

Warm-up: What to eat?

We are trying to stay healthy by finding the optimal amount of food to purchase.

We can choose the amount of **stir-fry** (ounce) and **boba** (fluid ounces).

Healthy Squad Goals

- $2000 \leq \text{Calories} \leq 2500$
- $\text{Sugar} \leq 100 \text{ g}$
- $\text{Calcium} \geq 700 \text{ mg}$

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

What is the cheapest way to stay “healthy” with this menu?

How much **stir-fry** (ounce) and **boba** (fluid ounces) should we buy?

Reminders

P1 and HW3 Online are both due Feb 6th

P2 and HW4 Written + Online will be released Feb 6th

HW4 due Feb 13th, P2 due Feb 20th (day after the midterm so budget time)

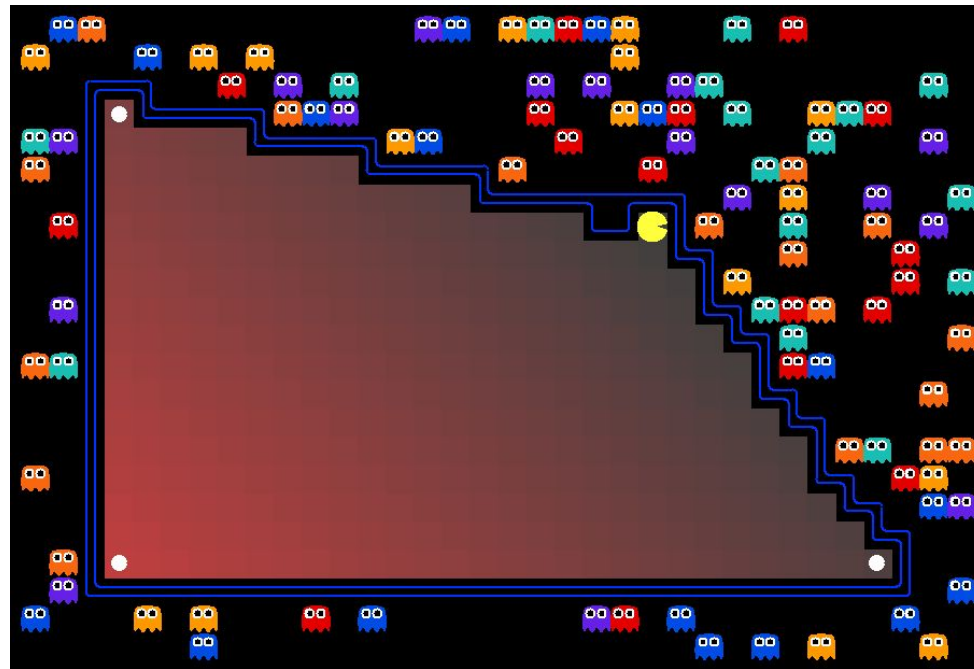
You must attend the recitation you chose to commit to on the recitation selection form for the rest of the semester

Midterm is Feb 19th so we will release practice material/review session details early next week

When asking for programming help on Piazza please format code in codeblock!

AI: Representation and Problem Solving

Linear Programming



Instructors: Tuomas Sandholm and Vincent Conitzer

Slide credits: CMU AI with drawings from <http://ai.berkeley.edu>

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JULES

This is the entire taco bell menu with calories - can you craft me an order that is about 1000 calories (give or take 100) that feels balanced and tasty?



CHATGPT

First, I'll need to extract the contents of the provided PDF to understand the Taco Bell menu and the calorie content of each item. Let's proceed with that.



CODE INTERPRETER

```
import PyPDF2

# Open the PDF file
with open("/mnt/data/premium.pdf",
"rb") as file:
    # Initialize PDF reader
    reader =
PyPDF2.PdfFileReader(file)
```

4:10



CHATGPT

Here's a crafted order totaling approximately 920 calories:

