Lazy Programming

15-150

Lecture 20: November 19, 2024

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Today

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To facilitate programming infinite data structures, we use the notion of a **delayed computation**.

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➡ To facilitate programming infinite data structures, we use the notion of a **delayed computation**.

➡ The notion of a delayed computation also facilitates **demand-driven** (aka **lazy**) programming in a call-by-value language.

Delayed computation

Delayed computation

Idea:

Delayed computation

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Encapsulate computation to suspend it.

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- Execute computation by explicitly forcing it.

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What is the difference between the following two expressions?

`e` and `fn x => e x`

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Here, SML will evaluate **e**.

Here, SML will only evaluate **e**, when the lambda is applied to an argument.

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- Lambdas allow us to suspend computation.

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Let's take a step back and ask ourselves the following question:

What is the difference between the following two expressions?

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- Lambdas allow us to suspend computation.
- Lambdas are values (even if encapsulated computation diverges).

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```
fun g x = g x
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```
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`g 3` loops, but `fn x => (g 3) x` is a value

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Delayed computation

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- Yes, using lambdas to represent infinite, possibly diverging computations.

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- We call such lambdas **suspensions**:

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A **suspension** of type `t` is a function `f` of type

Delayed computation

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f: unit -> t
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Delayed computation

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A suspension is **forced**, when it is applied, i.e., `f ()`.

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The suspension `f` is a **lazy** representation of `e` because `e` won't be evaluated until `f` is forced.

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- A suspension is **forced**, when it is applied, i.e., `f ()`.
- The suspension `f` is a **lazy** representation of `e` because `e` won't be evaluated until `f` is forced.
- Let's use suspensions to represent (possibly infinite) **streams** of data.

Streams*

* (Note, different from SML's built-in I/O streams.)

Streams*

Streams are data structures that are being continuously created, e.g.,

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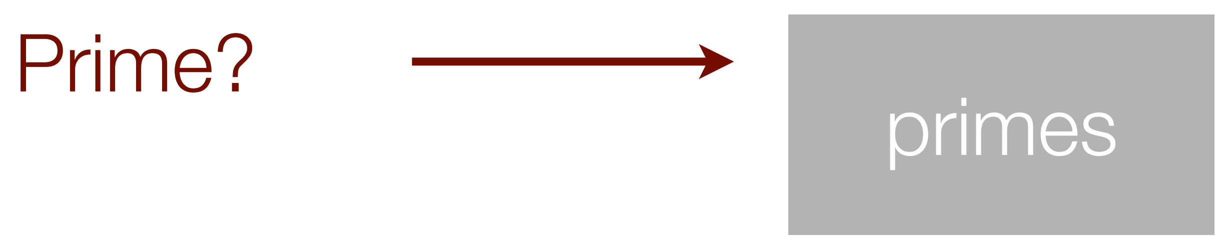

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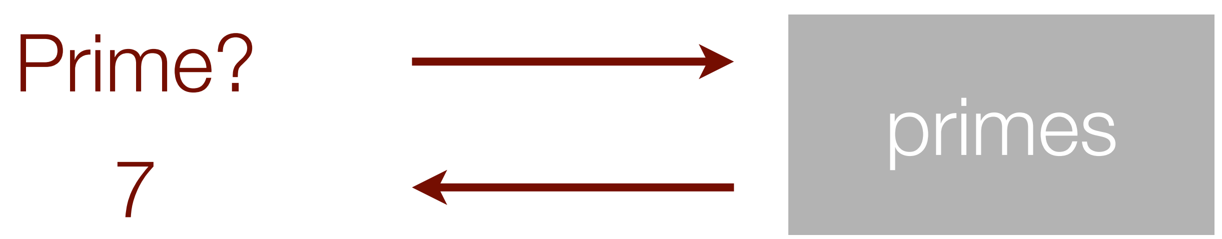

