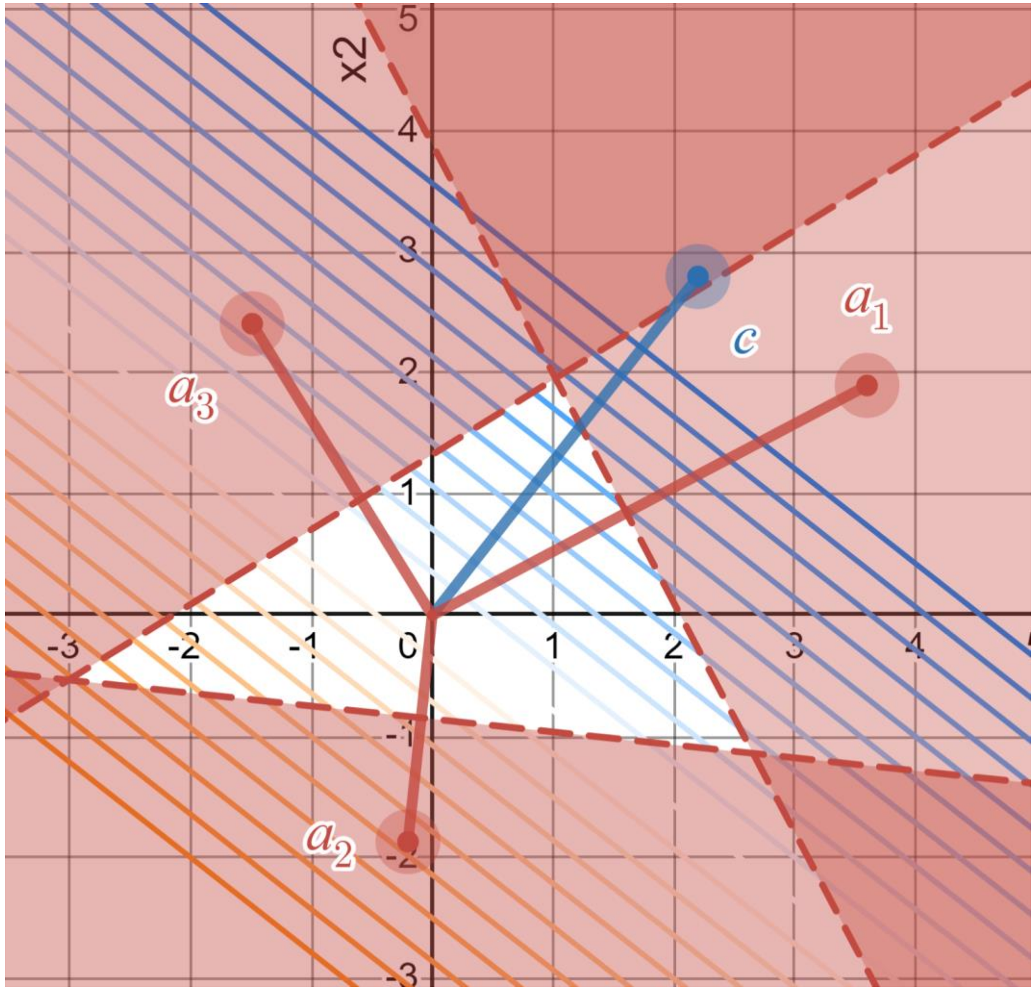
# Warm-up as you walk in

## What is the solution to this LP?




https://www.desmos.com/calculator/tnlo7p5plp

# Plan

## Last Time

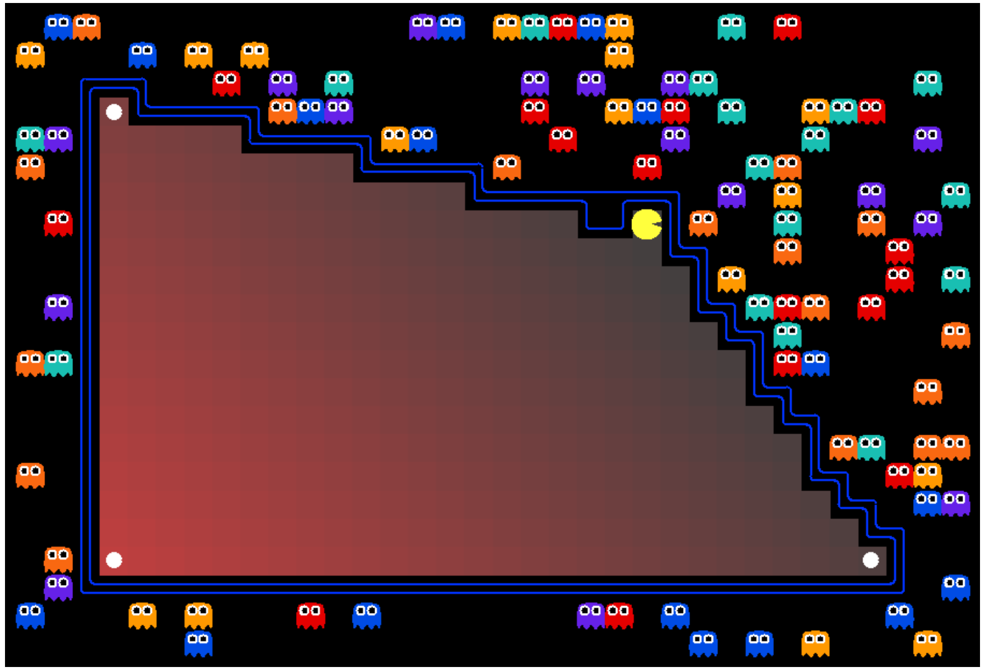
- Linear programming formulation
  - Problem description
  - Graphical representation
  - Optimization representation

