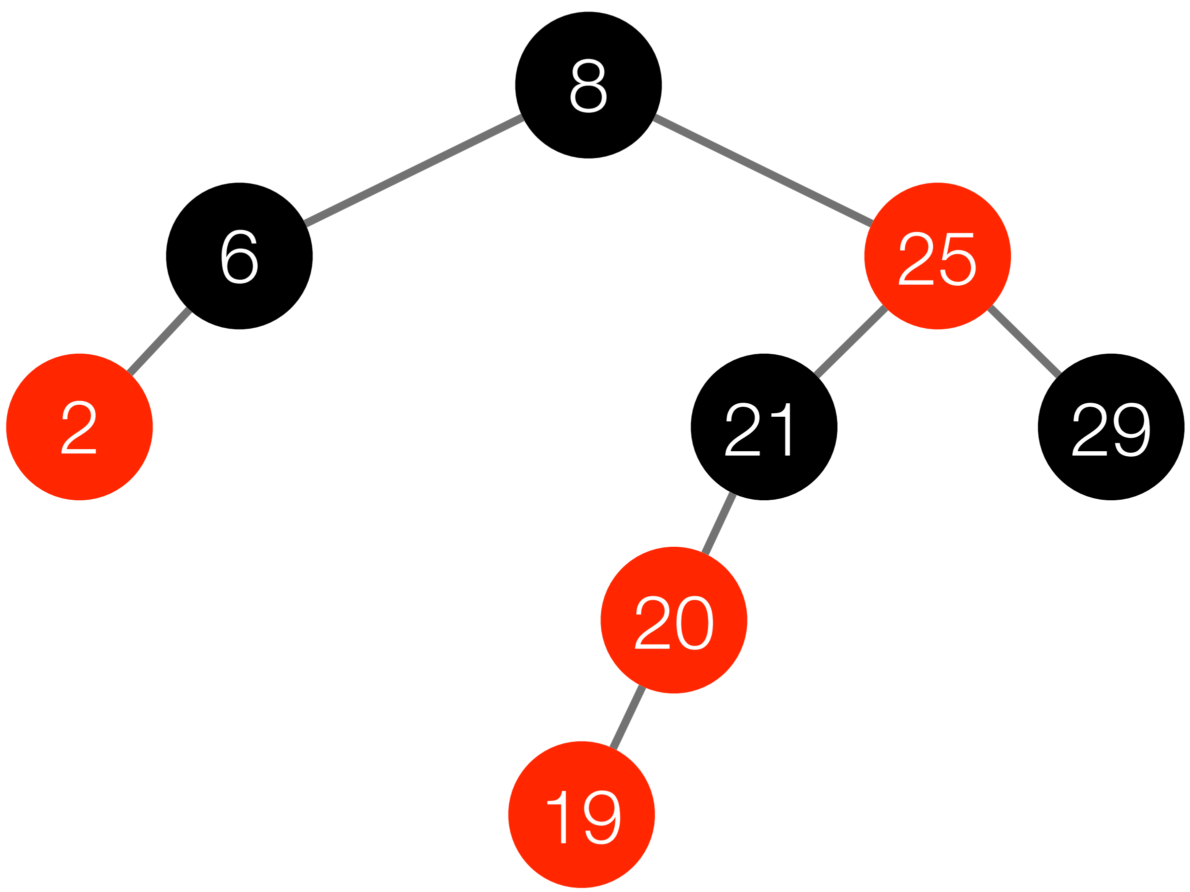

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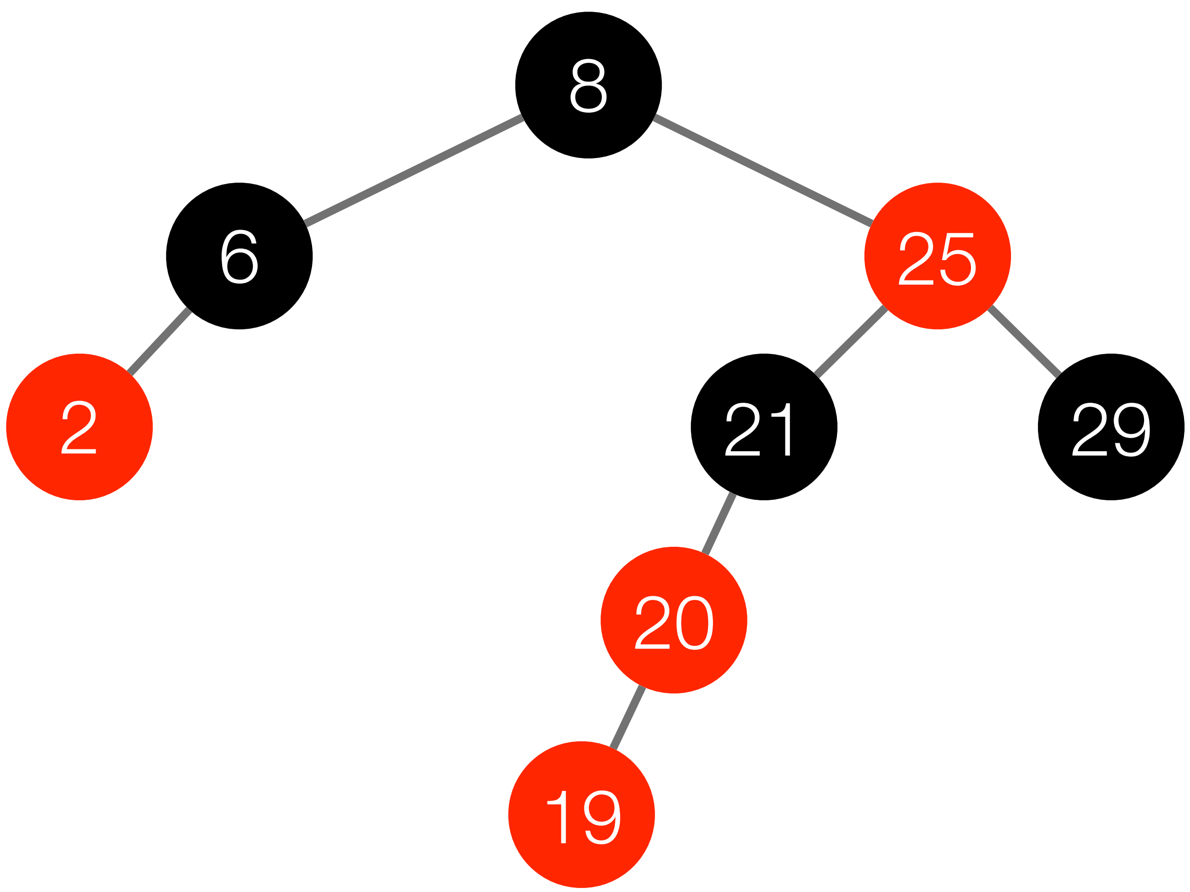

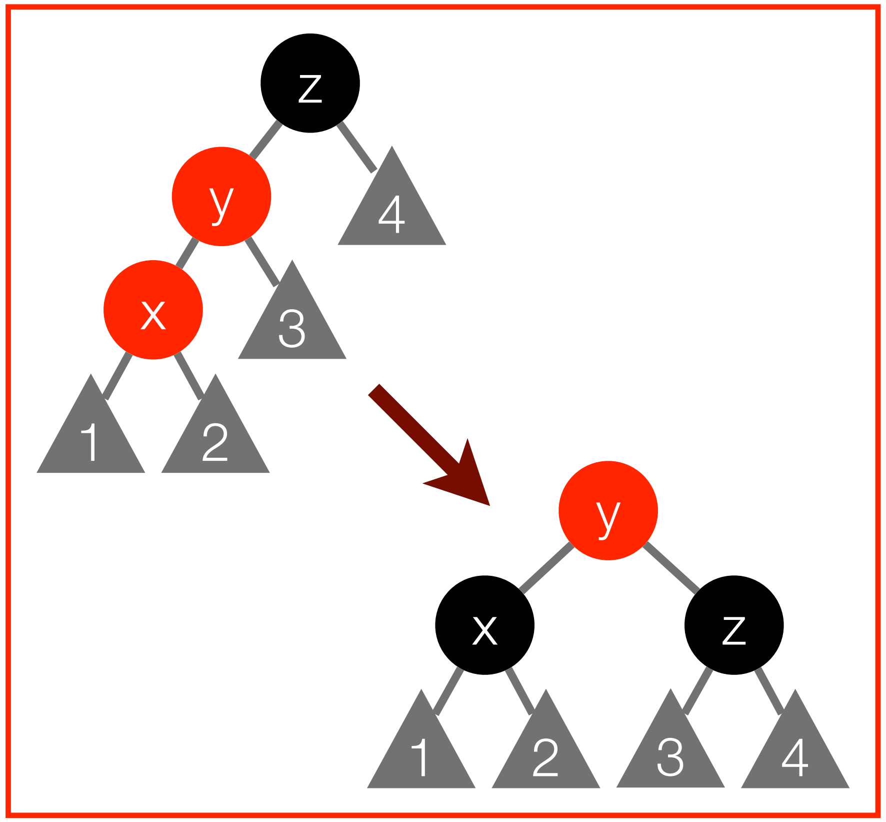

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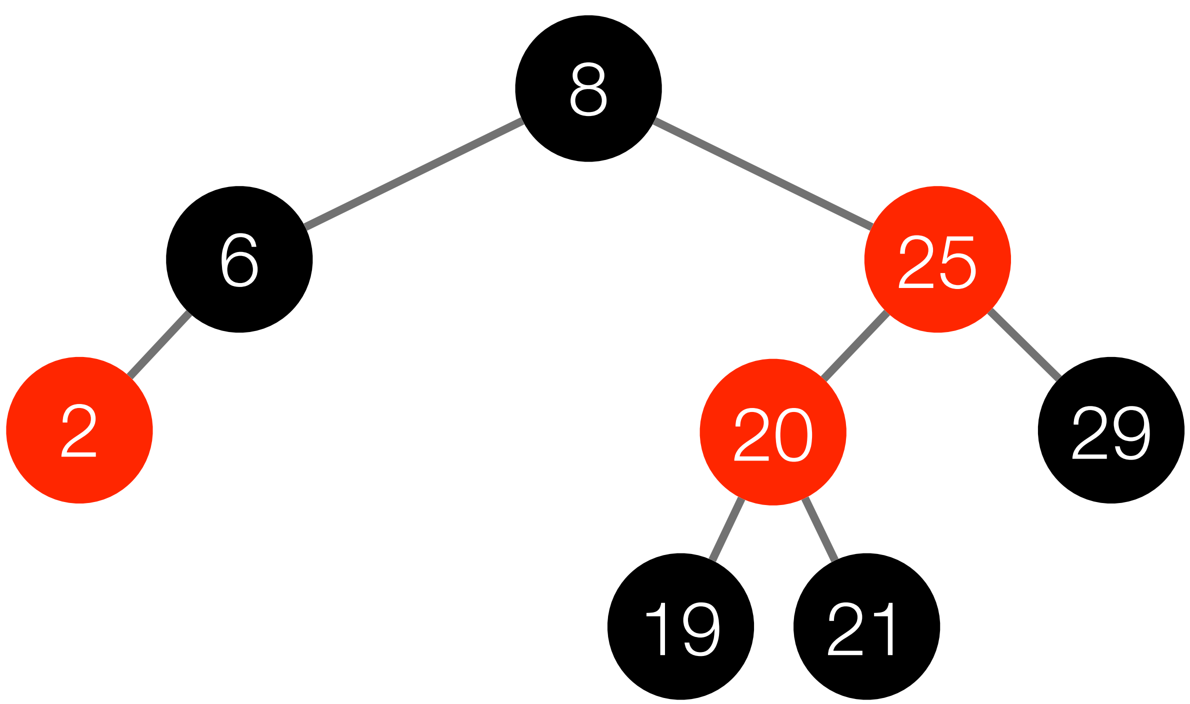