- Premium Roast Coffee - Hot
- Diet Pepsi® (20 fl oz)
- Diet Pepsi® (30 fl oz)
- Diet Dr Pepper® (20 fl oz)
- Brisk® Unsweetened No Lemon Iced Tea (20 fl oz)
- Diet Dr Pepper® (30 fl oz)
- Brisk® Unsweetened No Lemon Iced Tea (30 fl oz)
- Pepsi® Zero Sugar (20 fl oz)
- Mild Sauce Packet (V)
- Hot Sauce Packet (V)
- Breakfast Salsa Packet (V)
- Brisk® Unsweetened No Lemon Iced Tea (16 fl oz)
- Fire Sauce Packet (V)
- Diablo Sauce Packet (V)
- Mtn Dew® Zero Sugar (20 fl oz)
- Diet Dr Pepper® (16 fl oz)
- Diet Pepsi® (16 fl oz)
- Premium Roast Coffee - Iced
- Pepsi® Zero Sugar (16 fl oz)
- Premium Roast Coffee - Hot
- Mtn Dew® Zero Sugar (16 fl oz)
- Pepsi® Zero Sugar (30 fl oz)
- Premium Roast Coffee - Iced

Message

4:10

- Water
- Mad Tasty Yuzu Citrus Sparkling Water
- Mad Tasty Unicorn Tears Sparkling Water
- Mad Tasty Grapefruit Sparkling Water
- Mtn Dew® Baja Blast Zero Sugar (16 fl oz)
- Mtn Dew® Zero Sugar (30 fl oz)
- Mtn Dew® Baja Blast Zero Sugar (20 fl oz)
- Mad Tasty Watermelon Kiwi Sparkling Water
- Mtn Dew® Baja Blast Zero Sugar (30 fl oz)
- Red Sauce
- Vanilla Creamer
- Reduced-Fat Sour Cream
- G2 - Fruit Punch (16 fl oz)
- Hot Coffee with Creamer
- Hot Coffee with Creamer
- Iced Coffee with Creamer
- Iced Coffee with Creamer
- Black Beans (V)
- G2 - Fruit Punch (20 fl oz)
- Black Beans
- Brisk® Dragon Paradise Sparkling Iced Tea (16 fl oz)
- Nacho Cheese Sauce
- Vodka (80 proof)
- Tequila (80 proof)

Message

Optimization

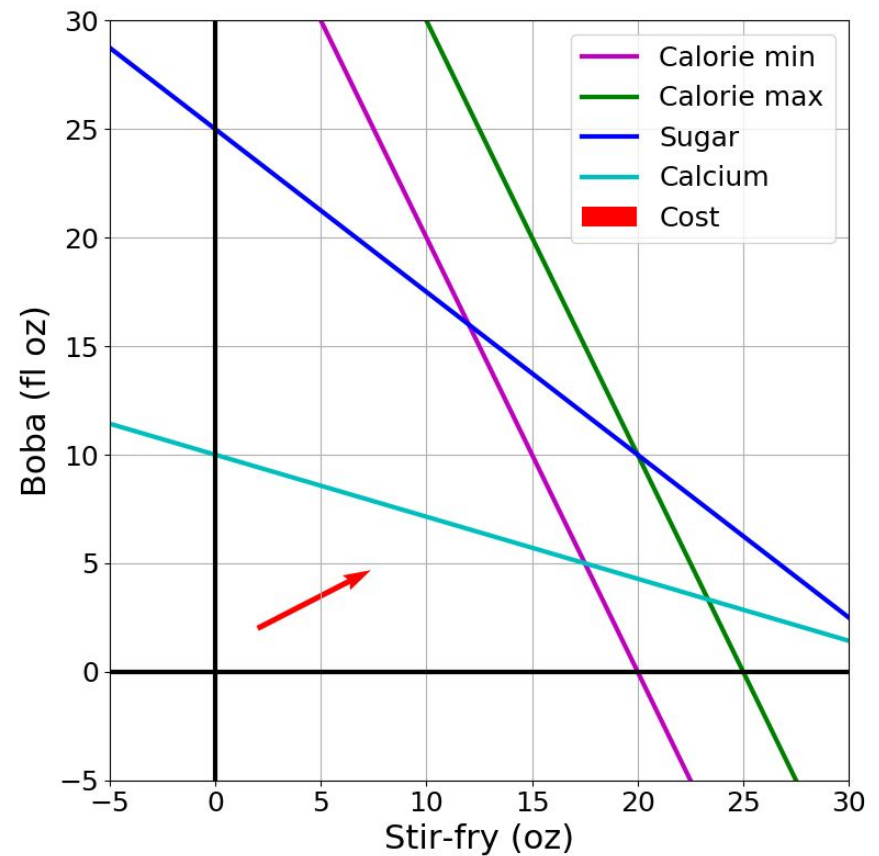
Problem
Description

Optimization
Representation

min. $\mathbf{c}'\mathbf{x}$
 \mathbf{x}

s.t. $A\mathbf{x} \leq \mathbf{b}$

Graphical Representation



Constraint Satisfaction Problems

Map coloring

Any \mathbf{x}
s.t. \mathbf{x} satisfies constraints
"such that"