## Today

- Solving linear programs
- Higher dimensions than just 2
- Integer programs

# AI: Representation and Problem Solving

## Integer Programming



Instructor: Pat Virtue

Slide credits: CMU AI with drawings from http://ai.berkeley.edu

# Reminder: Cost Contours

Given the cost vector $[c_1, c_2]^\top$ where will

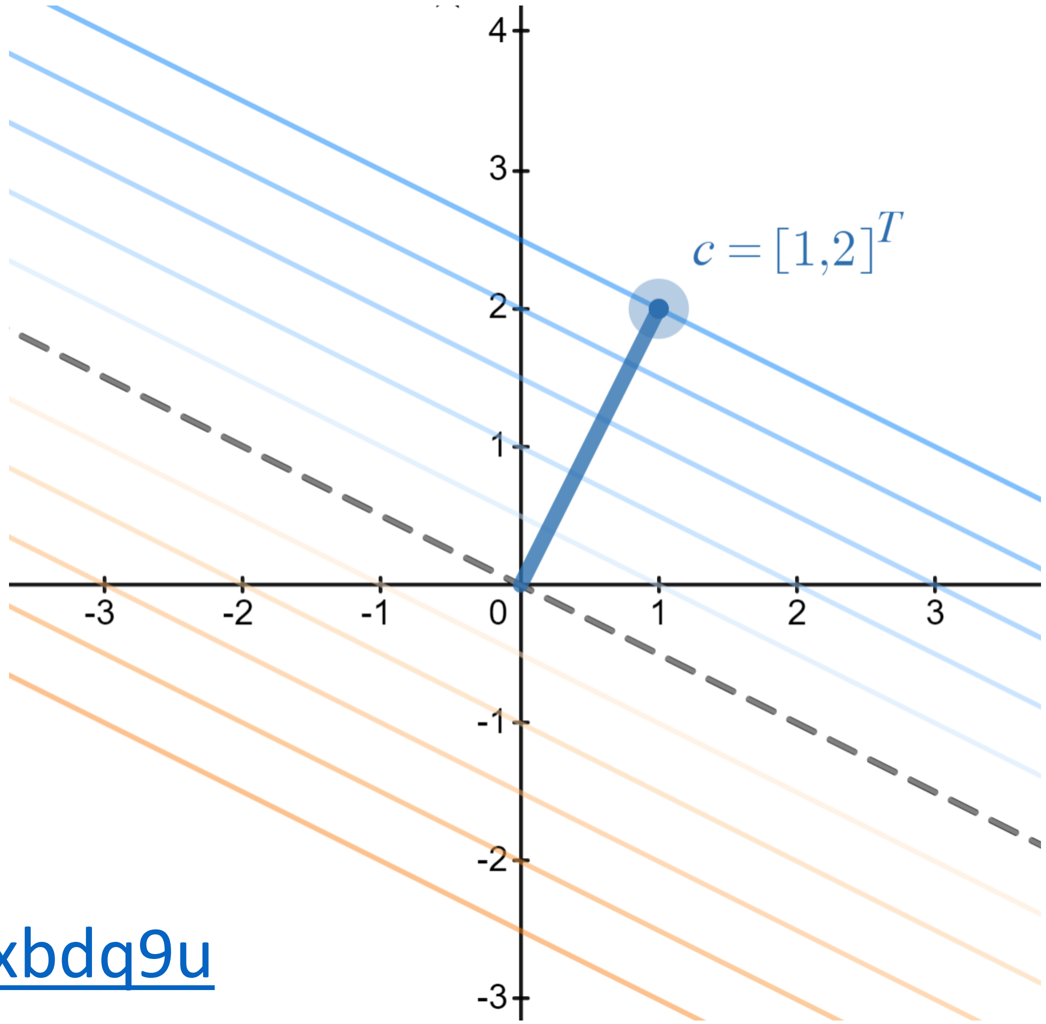
$\mathbf{c}^\top\mathbf{x} = 0$ ?

$\mathbf{c}^\top\mathbf{x} = 1$ ?

$\mathbf{c}^\top\mathbf{x} = 2$ ?

$\mathbf{c}^\top\mathbf{x} = -1$ ?

$\mathbf{c}^\top\mathbf{x} = -2$ ?

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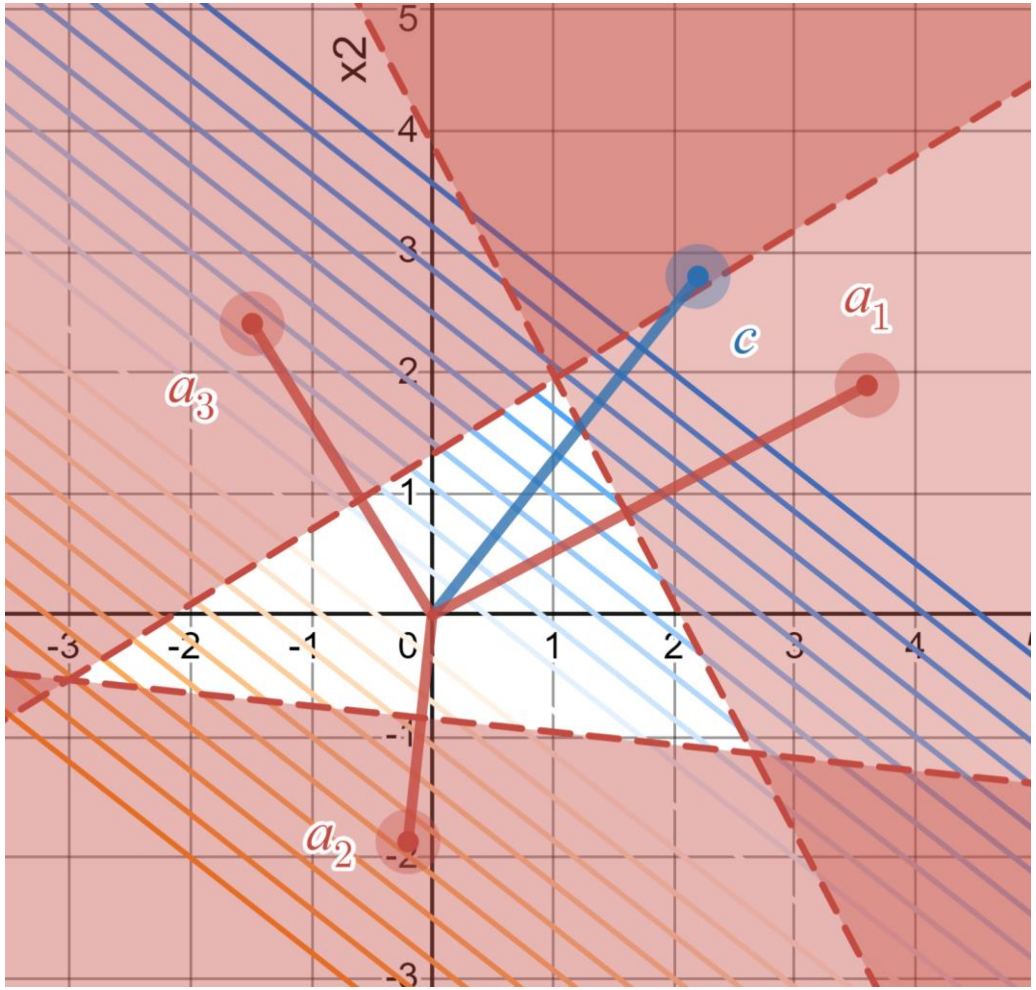
# Solving a Linear Program

## What is the solution to this LP?

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# Solving a Linear Program

## Inequality form, with no constraints

$$\min_{\mathbf{x}} . \quad \mathbf{c}^\top \mathbf{x}$$

# Solving a Linear Program

## Inequality form, with one constraint

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$
$$\text{s.t.} \quad a_1 x_1 + a_2 x_2 \leq b$$

# Poll 1

True or False: A minimizing LP with exactly on constraint, will always have a minimum objective at $-\infty$.

$$\begin{aligned} \min_{\mathbf{x}}. \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & a_1 x_1 + a_2 x_2 \leq b \end{aligned}$$