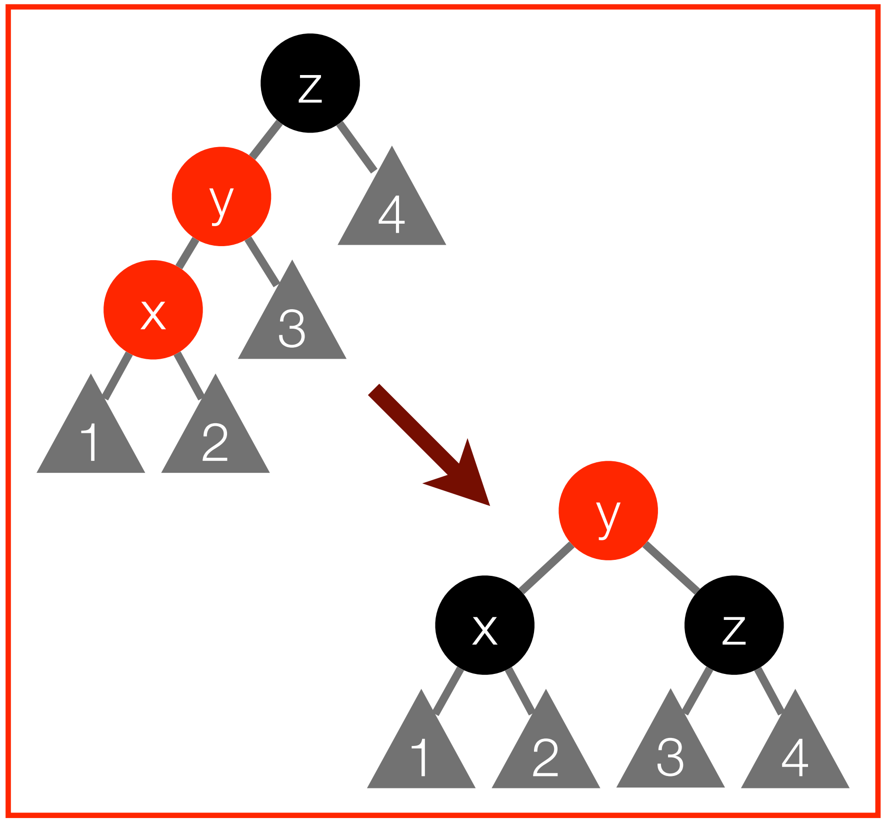

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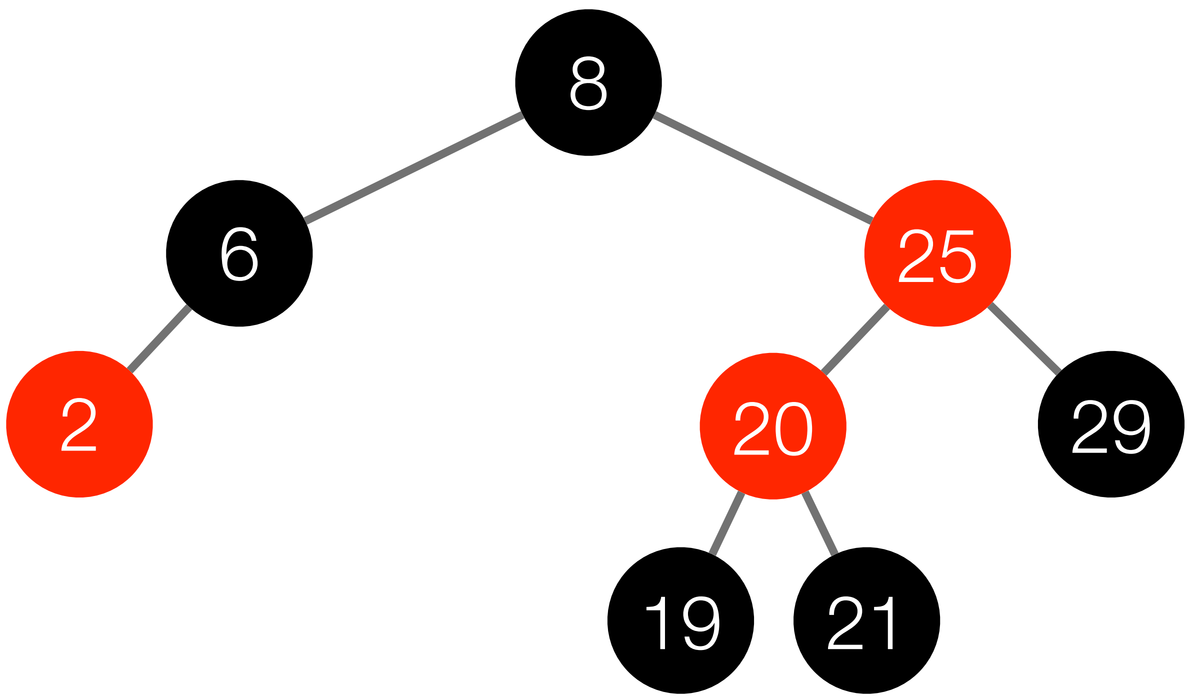

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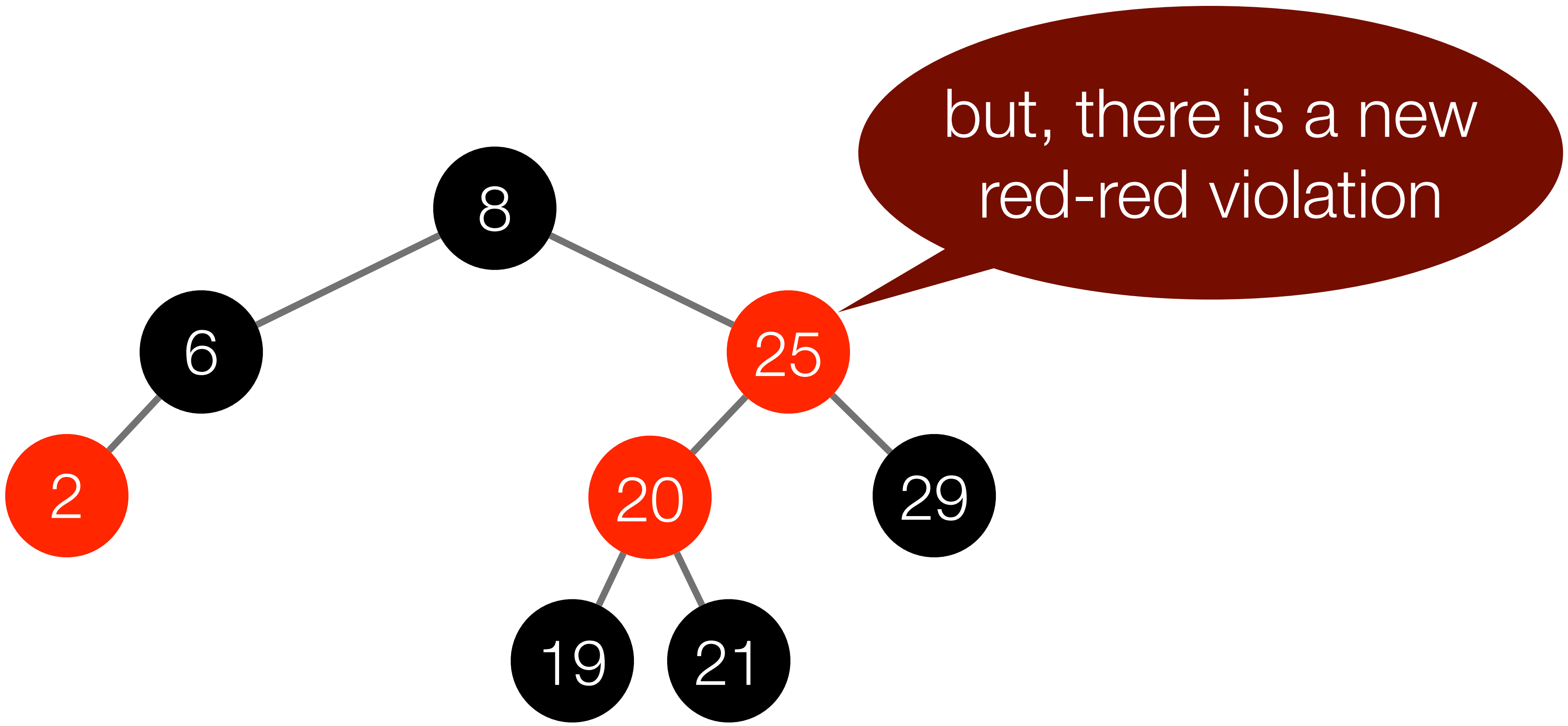

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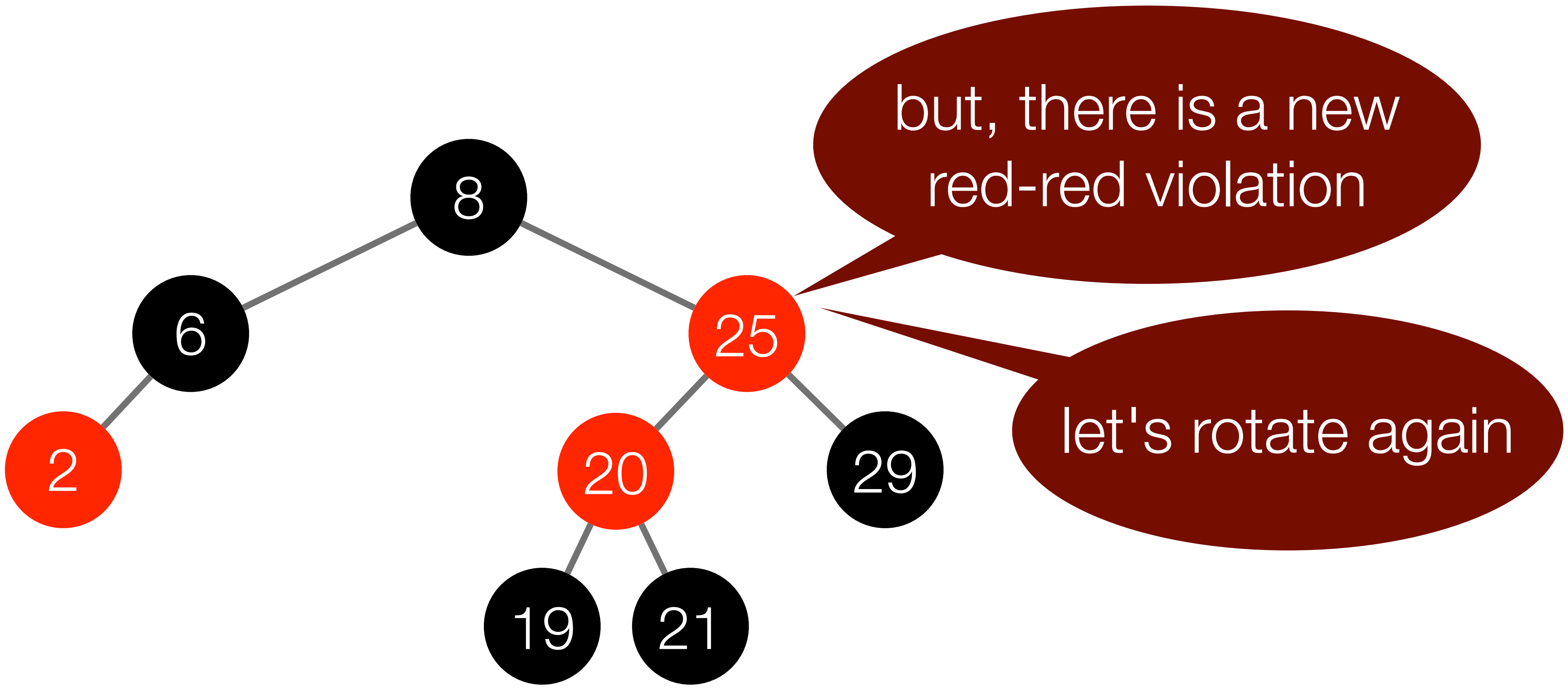