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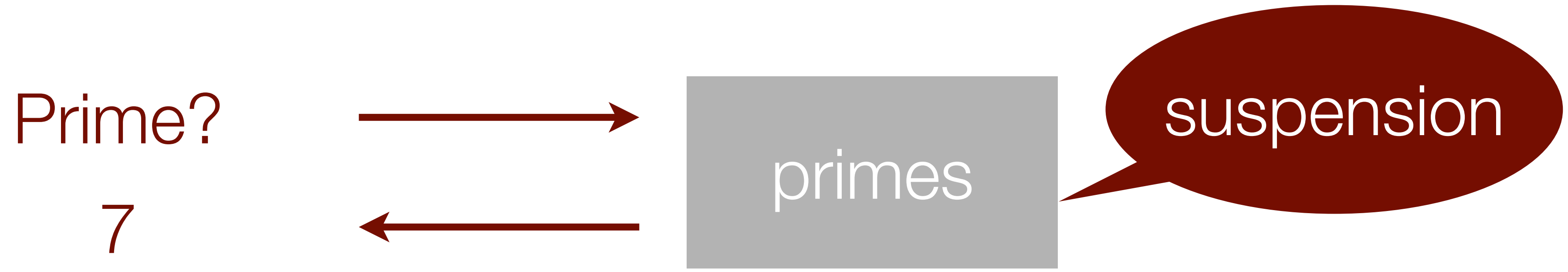

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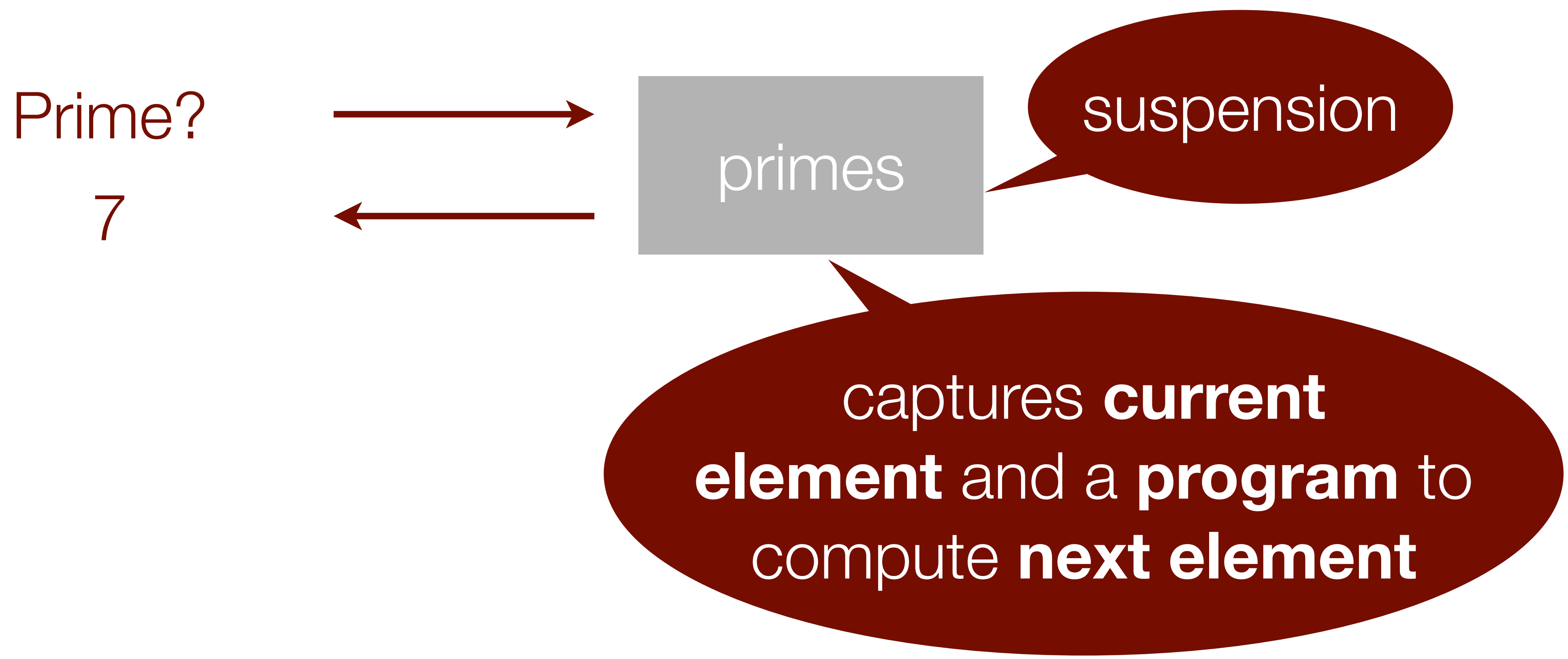