# Solving an LP

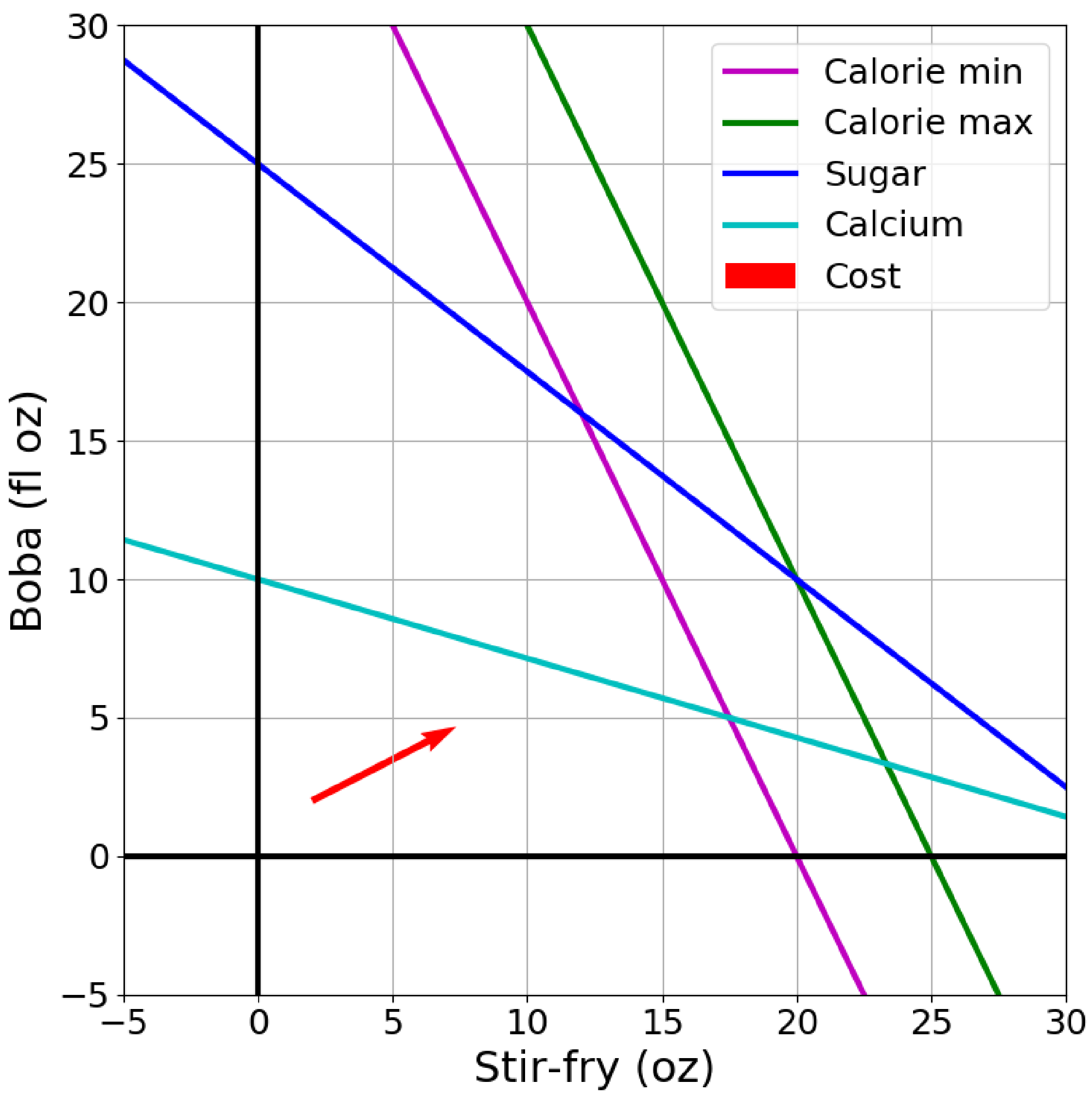
Solutions are at feasible intersections of constraint boundaries!!

Algorithms

- Check objective at all feasible intersections

In more detail:

1. Enumerate all intersections
2. Keep only those that are feasible (satisfy *all* inequalities)
3. Return feasible intersection with the lowest objective value



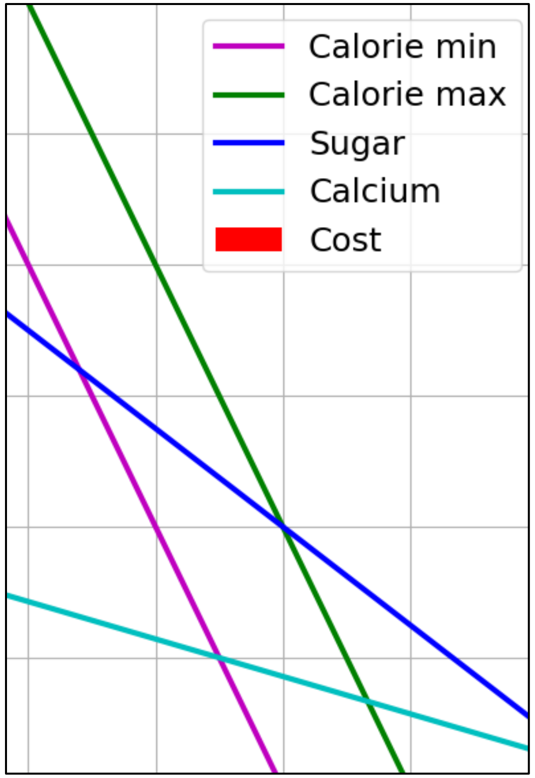
# Solving an LP

But, how do we find the intersection between boundaries?

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$
$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Calorie min
Calorie max
Sugar
Calcium