Assume vectors are column vectors

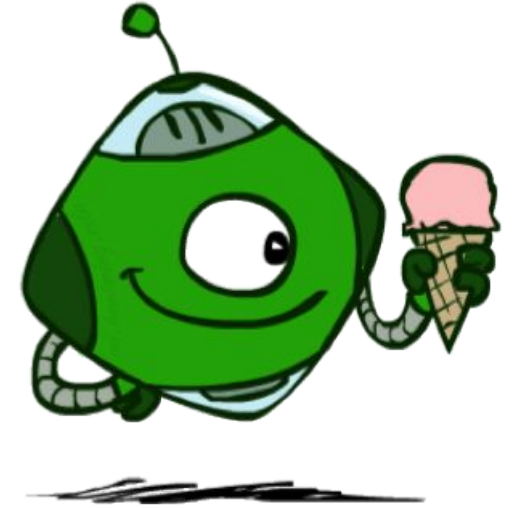
Notation Alert!

Optimization Formulation

Diet Problem

Any \mathbf{x}

s.t. \mathbf{x} satisfies constraints



Healthy Squad Goals

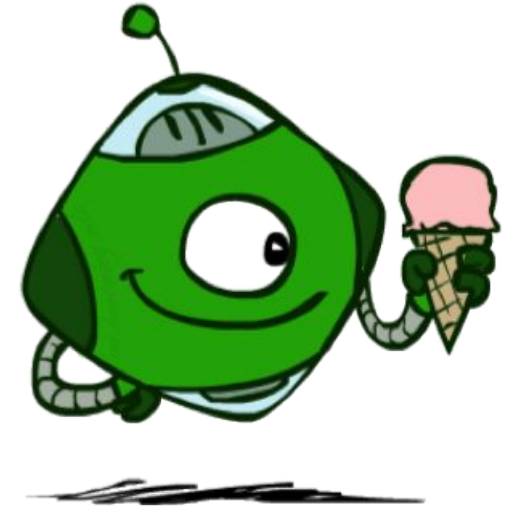
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Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

Optimization Formulation

Diet Problem

$$\begin{array}{lll} \min_{\mathbf{x}} & cost(\mathbf{x}) & \text{Objective function} \\ \text{s.t.} & \mathbf{x} \text{ satisfies} & \text{constraints} \end{array}$$



Healthy Squad Goals

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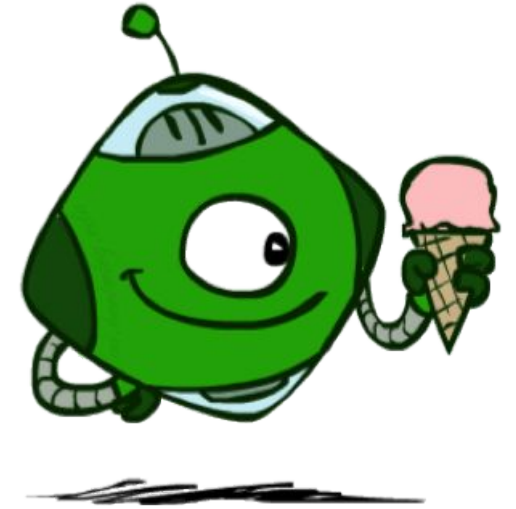
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Notation Alert!

Optimization Formulation

Diet Problem

$$\begin{array}{ll} \min_{\mathbf{x}} & \text{cost}(\mathbf{x}) \\ \text{s.t.} & \text{calories}(\mathbf{x}) \text{ contained} \\ & \text{sugar}(\mathbf{x}) \leq \text{sugarmax} \\ & \text{calcium}(\mathbf{x}) \geq \text{calciummin} \end{array}$$