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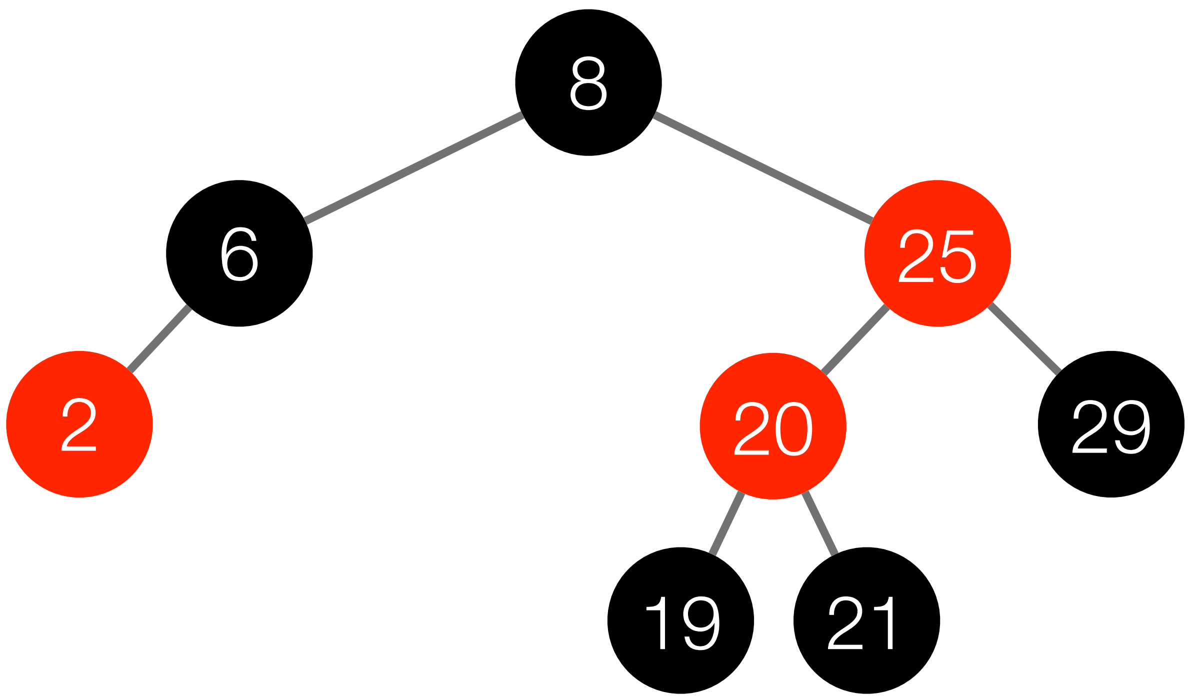

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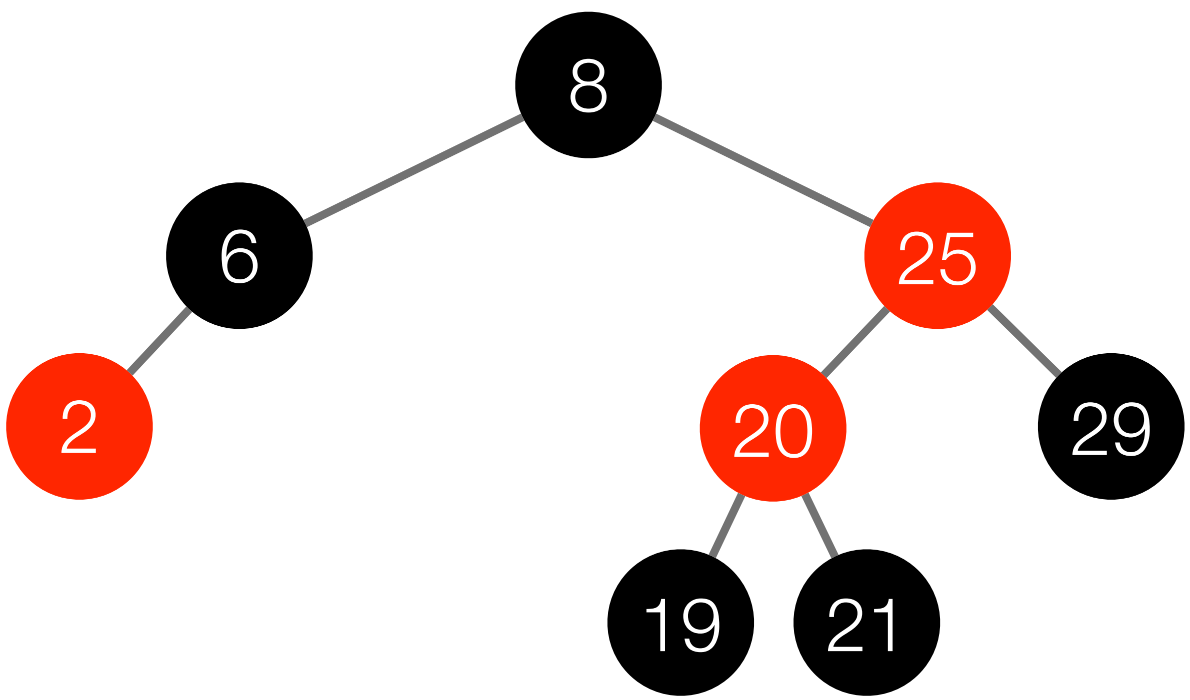

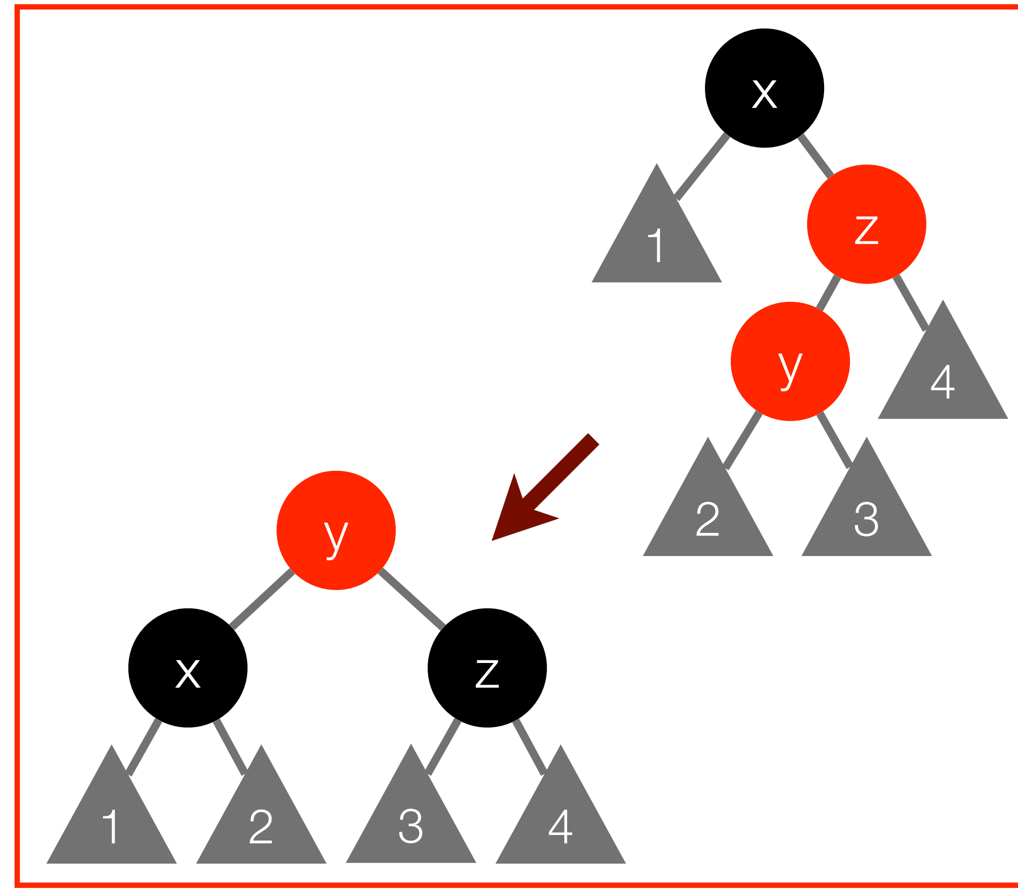

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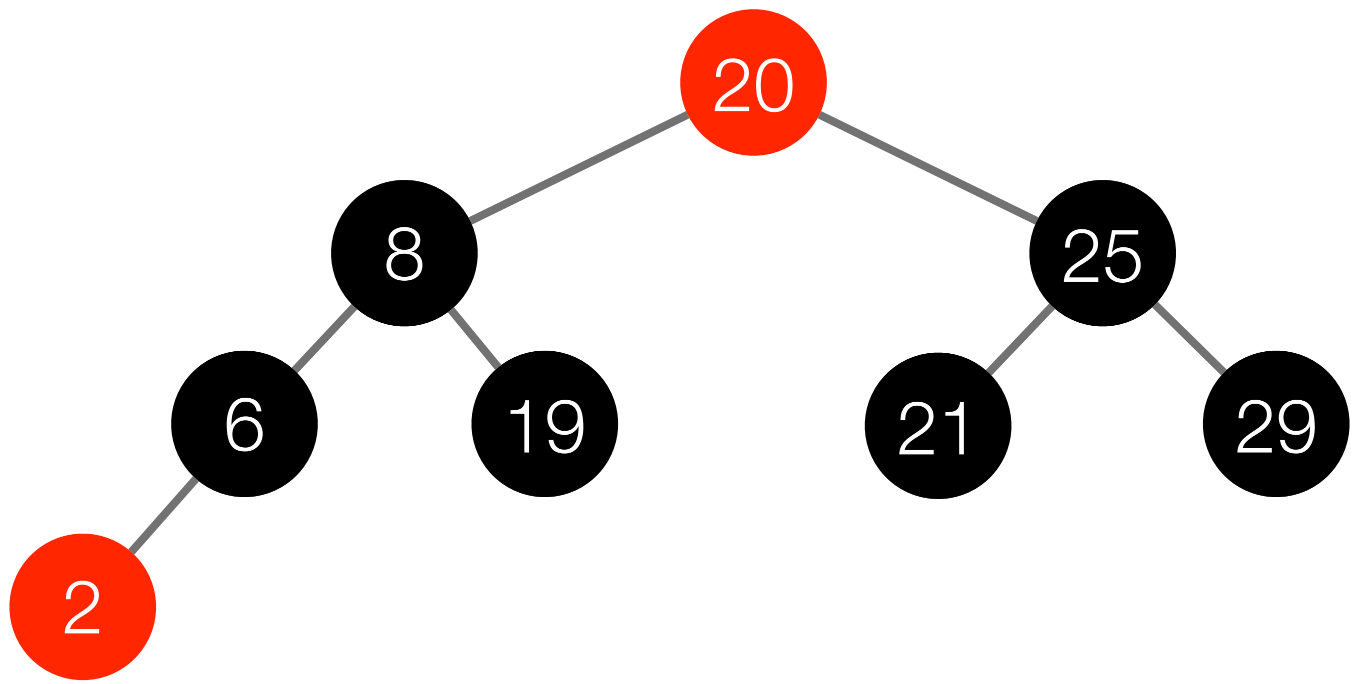