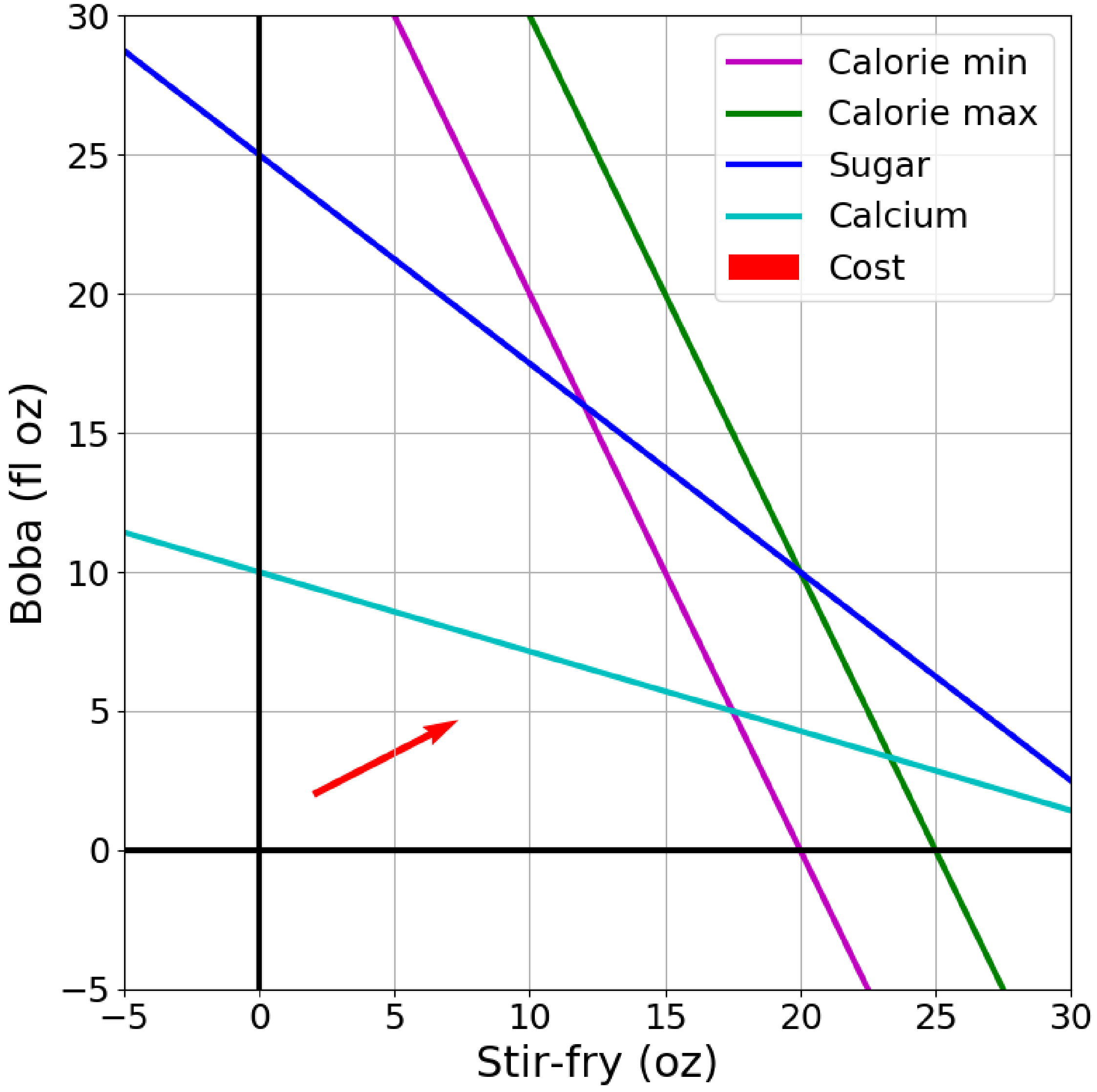


# Solving an LP

Solutions are at feasible intersections of constraint boundaries!!

Algorithms

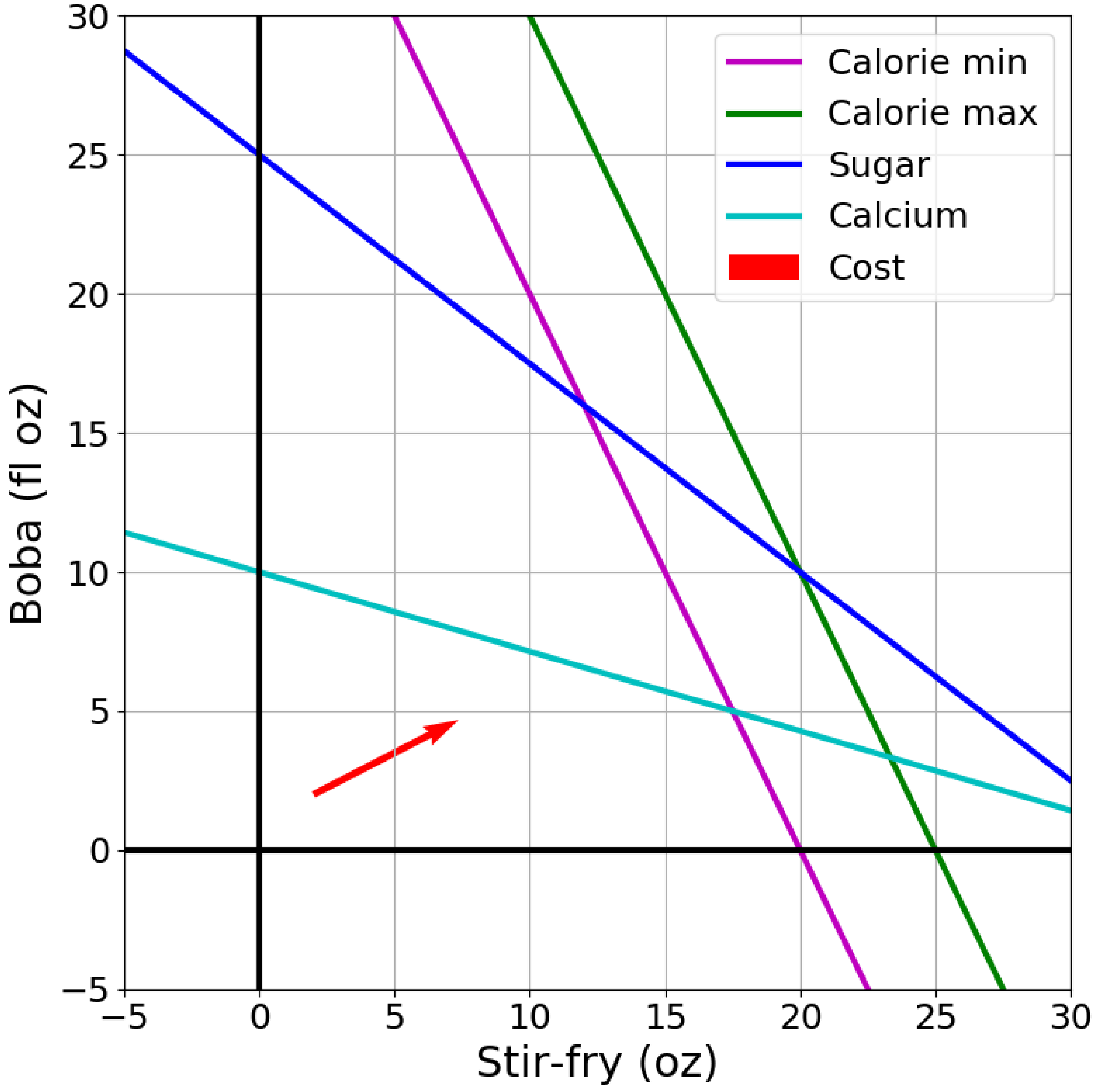
- Check objective at all feasible intersections
- Simplex



# Solving an LP

Simplex algorithm

- Start at a feasible intersection (if not trivial, can solve another LP to find one)
- Define successors as “neighbors” of current intersection
  - i.e., remove one row from our square subset of A, and add another row not in the subset; then check feasibility
- Move to any successor with lower objective than current intersection
  - If no such successors, we are done




Greedy local hill-climbing search! … but always finds *optimal* solution

# Solving an LP

Solutions are at feasible intersections of constraint boundaries!!