Healthy Squad Goals

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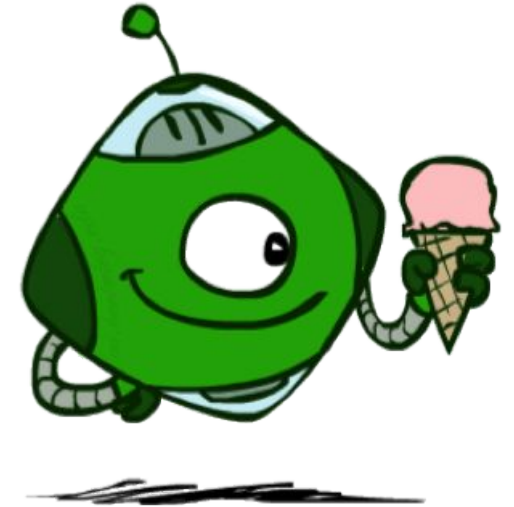
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Optimization Formulation

Diet Problem

$$\begin{aligned} \min_{x_1, x_2} \quad & 1 x_1 + 0.5 x_2 \\ \text{s.t.} \quad & 100 x_1 + 50 x_2 \geq 2000 \\ & 100 x_1 + 50 x_2 \leq 2500 \\ & 3 x_1 + 4 x_2 \leq 100 \\ & 20 x_1 + 70 x_2 \geq 700 \end{aligned}$$

Notation Alert!



Healthy Squad Goals

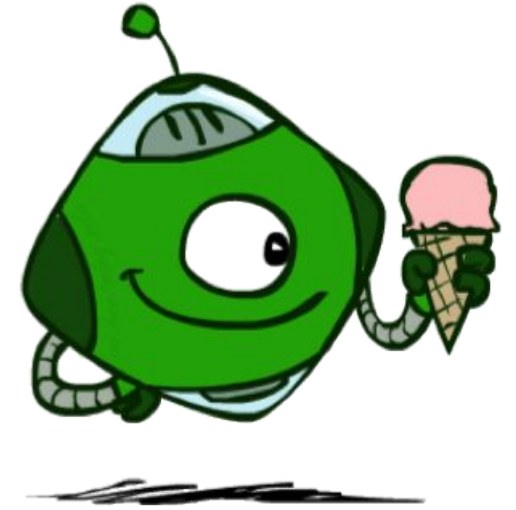
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Optimization Formulation

Diet Problem

$$\begin{aligned} \min_{x_1, x_2} \quad & c_1 x_1 + c_2 x_2 \\ \text{s.t.} \quad & a_{1,1} x_1 + a_{1,2} x_2 \geq b_1 \\ & a_{2,1} x_1 + a_{2,2} x_2 \leq b_2 \\ & a_{3,1} x_1 + a_{3,2} x_2 \leq b_3 \\ & a_{4,1} x_1 + a_{4,2} x_2 \geq b_4 \end{aligned}$$



Cost

$$\mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

Limit

$$\mathbf{b} = \begin{bmatrix} 2000 \\ 2500 \\ 100 \\ 700 \end{bmatrix} \begin{array}{l} \text{Calorie min} \\ \text{Calorie max} \\ \text{Sugar} \\ \text{Calcium} \end{array}$$

$$A = \begin{array}{cc} & \begin{array}{l} \text{Stir-fry} \\ \text{Boba} \end{array} \\ \begin{bmatrix} 100 & 50 \\ 100 & 50 \\ 3 & 4 \\ 20 & 70 \end{bmatrix} \end{array}$$

Notation Alert!

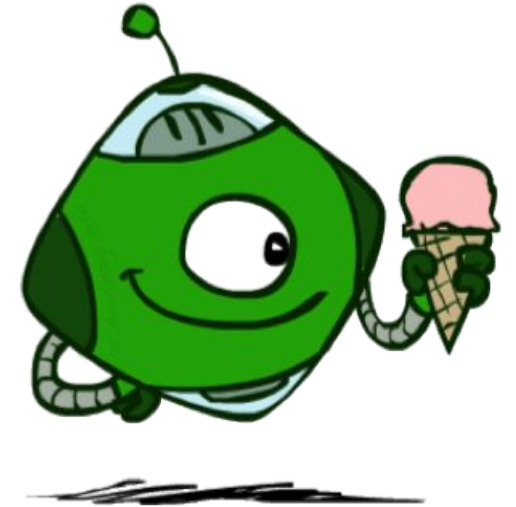
Optimization Formulation

Diet Problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} \quad & a_{1,1} x_1 + a_{1,2} x_2 \geq b_1 \\ & a_{2,1} x_1 + a_{2,2} x_2 \leq b_2 \\ & a_{3,1} x_1 + a_{3,2} x_2 \leq b_3 \\ & a_{4,1} x_1 + a_{4,2} x_2 \geq b_4 \end{aligned}$$

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$$\mathbf{b} = \begin{array}{c} \text{Limit} \\ \begin{bmatrix} 2000 \\ 2500 \\ 100 \\ 700 \end{bmatrix} \\ \begin{array}{l} \text{Calorie min} \\ \text{Calorie max} \\ \text{Sugar} \\ \text{Calcium} \end{array} \end{array}$$



Cost

$$\mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

Notation Alert!