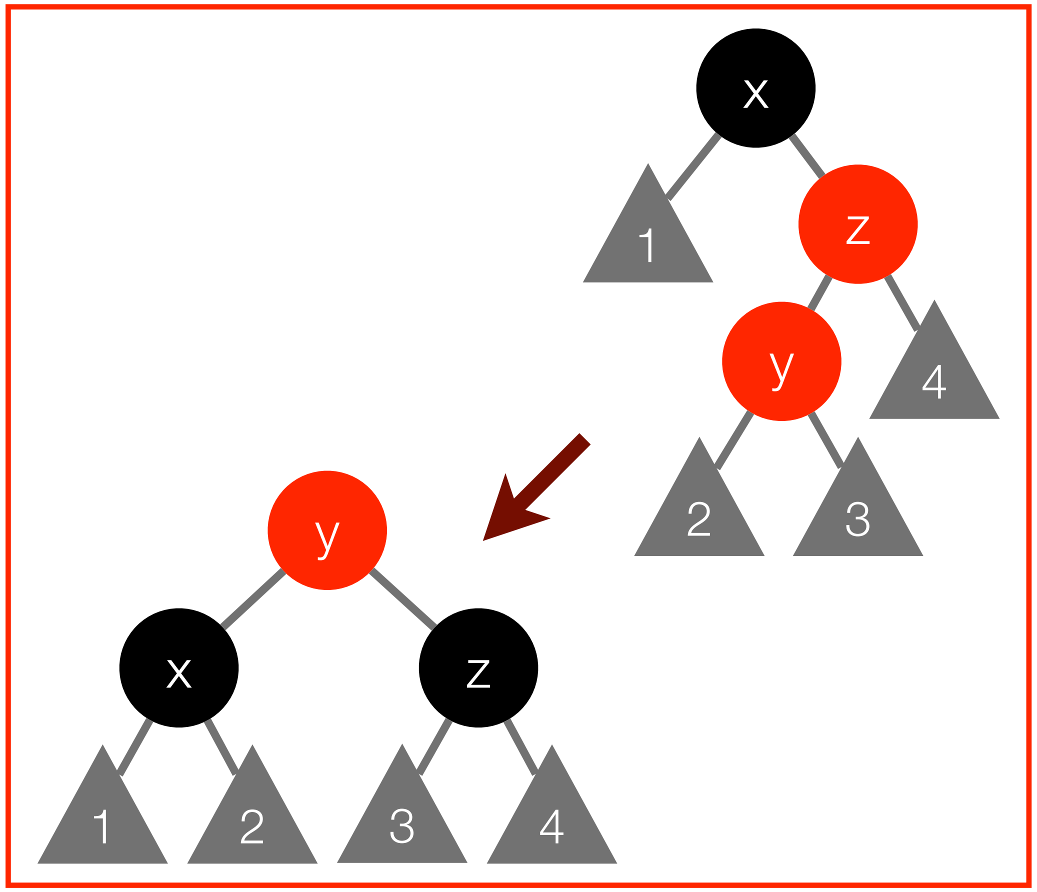

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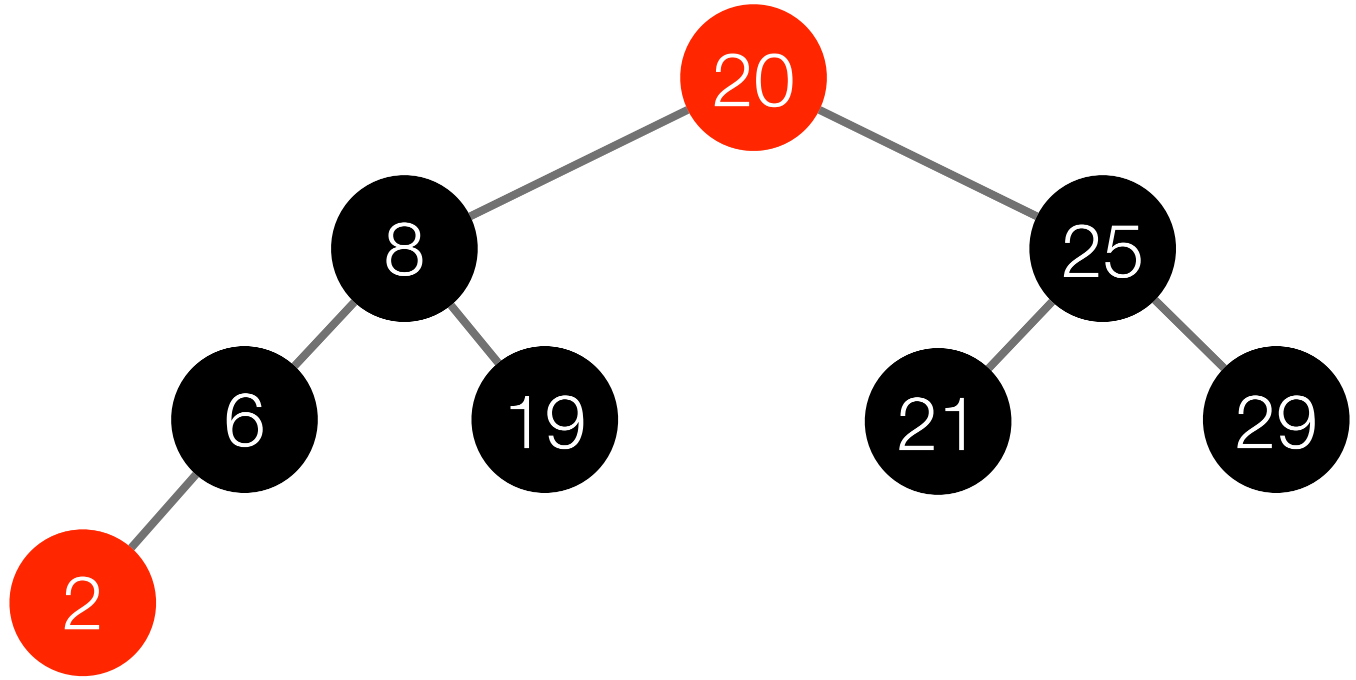
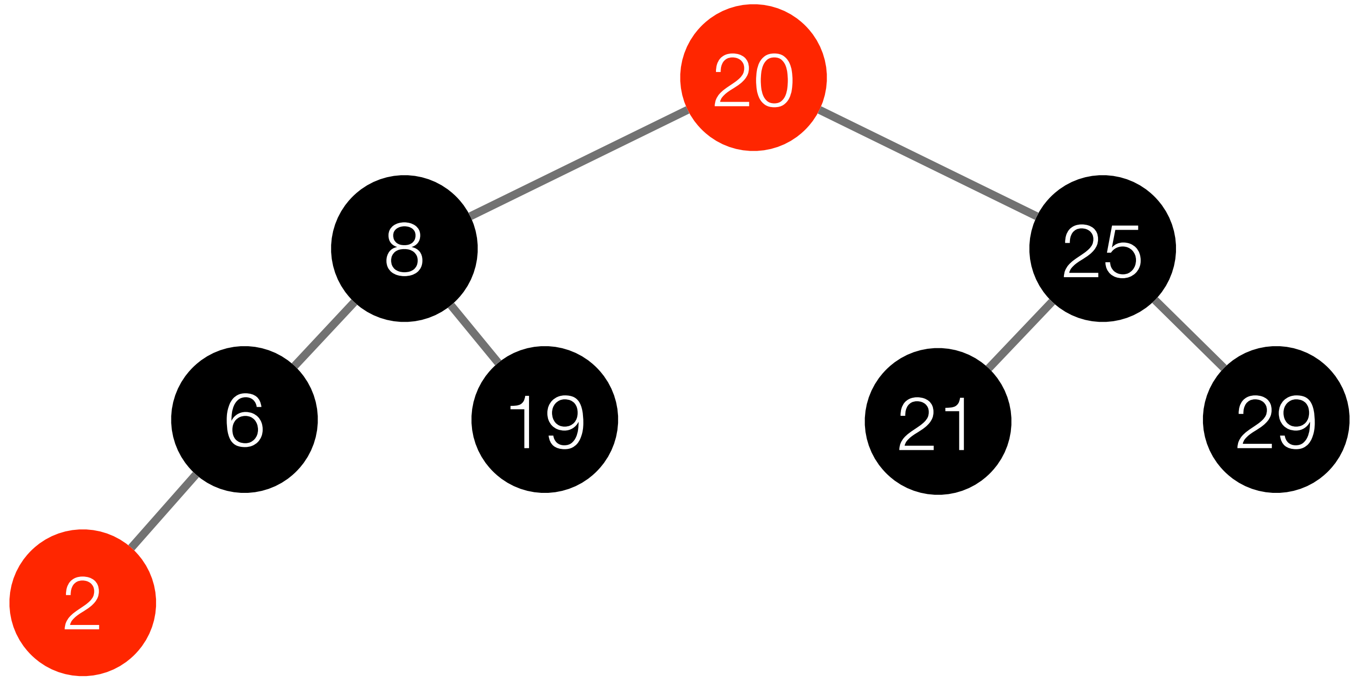


Let's look at another example

If we wanted, we could safely re-color the root:



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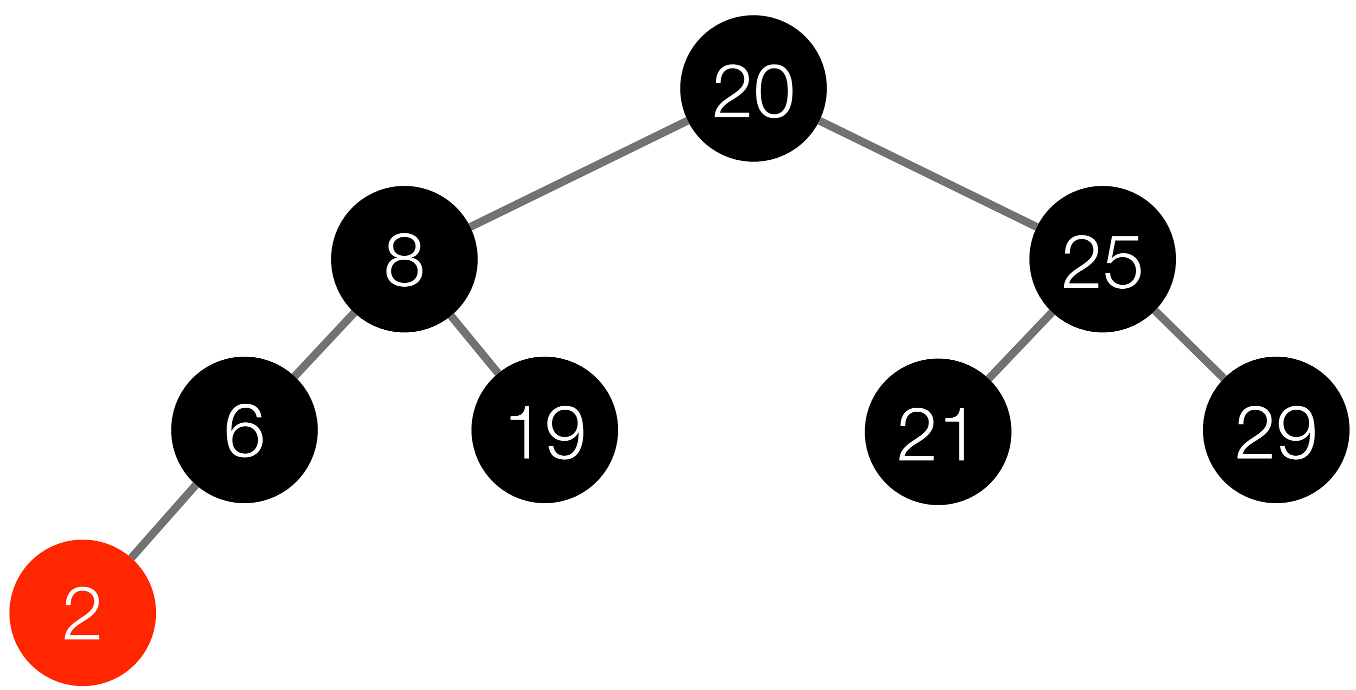

Now, let's implement our dictionary!

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```
datatype 'a dict =
  Empty
| Red of 'a dict * 'a entry * 'a dict
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A Tree is **sorted** according to an entry's key.

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A Tree is **sorted** according to an entry's key.

B A **red** node's children must be black.

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Red Black Tree (RBT) invariant:

A. Tree is **sorted** according to an entry's key.

B. A **red** node's children must be black.

C. **Black height**: for any node, the number of black nodes along any path from the node to a leaf (empty) is the same.

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→ RBT invariant will become **representation invariant** of structure.

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Recall, representation invariants are hidden consistency conditions, s.t.

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→ RBT invariant will become **representation invariant** of structure.

Recall, representation invariants are hidden consistency conditions, s.t.

→ All functions declared by structure
  → may **assume** representation invariant for input,
  → and must **assert** representation invariant for output.

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Our implementation will even make use of a weaker invariant, which can be locally and temporarily violated, but is restored in the end.