Algorithms

- Check objective at all feasible intersections
- Simplex
- Interior Point

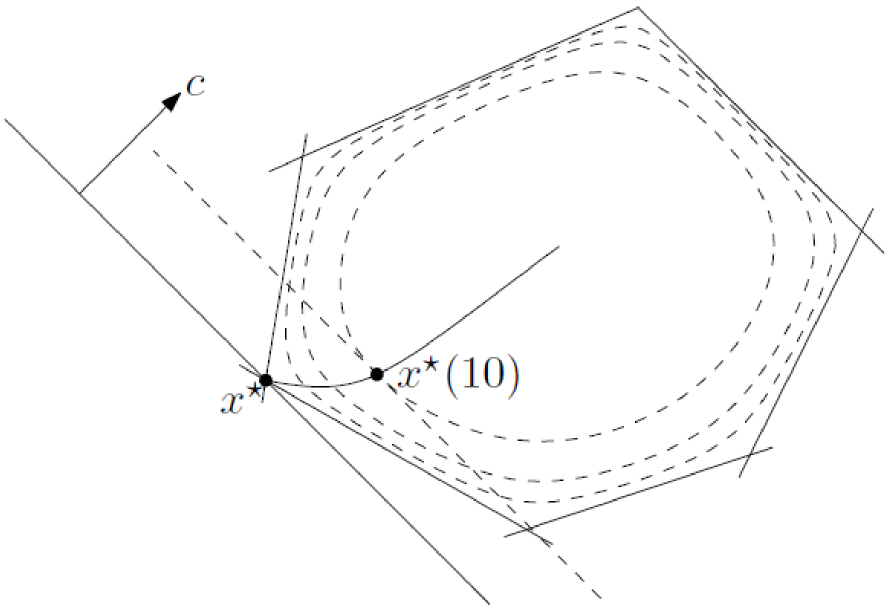



Figure 11.2 from Boyd and Vandenberghe, *Convex Optimization*

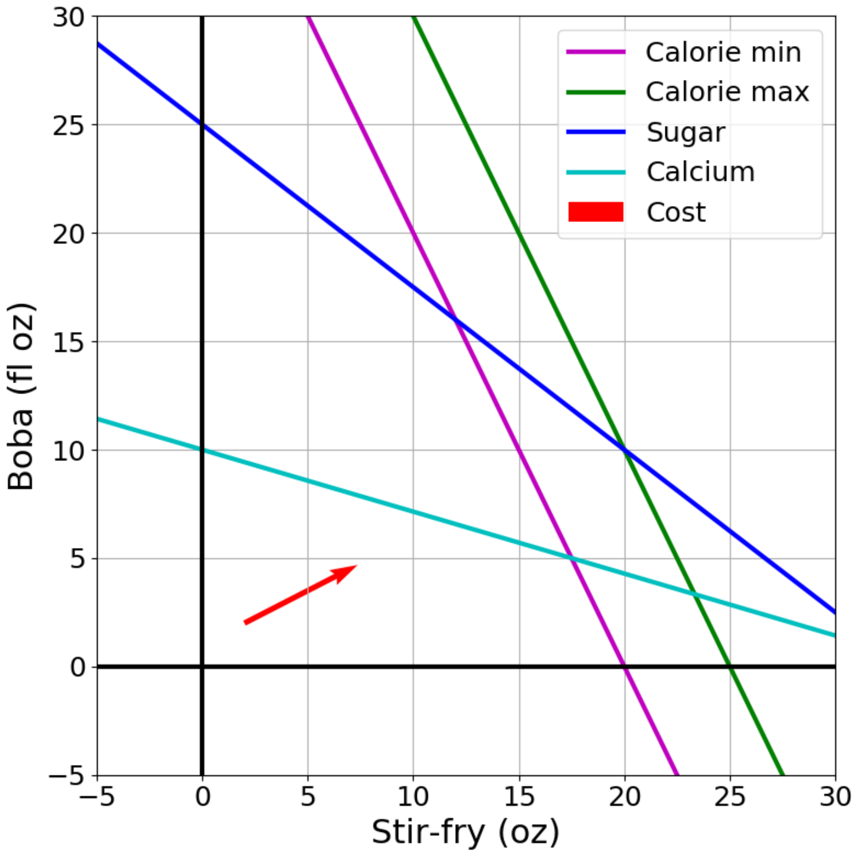
# What about higher dimensions?

## Problem Description

## Optimization Representation

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$
$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

## Graphical Representation



# “Marty, you’re not thinking fourth-dimensionally”


https://www.youtube.com/watch?v=CUcNM7OsdsY

# Shapes in higher dimensions

## How do these linear shapes extend to 3-D, N-D?

$$a_1\, x_1 + a_2\, x_2 = b_1$$

$$a_1\, x_1 + a_2\, x_2 \leq b_1$$

$$\begin{aligned} a_{1,1}\, x_1 + a_{1,2}\, x_2 &\leq b_1 \\ a_{2,1}\, x_1 + a_{2,2}\, x_2 &\leq b_2 \\ a_{3,1}\, x_1 + a_{3,2}\, x_2 &\leq b_3 \\ a_{4,1}\, x_1 + a_{4,2}\, x_2 &\leq b_4 \end{aligned}$$

# What are intersections in higher dimensions?

## How do these linear shapes extend to 3-D, N-D?

$$\min_{\mathbf{x}} \quad \mathbf{c}^{\top}\mathbf{x}$$
$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

$$A = \begin{bmatrix} -100 & -50 & \\ 100 & 50 & \\ 3 & 4 & \\ -20 & -70 & \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

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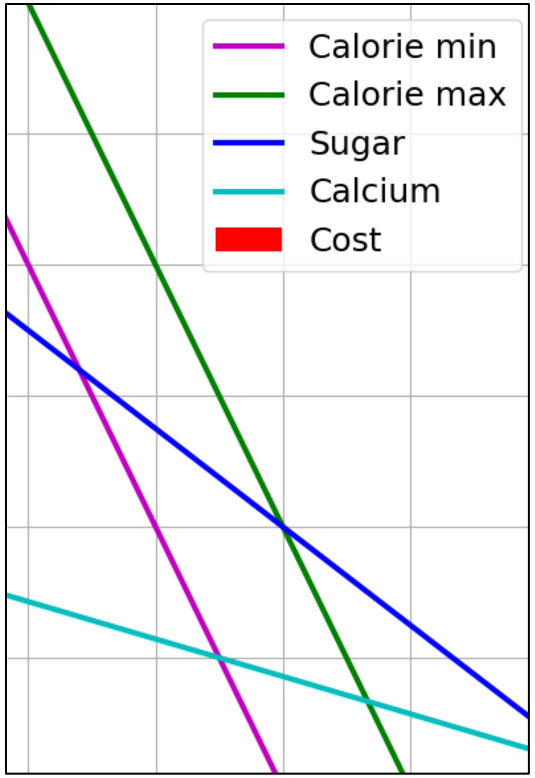

# How do we find intersections in higher dimensions?

## Still looking at subsets of $A$ matrix

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$
$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