Optimization Formulation

Diet Problem

$$\min_{\mathbf{x}} \quad \mathbf{c}^T \mathbf{x}$$

$$\text{s.t.} \quad -a_{1,1} x_1 - a_{1,2} x_2 \leq -b_1$$

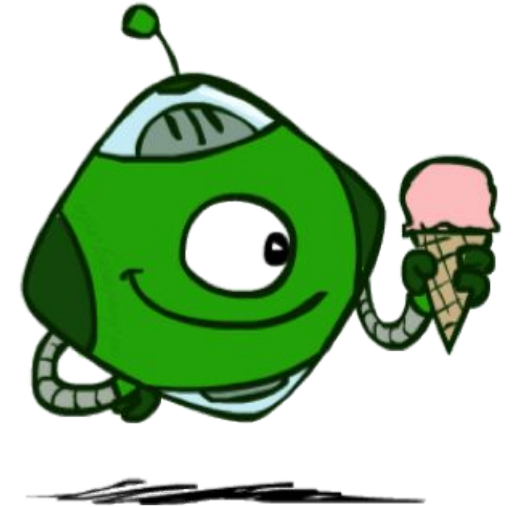
$$a_{2,1} x_1 + a_{2,2} x_2 \leq b_2$$

$$a_{3,1} x_1 + a_{3,2} x_2 \leq b_3$$

$$-a_{4,1} x_1 - a_{4,2} x_2 \leq -b_4$$

$$A = \begin{array}{cc} & \begin{array}{c} \text{Stir-fry} \\ \text{Boba} \end{array} \\ \begin{bmatrix} 100 & 50 \\ 100 & 50 \\ 3 & 4 \\ 20 & 70 \end{bmatrix} \end{array}$$

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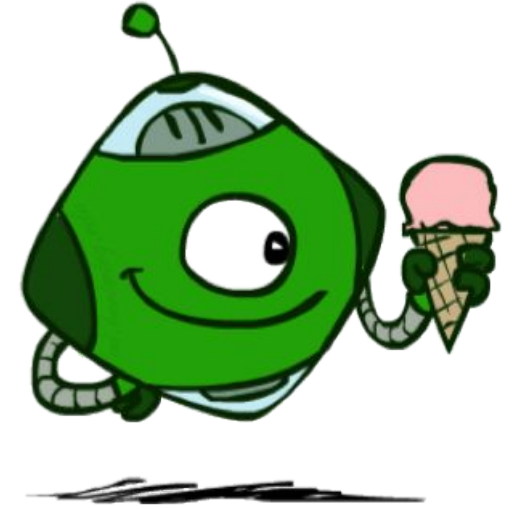
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Optimization Formulation

Diet Problem

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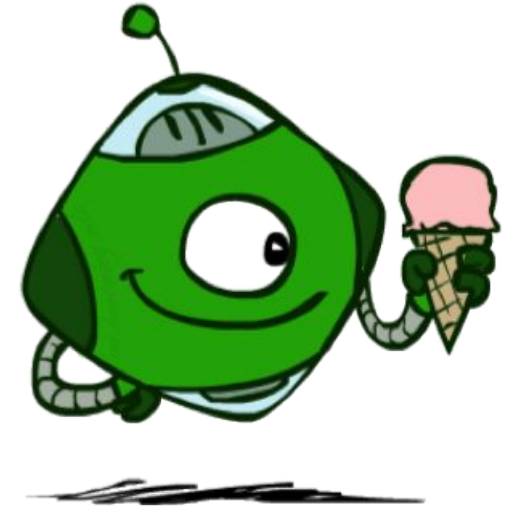
Limit

$$\mathbf{A} = \begin{array}{cc} \text{Stir-fry} & \text{Boba} \\ \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} & \mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix} \end{array} \begin{array}{l} \text{Calorie min} \\ \text{Calorie max} \\ \text{Sugar} \\ \text{Calcium} \end{array}$$