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Now, let's implement our dictionary!

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Now, let's implement our dictionary!

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Our implementation will even make use of a weaker invariant, which can be locally and temporarily violated, but is restored in the end.

Almost RBT (ARBT) invariant:

A and C as above,

B' A **red** node's children must be black, unless for a **red root** node, who may have **one** red child.

Specification for restoreLeft

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```
(*
  restoreLeft : 'a dict -> 'a dict

  REQUIRES: Either d is a RBT
            or d's root is black,
            its left child is an ARBT,
            and its right child a RBT.

  ENSURES:  restoreLeft(d) is a RBT,
            containing exactly the same entries as d,
            and with the same black height as d.
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input may only satisfy weaker invariant

representation invariant established for output

Picture-guided implementation of restoreLeft

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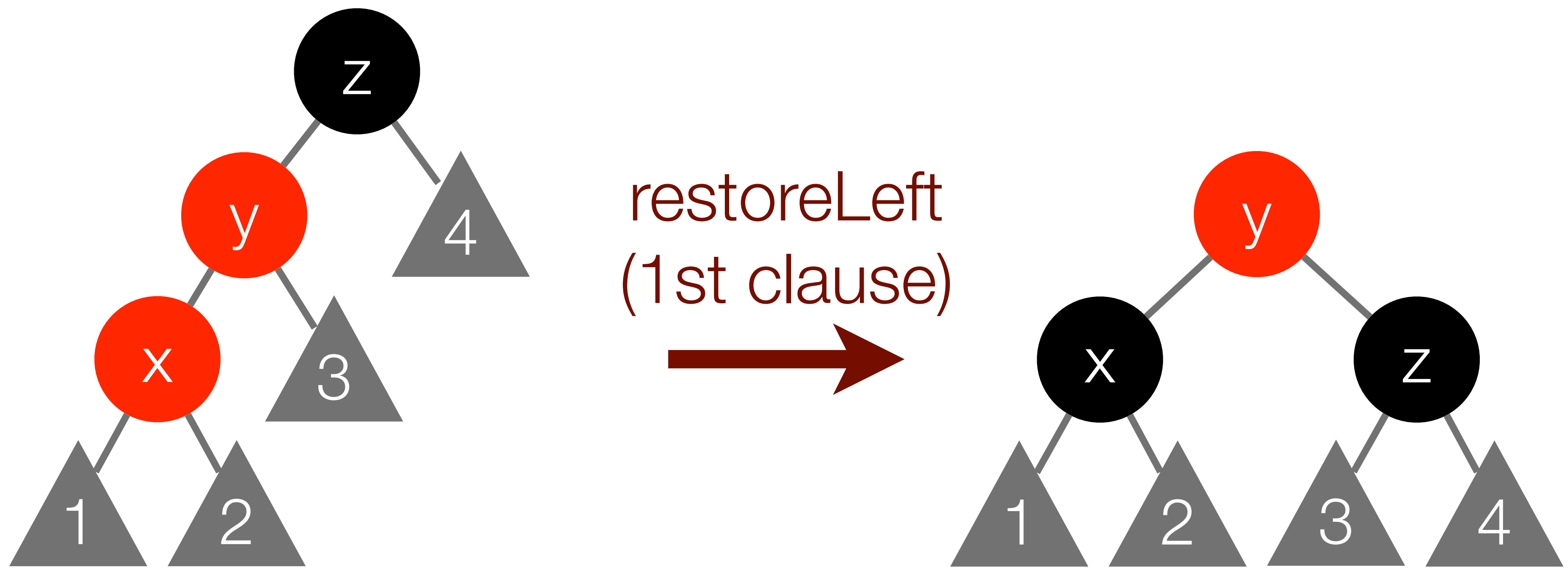

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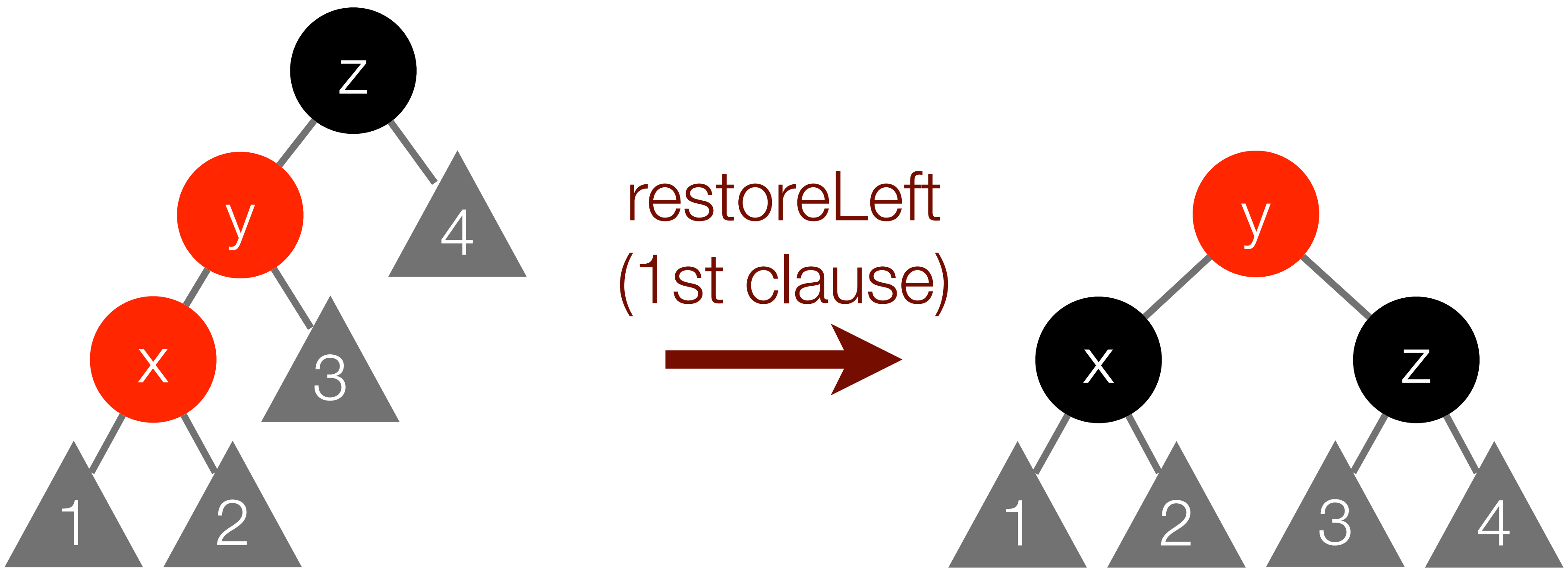


```
fun
  restoreLeft(Black(Red(Red(d1, x, d2), y, d3), z, d4)) =
```

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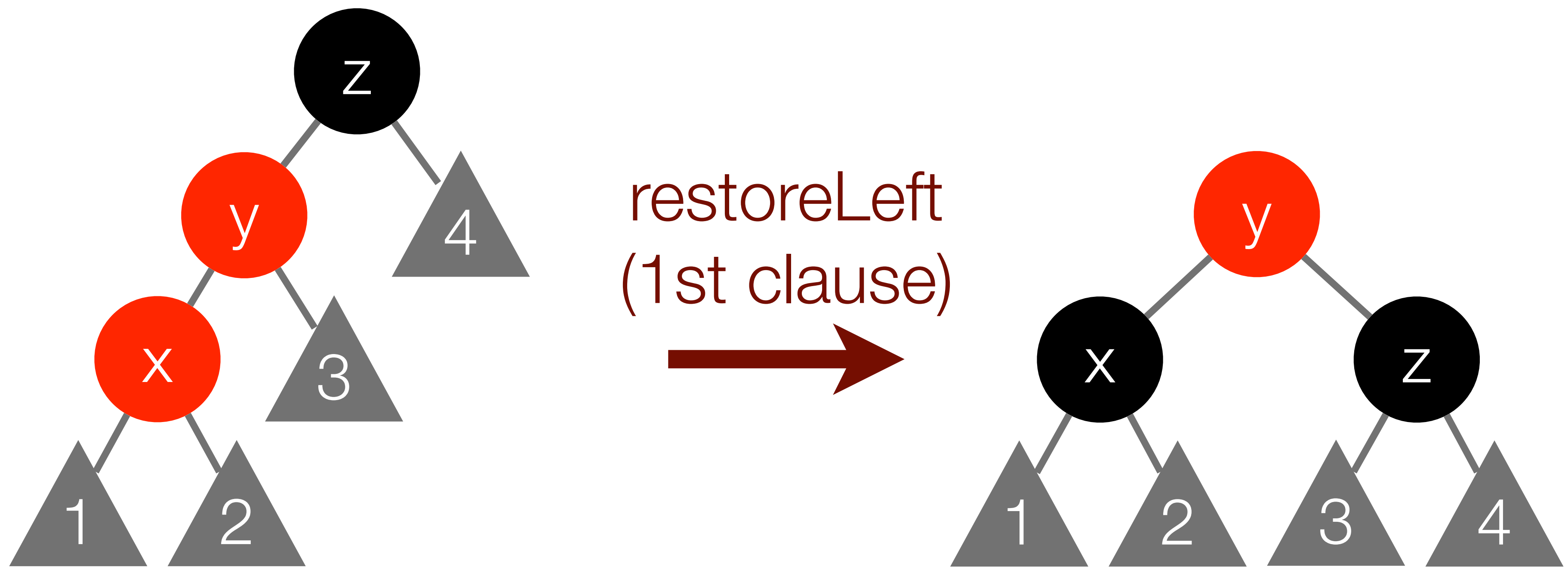


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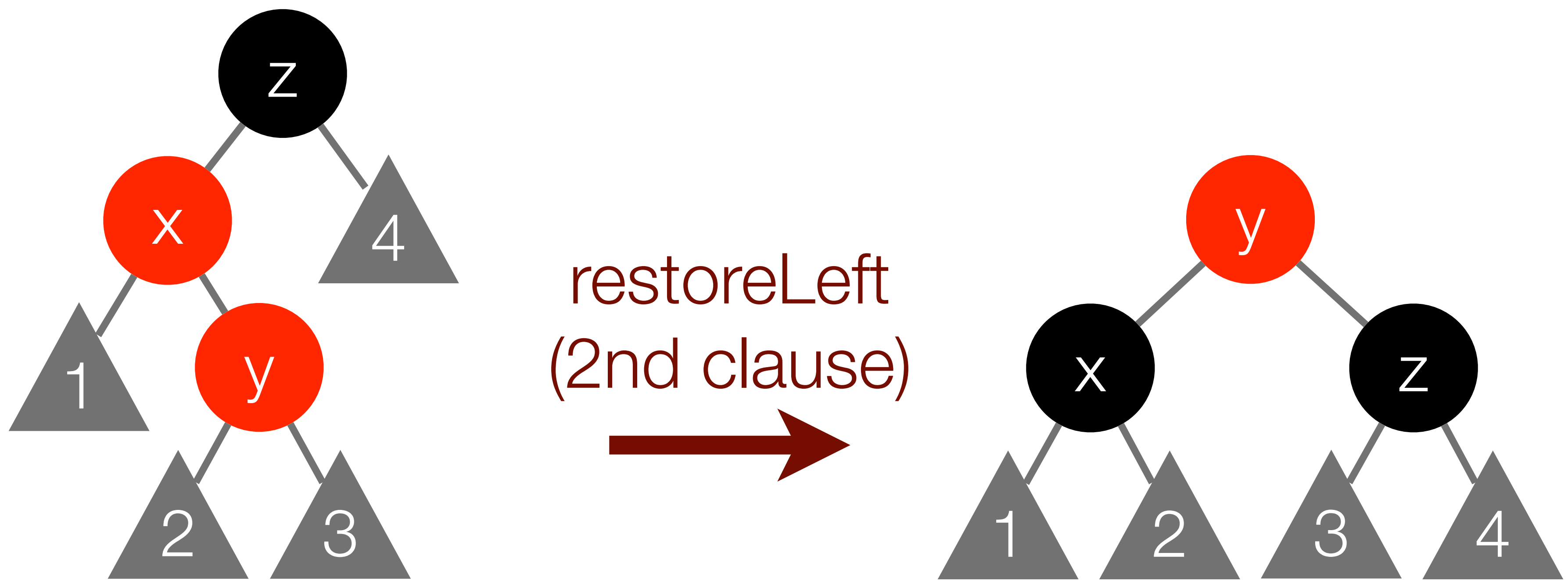