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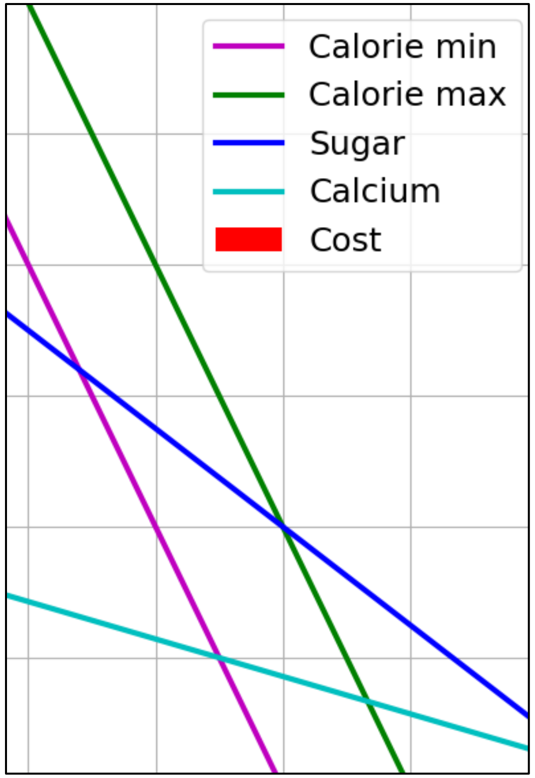


# Linear Programming

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

Healthy Squad Goals

- $2000 \leq$ Calories $\leq 2500$
- Sugar $\leq 100$ g
- Calcium $\geq 700$ mg

| Food | Cost | Calories | Sugar | Calcium |
|---|---|---|---|---|
| Stir-fry (per oz) | 1 | 100 | 3 | 20 |
| Boba (per fl oz) | 0.5 | 50 | 4 | 70 |

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

# Linear Programming → Integer Programming

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (bowls) and boba (glasses).

<u>Healthy Squad Goals</u>

- $2000 \leq$ Calories $\leq 2500$
- Sugar $\leq 100$ g
- Calcium $\geq 700$ mg

| Food | Cost | Calories | Sugar | Calcium |
|---|---|---|---|---|
| Stir-fry (per bowl) | 1 | 100 | 3 | 20 |
| Boba (per glass) | 0.5 | 50 | 4 | 70 |

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

# Linear Programming vs Integer Programming

Linear objective with linear constraints, but now with additional constraint that all values in $\boldsymbol{x}$ must be integers

$$\begin{aligned}\min_{\mathbf{x}}.\quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.}\quad & A\mathbf{x} \preceq \mathbf{b}\end{aligned}$$

$$\begin{aligned}\min_{\mathbf{x}}.\quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.}\quad & A\mathbf{x} \preceq \mathbf{b} \\ & \mathbf{x} \in \mathbb{Z}^N\end{aligned}$$

We could also do:

- Even more constrained: Binary Integer Programming
- A hybrid: Mixed Integer Linear Programming

Notation Alert!

# Integer Programming: Graphical Representation

Just add a grid of integer points onto our LP representation

$$\begin{aligned} \min_{\mathbf{x}}. \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} \preceq \mathbf{b} \\ & \mathbf{x} \in \mathbb{Z}^N \end{aligned}$$

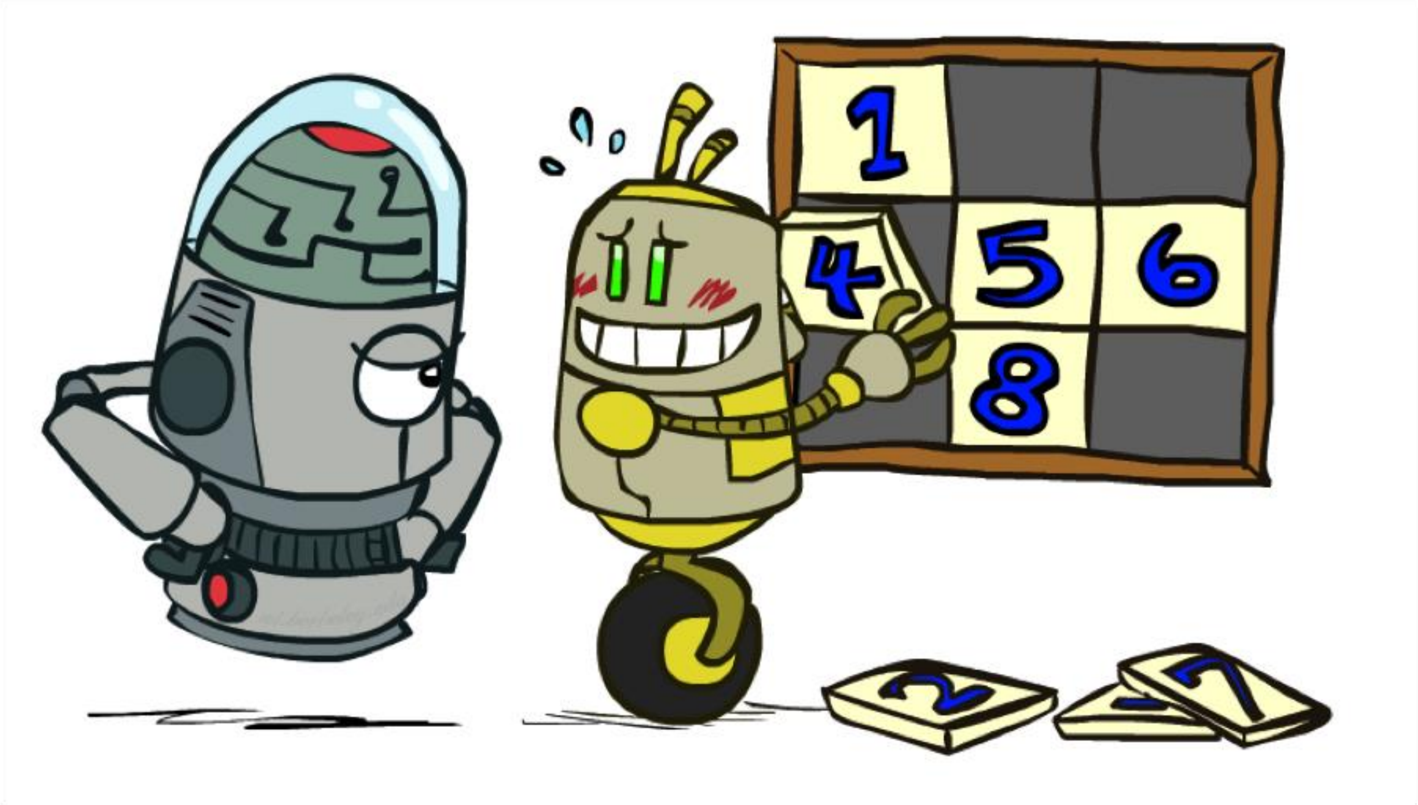
# Relaxation

## Relax IP to LP by dropping integer constraints

$$
\begin{aligned}
\min_{\mathbf{x}}. \quad & \mathbf{c}^T\mathbf{x} \\
\text{s.t.} \quad & A\mathbf{x} \preceq \mathbf{b}
\end{aligned}
$$

~~$\mathbf{x} \in \mathbb{Z}^N$~~

Remember heuristics?