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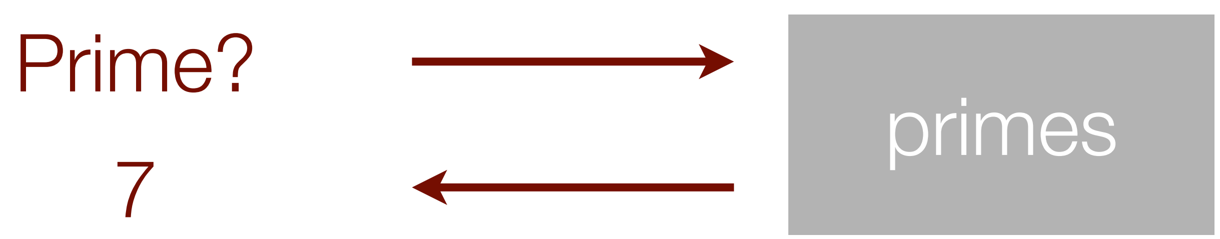

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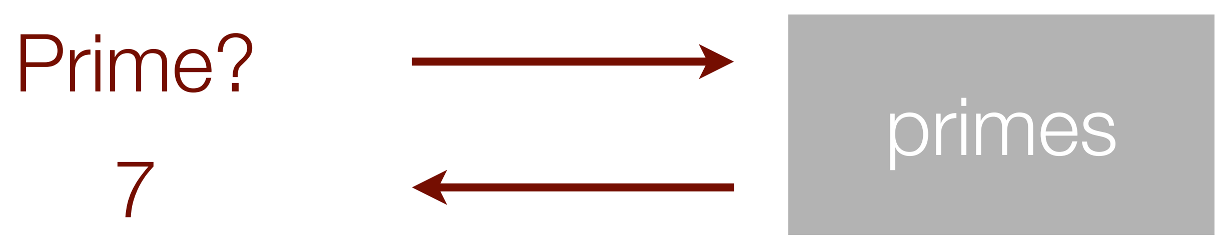

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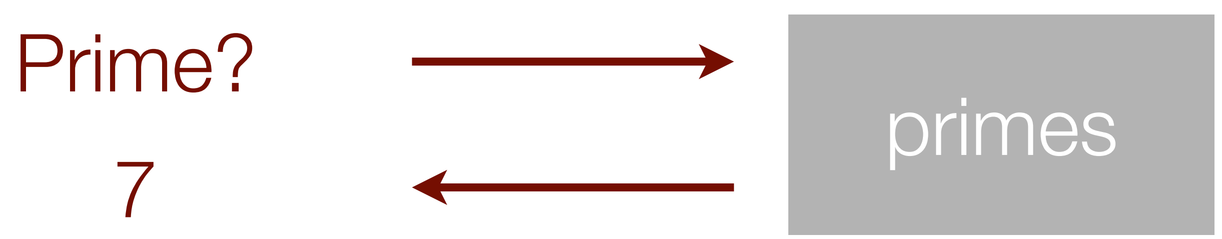

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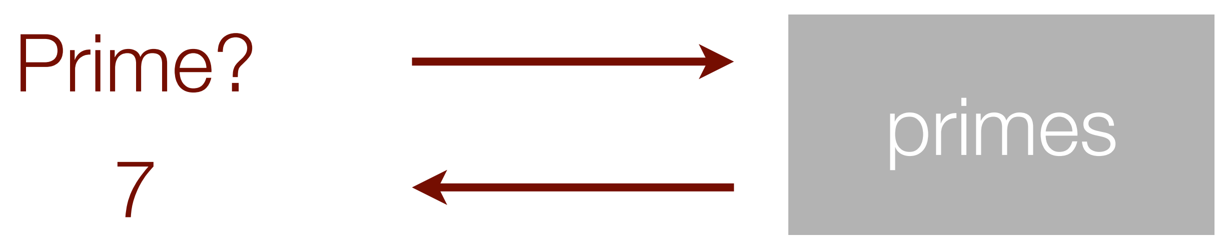