Optimization Formulation

Diet Problem

$$\begin{array}{ll} \min_{\mathbf{x}} & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq \mathbf{b} \end{array}$$



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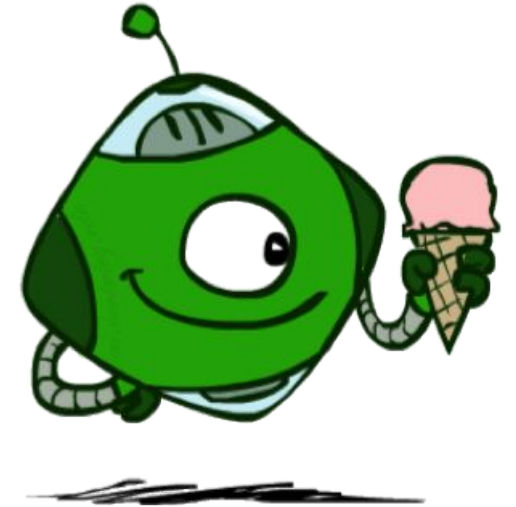
Poll 1

What has to increase to add more nutrition constraints?

$$\begin{array}{ll} \min_{\mathbf{x}} & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq \mathbf{b} \end{array}$$

Select all that apply

- A) length \mathbf{x}
- B) length \mathbf{c}
- C) height A



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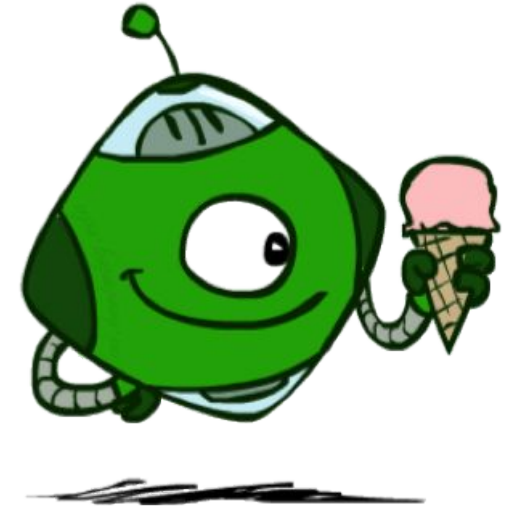
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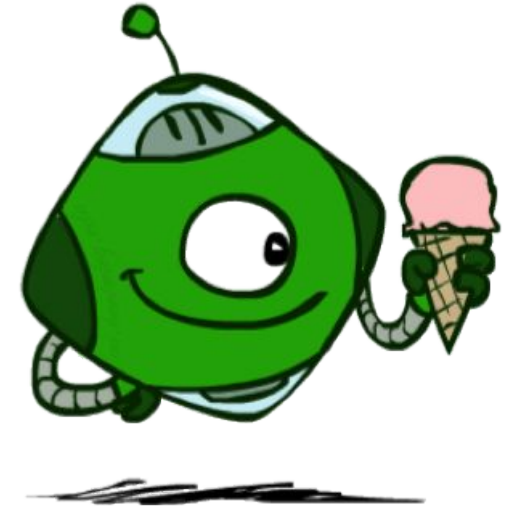
Poll 2

What has to increase to add more menu items?

$$\begin{array}{ll} \min_{\mathbf{x}} & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq \mathbf{b} \end{array}$$

Select all that apply

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- C) height A



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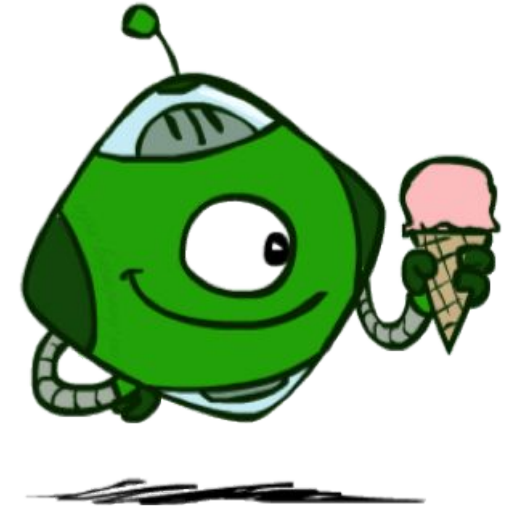
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$$\mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$