# Poll 2:

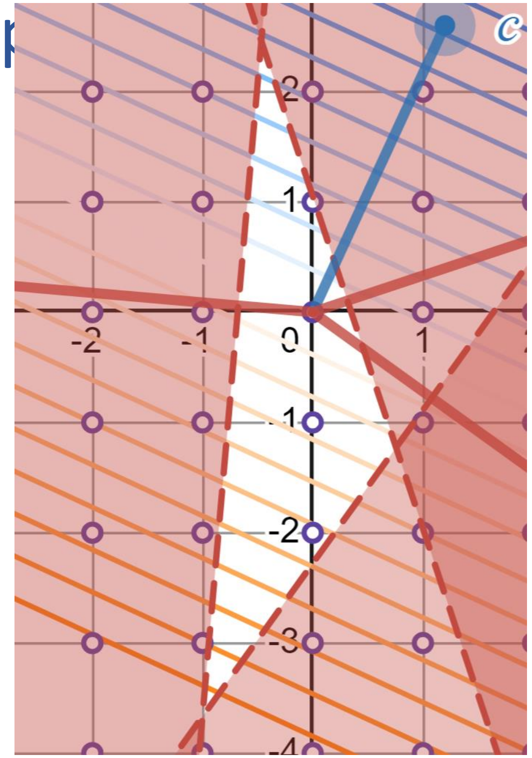
Let $y_{IP}^*$ be the optimal objective of an integer program $P$.

Let $\mathbf{x}_{IP}^*$ be an optimal point of an integer program $P$.

Let $y_{LP}^*$ be the optimal objective of the LP-relaxed version of $P$.

Let $\mathbf{x}_{LP}^*$ be an optimal point of the LP-relaxed version of $P$.

Assume that $P$ is a minimization p




$$y_{IP}^* = \min_{\mathbf{x}}. \quad \mathbf{c}^\top \mathbf{x}$$
$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$
$$\mathbf{x} \in \mathbb{Z}^N$$

$$y_{LP}^* = \min_{\mathbf{x}}. \quad \mathbf{c}^\top \mathbf{x}$$
$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

Which of the following are true?

A) $\mathbf{x}_{IP}^* = \mathbf{x}_{\text{LP}}^*$

B) $y_{IP}^* \leq y_{LP}^*$

C) $y_{IP}^* \geq y_{LP}^*$

# Poll 3:

True/False: It is sufficient to consider the integer points around the corresponding LP solution?

# Solving an IP

## Branch and Bound algorithm

1. Push LP solution of problem into priority queue,

   ordered by objective value of LP solution

2. Repeat:
   - If queue is empty, return IP is infeasible
   - Pop candidate solution $\mathbf{x}_{LP}^{\star}$ from priority queue ()
   - If $\mathbf{x}_{LP}^{\star}$ is all integer valued, we are done; return solution
   - Otherwise, select a coordinate $x_i$ that is not integer valued, and add two additional LPs to the priority queue:

     *Left branch*: Added constraint $x_i \leq floor(x_i)$

     *Right branch*: Added constraint $x_i \geq ceil(x_i)$

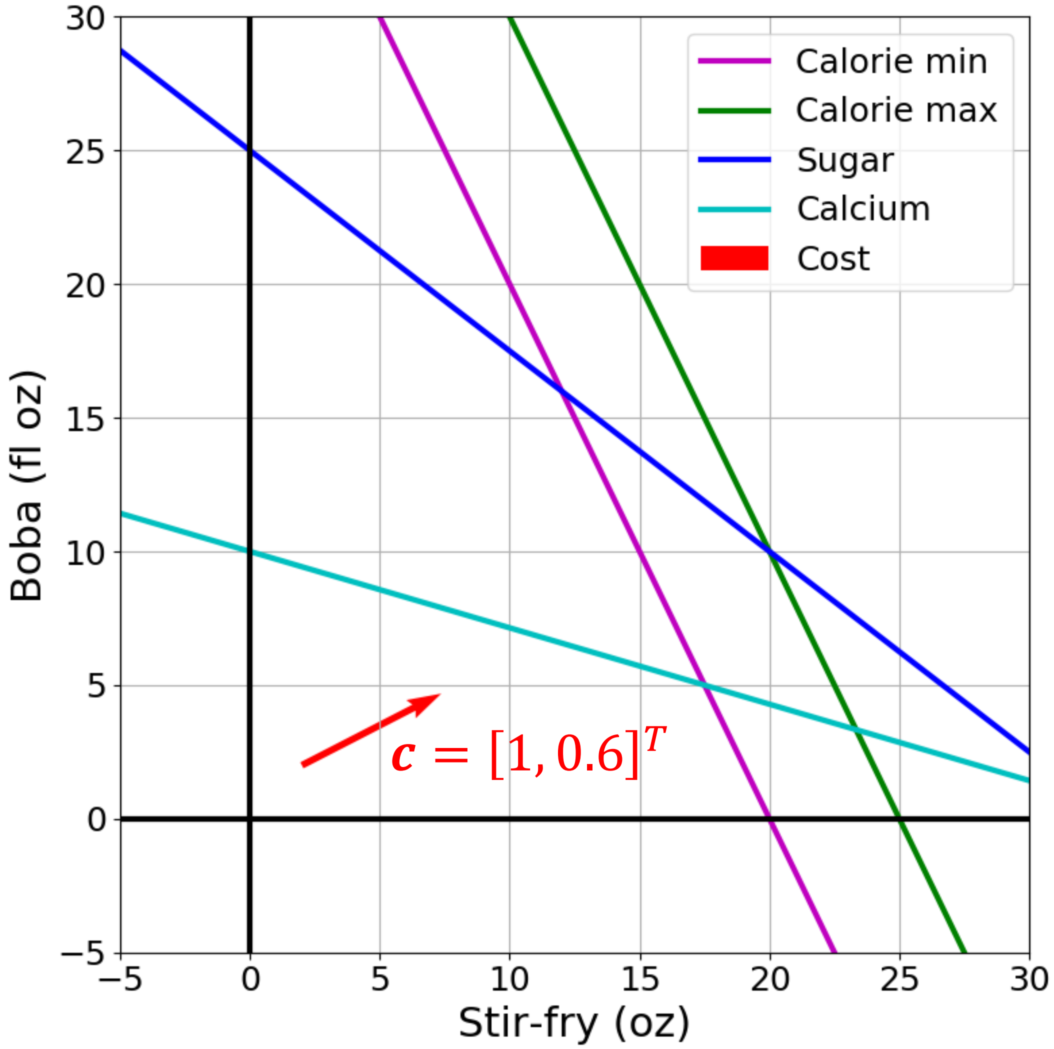
Note: Only add LPs to the queue if they are feasible