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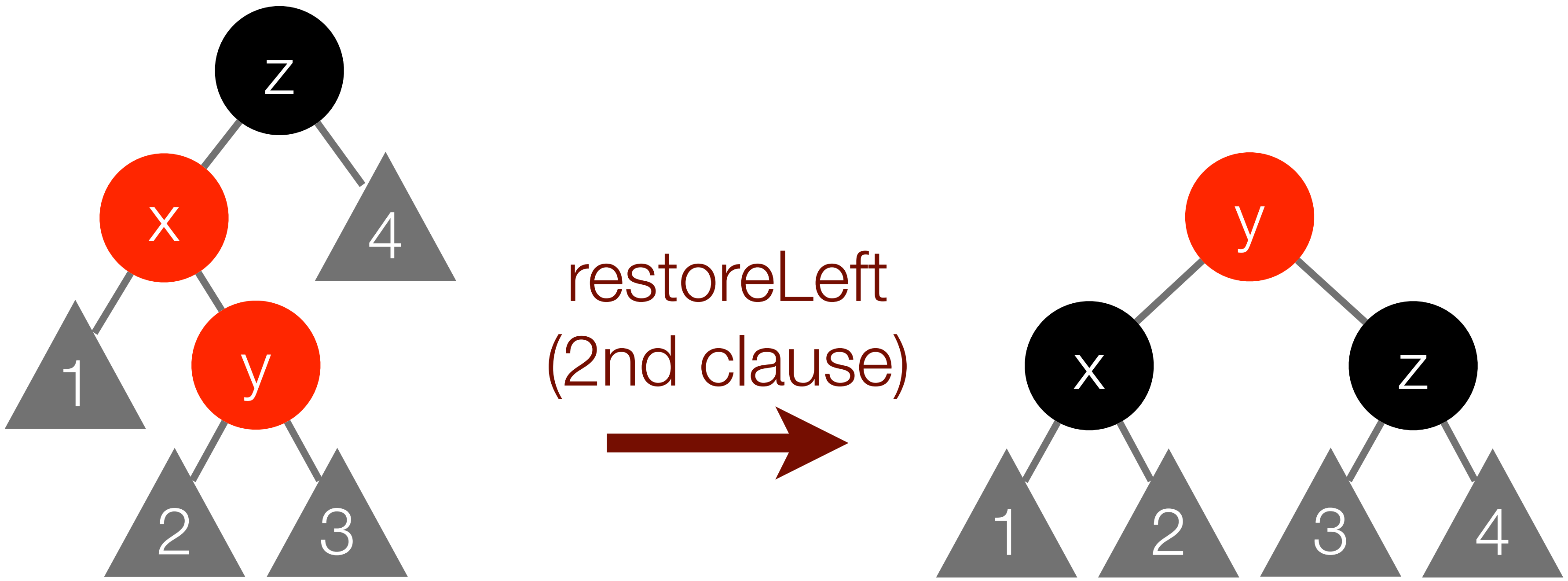


```
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 |restoreLeft(Black(Red(d1, x, Red(d2, y, d3)), z, d4)) =
```
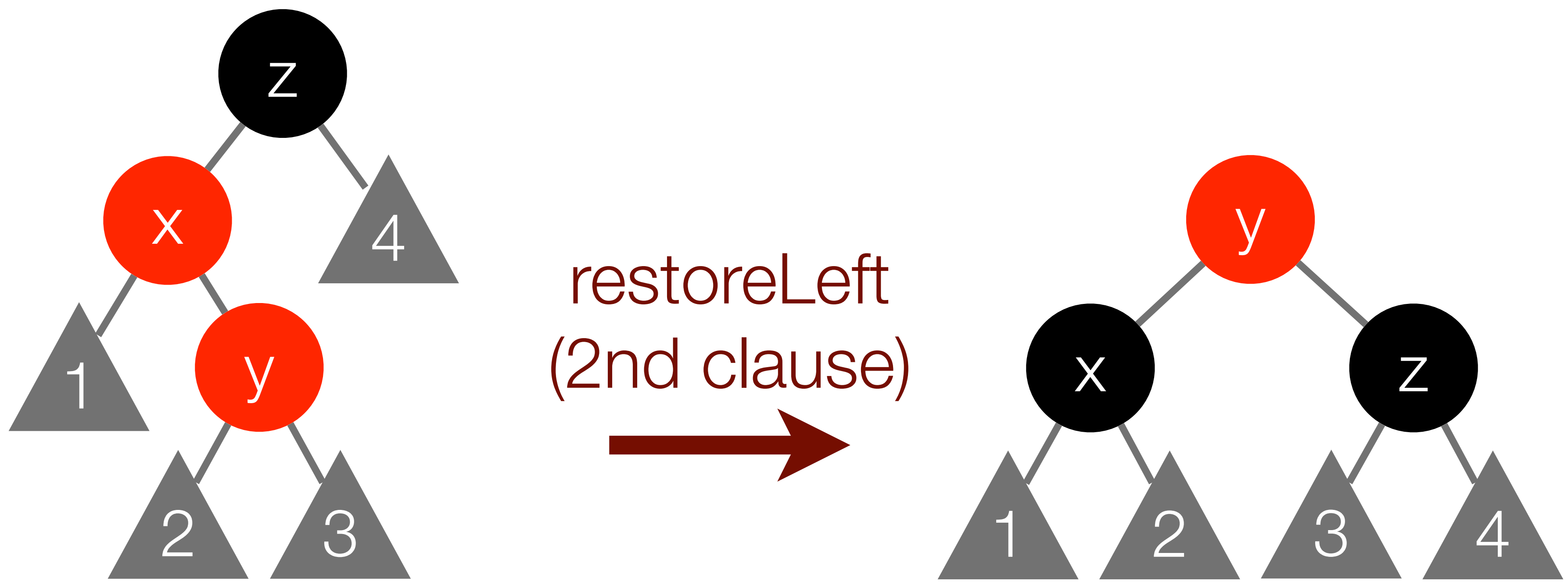
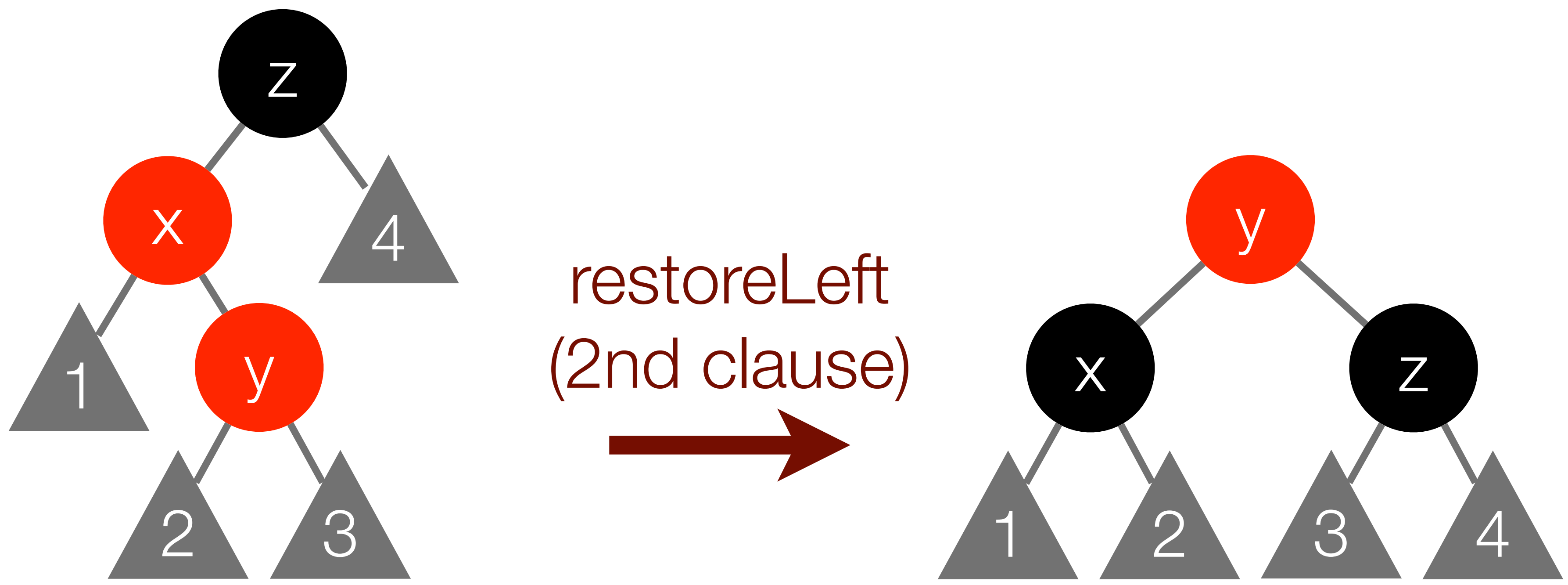

Picture-guided implementation of restoreLeft




```
fun
  restoreLeft(Black(Red(Red(d1, x, d2), y, d3), z, d4)) =
    Red(Black(d1, x, d2), y, Black(d3, z, d4))
 |restoreLeft(Black(Red(d1, x, Red(d2, y, d3)), z, d4)) =
    Red(Black(d1, x, d2), y, Black(d3, z, d4))
```

Picture-guided implementation of restoreLeft




```
fun
  restoreLeft(Black(Red(Red(d1, x, d2), y, d3), z, d4)) =
    Red(Black(d1, x, d2), y, Black(d3, z, d4))
 |restoreLeft(Black(Red(d1, x, Red(d2, y, d3)), z, d4)) =
    Red(Black(d1, x, d2), y, Black(d3, z, d4))
 |restoreLeft d = d
```

Specification for restoreRight

Specification for restoreRight

```
(*
  restoreRight : 'a dict -> 'a dict

  REQUIRES: Either d is a RBT
            or d's root is black,
            its right child is an ARBT,
            and its left child a RBT.

  ENSURES:  restoreRight(d) is a RBT,
            containing exactly the same entries as d,
            and with the same black height as d.
 *)
```

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input may only satisfy weaker invariant