- We can think of streams as being generated by state machines:
  - only when "kicked" (forcing suspension) they yield element
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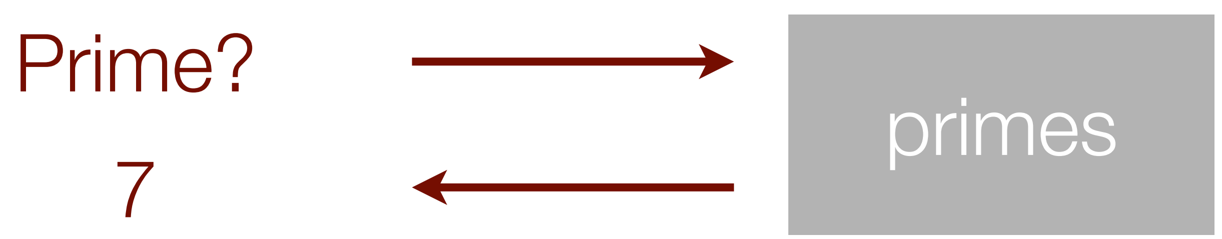

- We can think of streams as being generated by state machines:
  - only when "kicked" (forcing suspension) they yield element
  - advancing state for computation of next element.
- Streams are defined **coinductively**

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Intermezzo: induction versus coinduction

Intermezzo: induction versus coinduction

Intermezzo: induction versus coinduction

if you'd like to
know

Intermezzo: induction versus coinduction

if you'd like to know

aka, we don't expect you to know

Intermezzo: induction versus coinduction

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The data types (e.g., lists, trees) encountered so far were defined **inductively**.

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Inductive data types are constructed **upfront** and are thus **finite**.

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Intermezzo: induction

We can also define corresponding lazy versions!

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- Then, we implement them in a corresponding structure.

Stream signature

Stream signature

```
signature STREAM =
sig
  type 'a stream                                      (* abstract *)







end
```

Stream signature

```
signature STREAM =
sig
  type 'a stream                                       (* abstract *)
end
```

streams with
elements of type 'a

Stream signature

```
signature STREAM =
sig
  type 'a stream                                      (* abstract *)




end
```

Stream signature

```
signature STREAM =
sig
  type 'a stream                          (* abstract *)

  datatype 'a front = Cons of 'a * 'a stream
                                          (* concrete *)




end
```

Stream signature

```
signature STREAM =
sig
  type 'a stream                                     (* abstract *)

  datatype 'a front = Cons of 'a * 'a stream
                                                     (* concrete *)




end
```

Stream signature

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signature STREAM =
sig
  type 'a stream                                  (* abstract *)

  datatype 'a front = Cons of 'a * 'a stream
                                                  (* concrete *)
```

Forcing ("kicking") a stream yields a value of type `'a front`,

```
end
```

Stream signature

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signature STREAM =
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  type 'a stream                                    (* abstract *)

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                                                    (* concrete *)
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Forcing ("kicking") a stream yields a value of type `'a front`,

comprising the current element

```
end
```

Stream signature

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signature STREAM =
sig
  type 'a stream                                (* abstract *)

  datatype 'a front = Cons of 'a * 'a stream
                                                (* concrete *)
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Forcing ("kicking") a stream yields a value of type `'a front`,

comprising the current element and the rest of the stream,

```
end
```

Stream signature

```
signature STREAM =
sig
  type 'a stream                             (* abstract *)

  datatype 'a front = Cons of 'a * 'a stream
                    | Empty                  (* concrete *)
```

Forcing ("kicking") a stream yields a value of type `'a front`,

comprising the current element and the rest of the stream,

or `Empty`, in case the stream is finite.