Poll 3

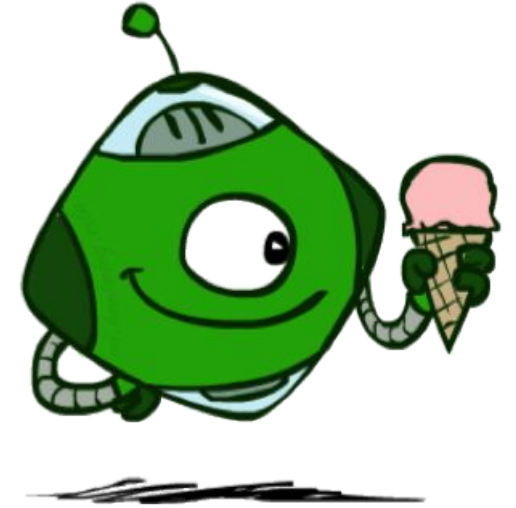
If $A \in \mathbb{R}^{M \times N}$, which of the following also equals N ?

$$\begin{array}{ll} \min_{\mathbf{x}} & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq \mathbf{b} \end{array}$$

Select all that apply

- A) length \mathbf{x}
- B) length \mathbf{c}
- C) length \mathbf{b}

Notation Alert!



Linear Programming

Linear objective with linear constraints

$$\begin{array}{ll} \min_{\mathbf{x}} & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq \mathbf{b} \end{array}$$

As opposed to general optimization

$$\begin{array}{ll} \min_{\mathbf{x}} & f_0(\mathbf{x}) \\ \text{s.t.} & f_i(\mathbf{x}) \leq 0, \quad i = 1 \dots M \\ & \mathbf{a}_i^T \mathbf{x} = \mathbf{b}_i, \quad i = 1 \dots P \end{array}$$

Linear Programming

Different formulations

Inequality form

$$\begin{array}{ll} \min. & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} & \mathbf{Ax} \leq \mathbf{b} \end{array}$$

General form

$$\begin{array}{ll} \min. & \mathbf{c}^\top \mathbf{x} + \mathbf{d} \\ \text{s.t.} & \mathbf{Gx} \leq \mathbf{h} \\ & \mathbf{Ax} = \mathbf{b} \end{array}$$

Standard form

$$\begin{array}{ll} \min. & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array}$$

Important to pay attention to form!

Can switch between formulations!

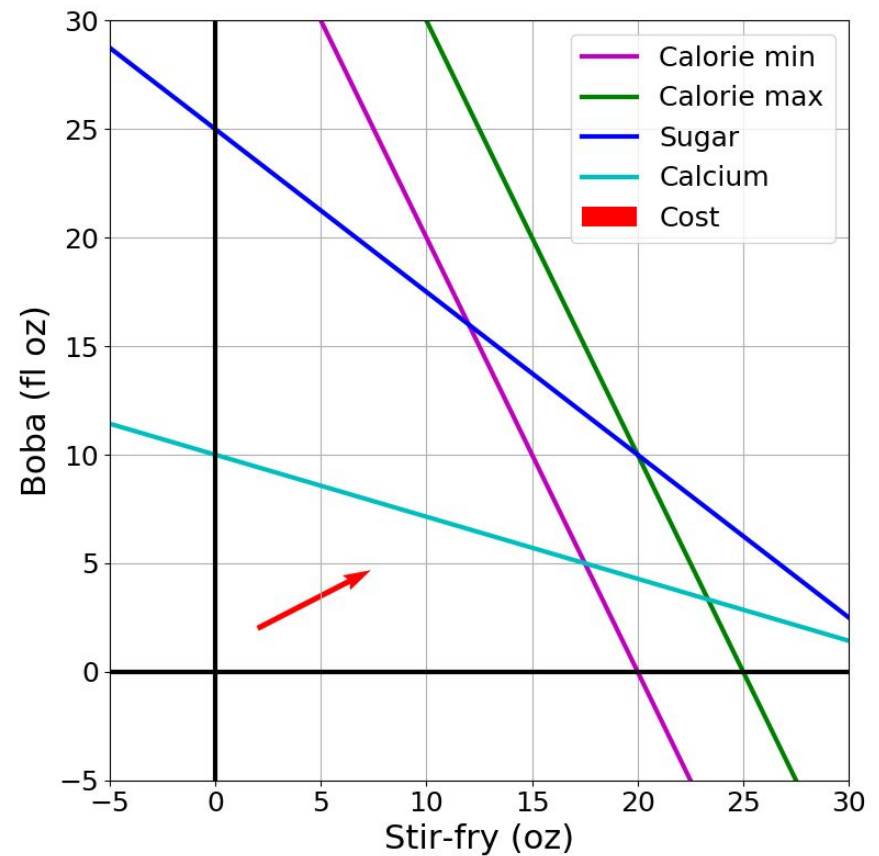
Optimization

Problem
Description

Optimization
Representation

$$\begin{array}{ll} \min & \mathbf{c}'\mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq \mathbf{b} \end{array}$$