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*)
```

input may only satisfy weaker invariant

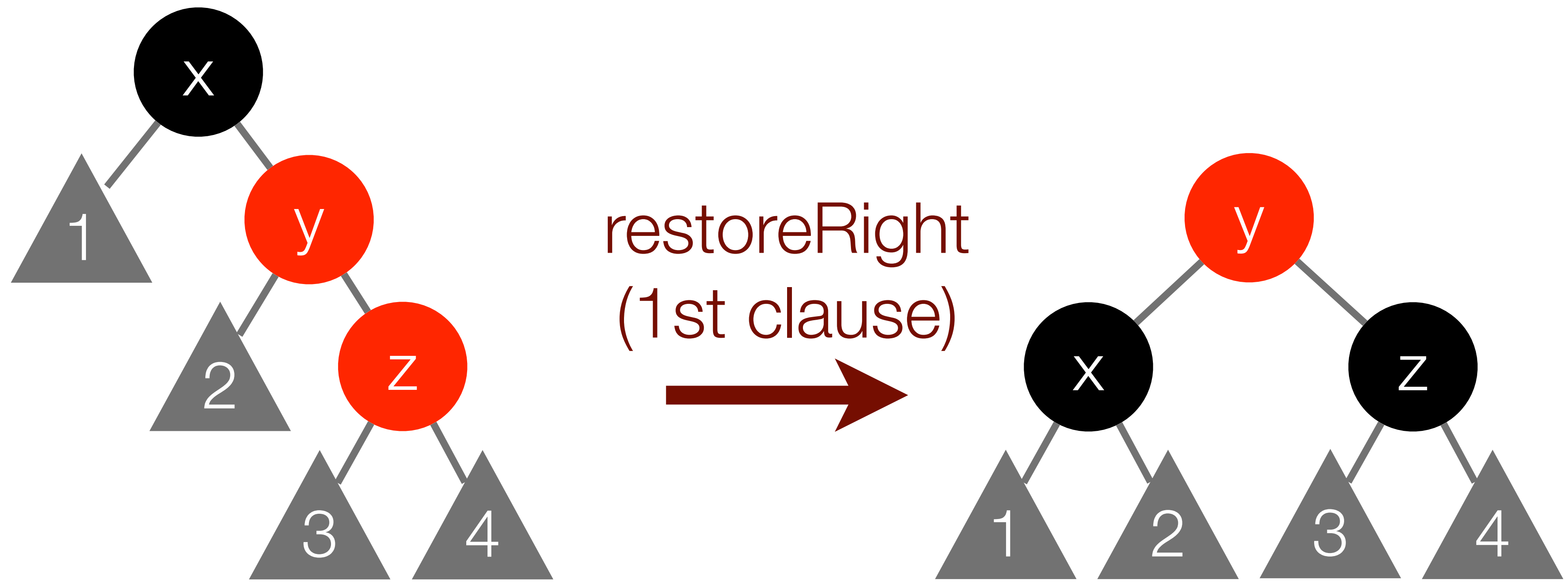
representation invariant established for output

Picture-guided implementation of restoreR

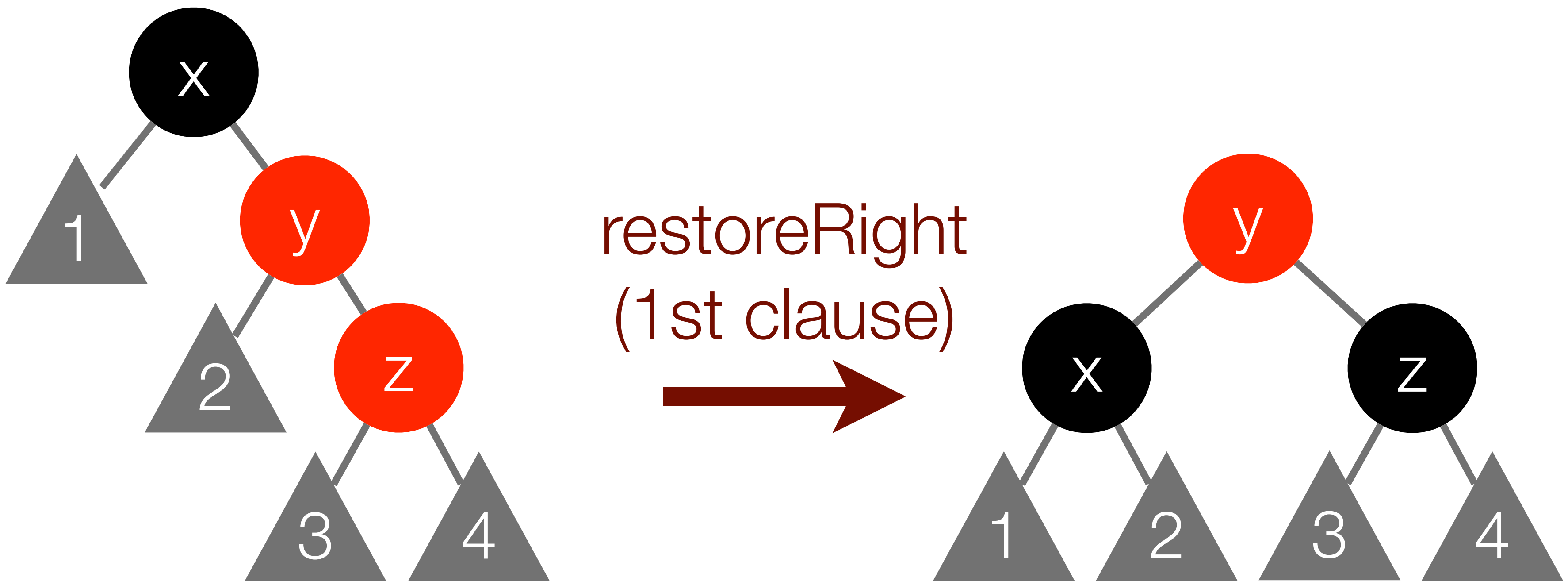
:

:

Picture-guided implementation of restoreR



Picture-guided implementation of restoreR

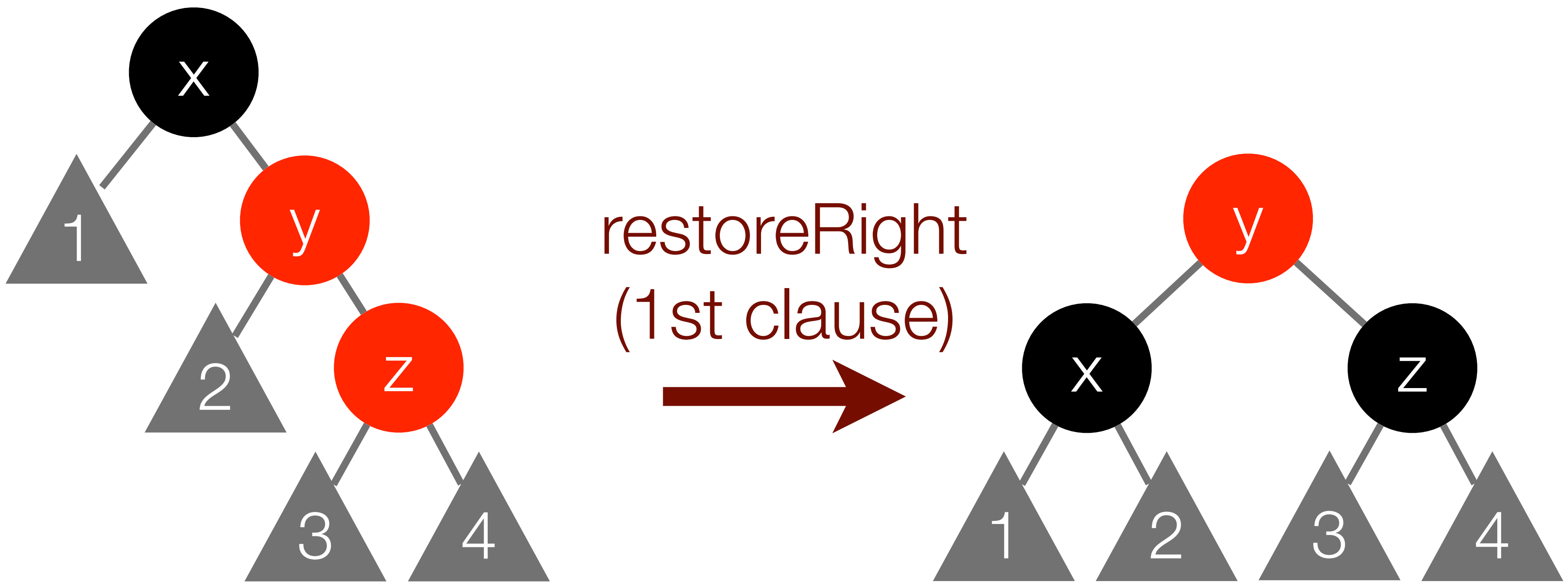



```
fun
  restoreRight(Black(d1, x, Red(d2, y, Red(d3, z, d4)))) =




                                                              :
```

Picture-guided implementation of restoreR

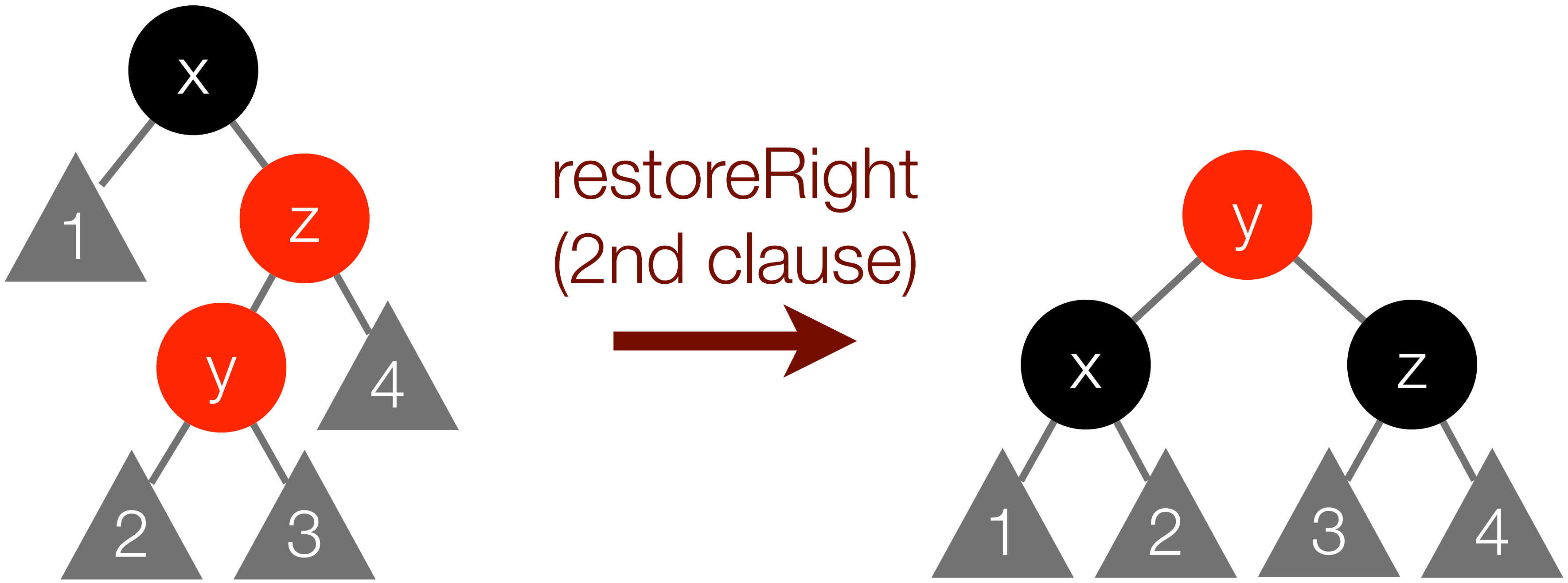



```
fun
  restoreRight(Black(d1, x, Red(d2, y, Red(d3, z, d4)))) =
    Red(Black(d1, x, d2), y, Black(d3, z, d4))
                                                           :
```

Picture-guided implementation of restoreR

```
fun
  restoreRight(Black(d1, x, Red(d2, y, Red(d3, z, d4)))) =
    Red(Black(d1, x, d2), y, Black(d3, z, d4))
                                                          :
```

Picture-guided implementation of restoreR