```
end
```

Stream signature

```
signature STREAM =
sig
  type 'a stream                                    (* abstract *)

  datatype 'a front = Cons of 'a * 'a stream
                            | Empty                 (* concrete *)

end
```

Stream signature

```
signature STREAM =
sig
  type 'a stream                                (* abstract *)

  datatype 'a front = Cons of 'a * 'a stream
                    | Empty                     (* concrete *)

  val expose : 'a stream -> 'a front




end
```

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signature STREAM =
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Function `expose` forces the computation yielding the current element and the remainder of the stream.

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Function **`expose`** forces the computation yielding the current element and the remainder of the stream.

Caution: **`expose`** may loop!

```
end
```

Stream signature

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signature STREAM =
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                    | Empty                 (* concrete *)

  val expose : 'a stream -> 'a front

  val delay : (unit -> 'a front) -> 'a stream


end
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Function `delay` creates a stream, given a suspension for computing the stream.

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Function `delay` creates a stream, given a suspension for computing the stream.

Suspension required, otherwise SML will evaluate argument!

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  val delay : (unit -> 'a front) -> 'a stream

  (* more functions (see accompanying code) *)
end
```

Stream structure

Stream structure

```
structure Stream : STREAM =
struct
  datatype 'a stream = Stream of unit -> 'a front









end
```

Stream structure

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end
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Stream structure

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structure Stream : STREAM =
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We find it convenient to wrap a `Stream` constructor around the suspension of an `'a front`.

```
end
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Stream structure

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structure Stream : STREAM =
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  datatype 'a stream = Stream of unit -> 'a front
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We find it convenient to wrap a `Stream` constructor around the suspension of an `'a front`.

The use of the constructor `Stream`, instead of the plain suspension, conveys more readily what the function is about.

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Recall: `'a front` refers to `'a stream`.

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The use of the constructor `Stream`, instead of the plain suspension, conveys more readily what the function is about.

Recall: `'a front` refers to `'a stream`.

How do we handle that?

```
end
```

Stream structure

```
structure Stream : STREAM =
struct
  datatype 'a stream = Stream of unit -> 'a front
  and 'a front = Cons of 'a * 'a stream | Empty







end
```

Stream structure

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Define mutually recursive data structures with keyword `and`.

```
end
```

Stream structure

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- Define mutually recursive data structures with keyword `and`.
- Recall: `'a front` is already defined as such in signature.

```
end
```

Stream structure

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  datatype 'a stream = Stream of unit -> 'a front
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```

Stream structure

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structure Stream : STREAM =
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  datatype 'a stream = Stream of unit -> 'a front
  and 'a front = Cons of 'a * 'a stream | Empty

  (* delay : (unit -> 'front) -> 'a stream *)
  fun delay (d) = Stream(d)





end
```

Stream structure

```
structure Stream : STREAM =
struct
  datatype 'a stream = Stream of unit -> 'a front
  and 'a front = Cons of 'a * 'a stream | Empty

  (* delay : (unit -> 'front) -> 'a stream *)
  fun delay (d) = Stream(d)
```

Wraps `Stream` constructor around suspension of `'a front`.

```
end
```

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Forces underlying suspension in input stream.

```
end
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  (* more functions (see accompanying code) *)
end
```

Let's practice: stream of 1s

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remains the same in tail

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current element

next element

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x is bound to 0 and y to 1

`tail` denotes the stream of all natural numbers greater than 0

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What value is z bound to?

z is bound to 0

Memoization for efficiency

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  - initially, reference cell contains suspension
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- `take(s,n)` returns the first `n` elements of stream `s` as a list.
- May loop or raise an exception if stream is empty.

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```
Stream.take(X,n) ≅ Stream.take(Y,n)
```

Another example: prime numbers

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Inspired by the Sieve of Eratosthenes.

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```
2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,...
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`2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,...`

Write down all the natural numbers greater than `1`.

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Cross off all multiples of that leftmost element.

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Repeat the process with the remaining numbers.

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The diagonal of leftmost elements constitutes all primes.

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Now, the algorithm:

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val notDivides p q = (q mod p <> 0)
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delays actual sieving

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not really needed because primes are infinite

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```

recursively constructs stream of larger primes, with p at front

filters multiples of current element p

Recall: `(* delay` … `->` … `*)`
`fun delay`

That's all for today.