# Branch and Bound Example

$$\mathbf{x}^\star = (17.5, 5)$$
$$y^\star = 20.5$$

$x_1 \leq 17$

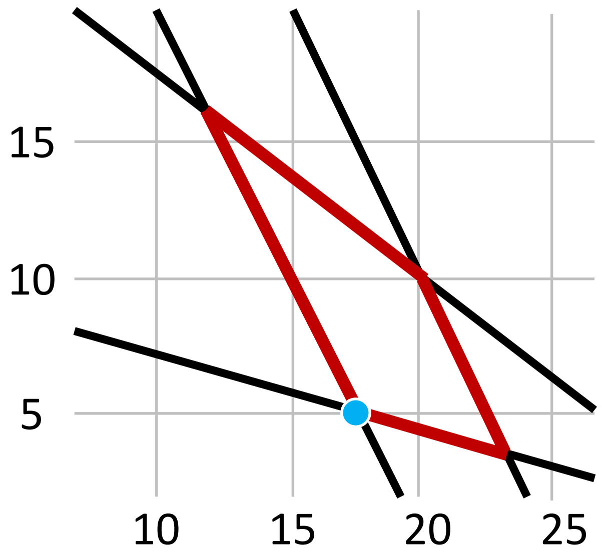
$x_1 \geq 18$



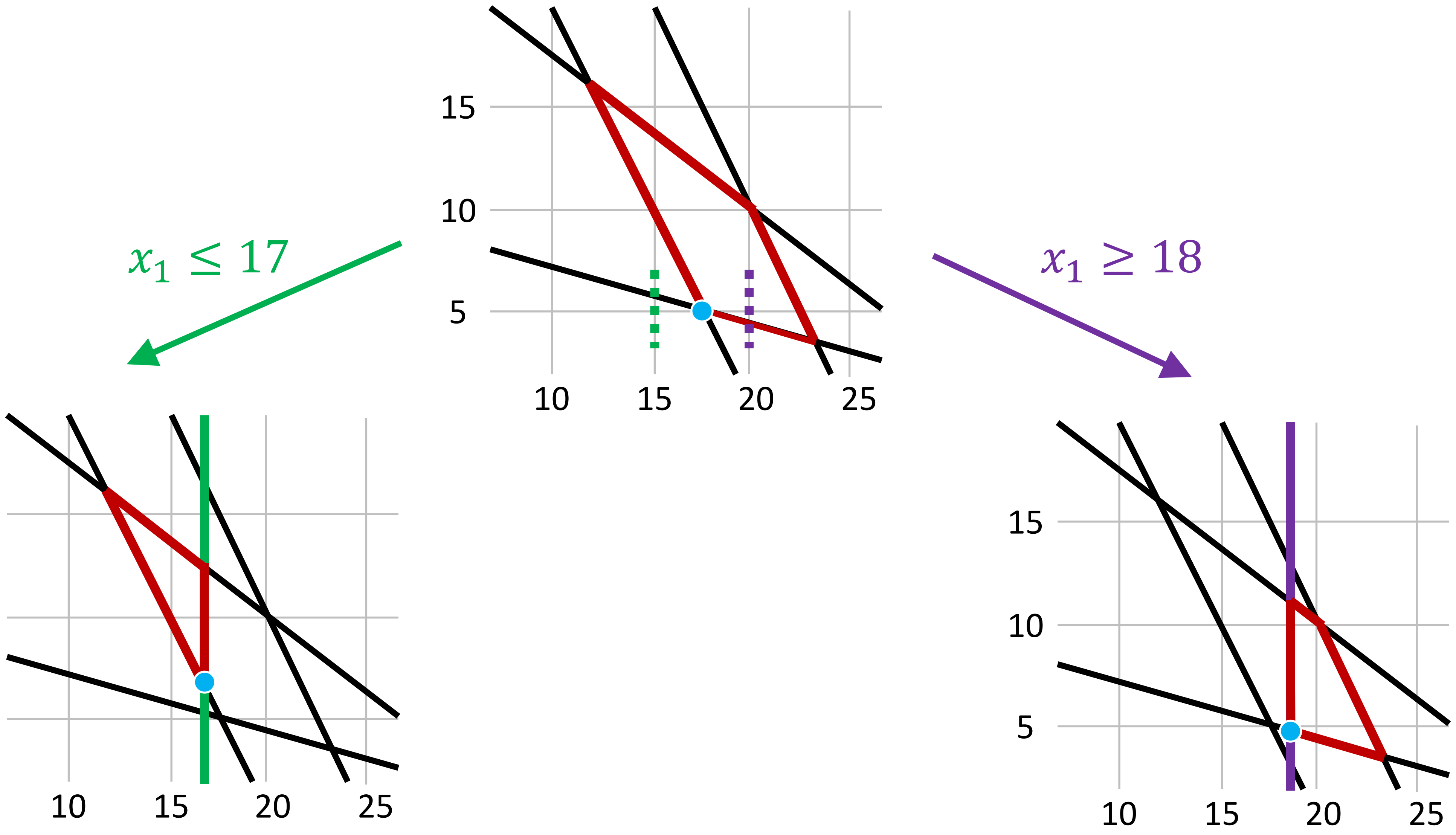

Priority Queue:

1. $\mathbf{x}^\star = (17.5, 5),\ y^\star = 20.5$

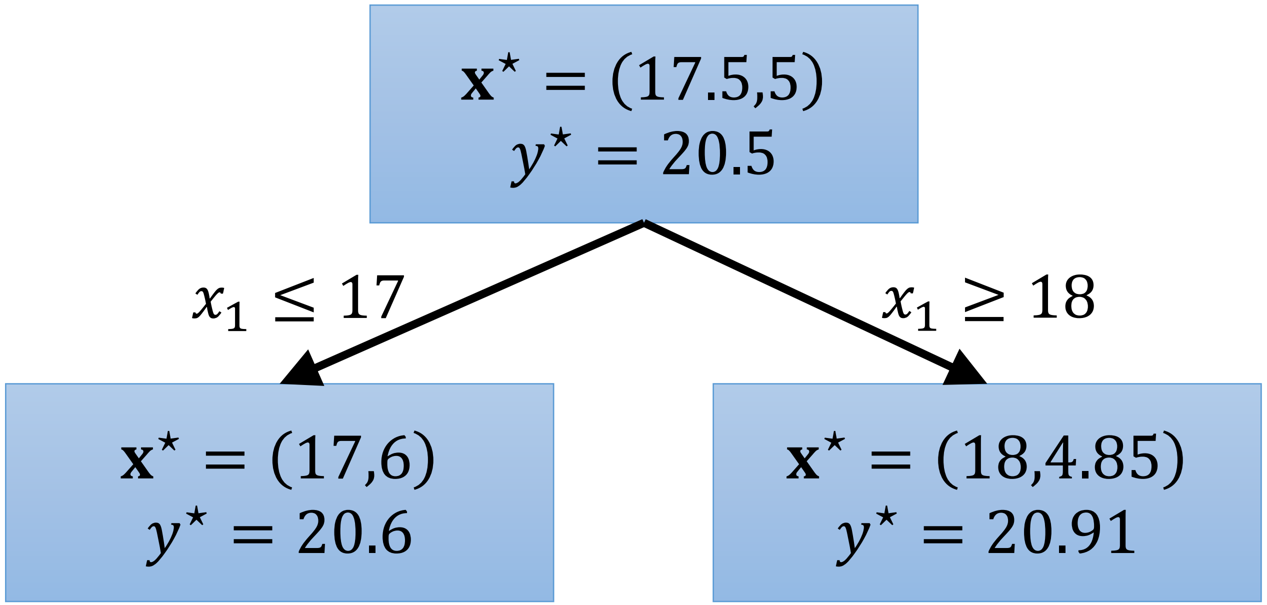
# Branch and Bound Example



# Branch and Bound Example



# Branch and Bound Example




Priority Queue:

1. $\mathbf{x}^\star = (17,6), \quad y^\star = 20.6$
2. $\mathbf{x}^\star = (18,4.85), \quad y^\star = 20.91$