```
fun
  restoreRight(Black(d1, x, Red(d2, y, Red(d3, z, d4)))) =
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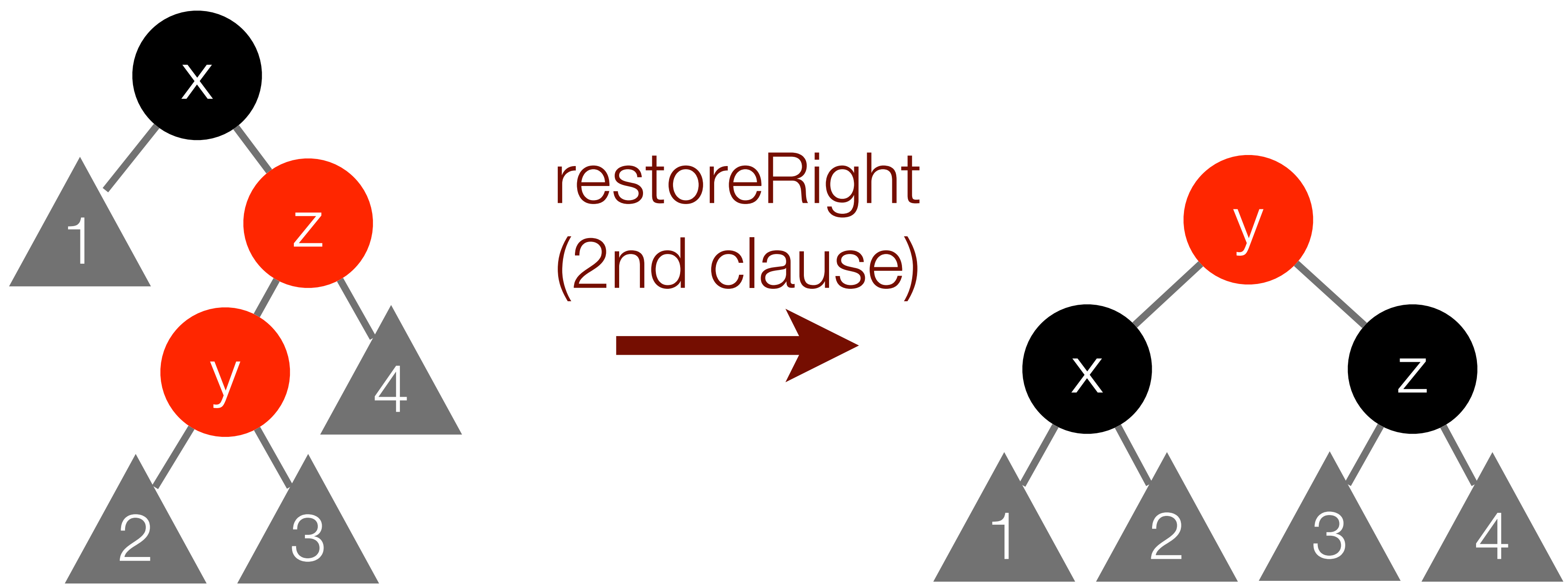
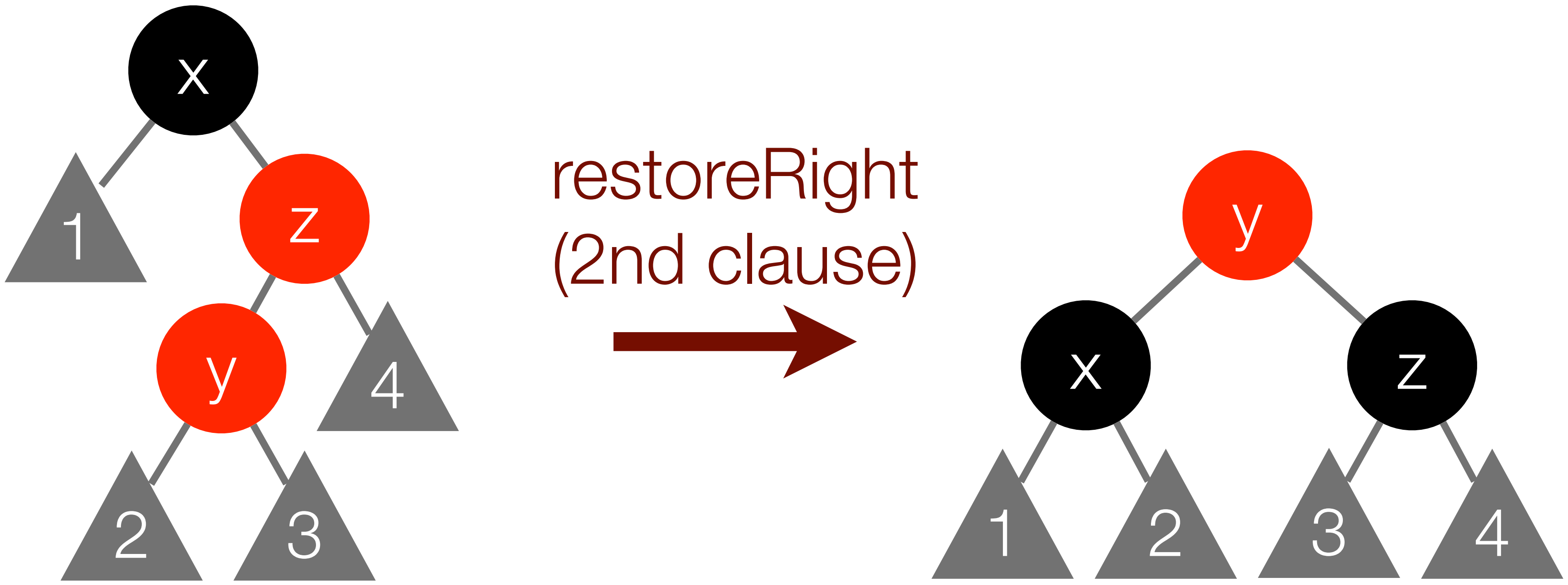
                                                        :
```

Picture-guided implementation of restoreR




```
fun
  restoreRight(Black(d1, x, Red(d2, y, Red(d3, z, d4)))) =
    Red(Black(d1, x, d2), y, Black(d3, z, d4))
 |restoreRight(Black(d1, x, Red(Red(d2, y, d3), z, d4))) =
```

Picture-guided implementation of restoreR

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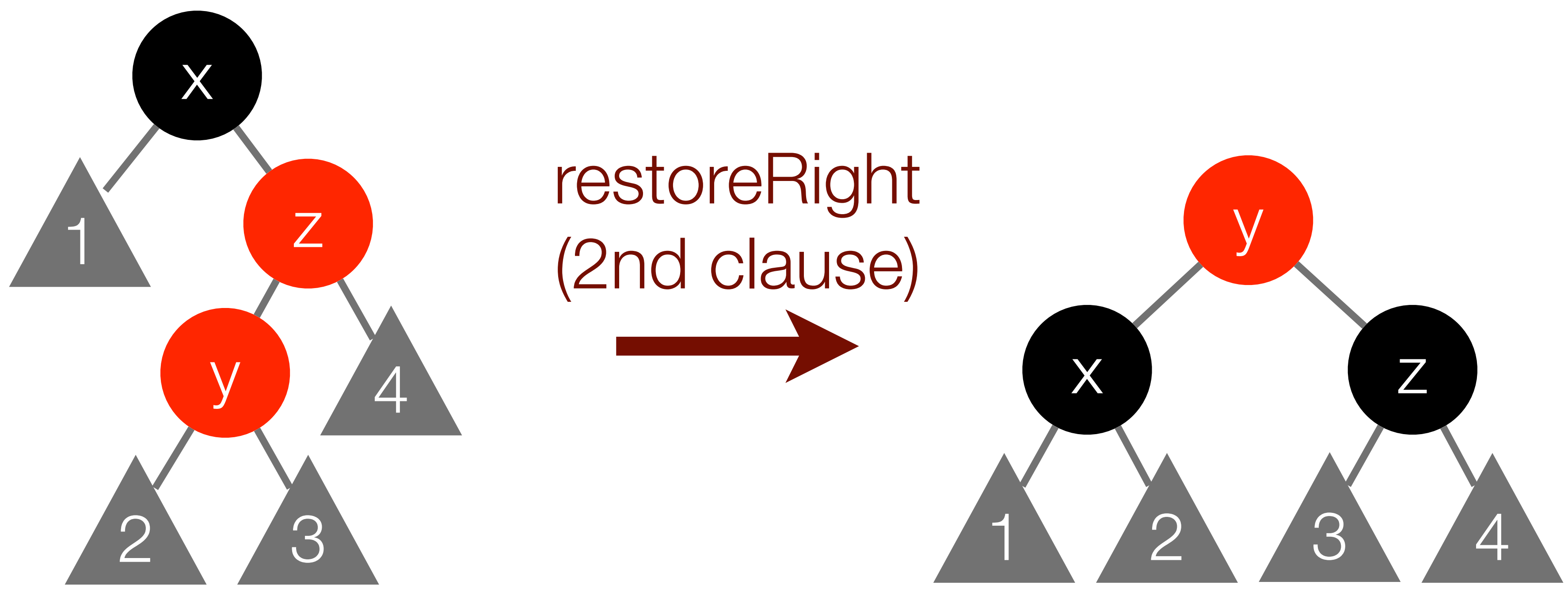


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fun
  restoreRight(Black(d1, x, Red(d2, y, Red(d3, z, d4)))) =
    Red(Black(d1, x, d2), y, Black(d3, z, d4))
 |restoreRight(Black(d1, x, Red(Red(d2, y, d3), z, d4))) =
    Red(Black(d1, x, d2), y, Black(d3, z, d4))
 |restoreRight d = d
```

What else?

What else?

```
signature DICT =
sig
  type key = string                       (* concrete  *)
  type 'a entry = key * 'a                (* concrete  *)
  type 'a dict                            (* abstract  *)
  val empty : 'a dict
  val lookup : 'a dict -> key -> 'a option
  val insert : 'a dict * 'a entry -> 'a dict
end
```

What else?

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signature DICT =
sig
  type key = string                     (* concrete  *)
  type 'a entry = key * 'a              (* concrete  *)
  type 'a dict                          (* abstract  *)
  val empty : 'a dict
  val lookup : 'a dict -> key -> 'a option
  val insert : 'a dict * 'a entry -> 'a dict
end
```

Note: restoreLeft and restoreRight are not externally visible!

What else?

```
signature DICT =
sig
  type key = string                       (* concrete  *)
  type 'a entry = key * 'a                (* concrete  *)
  type 'a dict                            (* abstract  *)
  val empty : 'a dict
  val lookup : 'a dict -> key -> 'a option
  val insert : 'a dict * 'a entry -> 'a dict
end
```

Note: restoreLeft and restoreRight are not externally visible!

Let's implement insert next.

Specification for insert

Specification for insert

```
(* insert: 'a dict * 'a entry -> 'a dict
   REQUIRES: d is a RBT.
   ENSURES:  insert(d,e) is a RBT containing exactly
             all the entries of d plus e,
             with e replacing an entry of d,
             if the keys are EQUAL.
```

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expects
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expects representation invariant

establishes representation invariant

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   ins: 'a dict -> 'a dict
   REQUIRES: d is a RBT.
   ENSURES:  ins(d) is a tree containing exactly
             all the entries of d plus e,
             with e replacing an entry of d,
             if the keys are EQUAL.
```

Specification for insert

```
(* insert: 'a dict * 'a entry -> 'a dict
   REQUIRES: d is a RBT.
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```

insert makes use of a locally defined helper function

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   REQUIRES: d is a RBT.
   ENSURES:  ins(d) is a tree containing exactly
             all the entries of d plus e,
             with e replacing an entry of d,
             if the keys are EQUAL.
             ins(d) has the same black height as d.
             Moreover, ins(Black(t)) is a RBT
                       ins(Red(t)) is an ARBT. *)
```

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                       ins(Red(t)) is an ARBT. *)
```

may
temporarily violate
representation invariant

Let's implement insert

Let's implement insert

```
fun insert (d, e as (k, v)) =
  let
    fun ins ... (* will write shortly *)
  in
    (case ins d of
       Red(t as (Red (_), _, _)) => Black t
     | Red(t as (_, _, Red(_)))  => Black t
     | d' => d')
  end
```

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re-color in case of a red-red violation at the root

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re-color in
case of a red-red violation
at the root

RBT representation invariant preserved.

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     | d' => d')
  end
```

recall layered pattern matching!

Let's implement ins

Let's implement ins

```
fun ins (Empty) = Red(Empty, e, Empty)
  | ins (Black(l, e' as (k',_), r)) =
    (case String.compare (k, k') of
      EQUAL =>
    | LESS =>
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only choice,
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only choice, otherwise we destroy black height

recall: weaker invariant still guarantees black height

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➡ Is that really it?

Let's implement ins

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Is that really it?

No, we have to invoke restore functions because `ins` may return a tree that only satisfies ARBT!

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    | LESS => restoreLeft(Black(ins l, e', r))
    | GREATER => restoreRight(Black(l, e', ins r)))
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```

Should we call the restore functions here too?

Let's implement ins

```
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- Moreover, `l` and `r` must have black roots by the pre-condition.
- And, we get back an RBT by the post-condition.

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  - Experiment with the code to see hiding at play!

That's all for today.