Graphical Representation



Graphical Representation

Geometry / Algebra I Quiz

What shape does this inequality represent?

$$a_1 x_1 + a_2 x_2 \leq b_1$$

Graphics Representation

Geometry / Algebra I Quiz

What shape do these represent?

1. $a_1 x_1 + a_2 x_2 = b_1$

2. $a_1 x_1 + a_2 x_2 \leq b_1$

3. $a_{1,1} x_1 + a_{1,2} x_2 \leq b_1$
 $a_{2,1} x_1 + a_{2,2} x_2 \leq b_2$
 $a_{3,1} x_1 + a_{3,2} x_2 \leq b_3$
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Feasible region:

All points x that satisfy the constraints

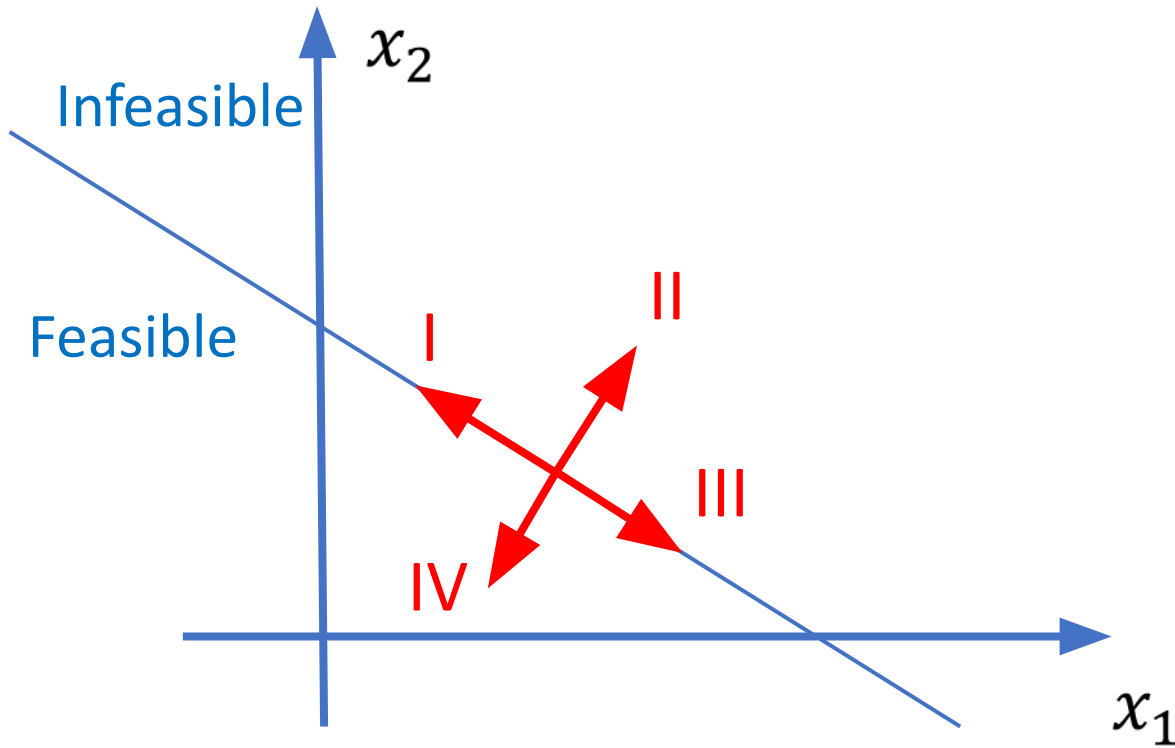
Poll 4

What is the relationship between the half plane:

$$a_1 x_1 + a_2 x_2 \leq b_1$$

and the vector:

$$[a_1, a_2]^T$$



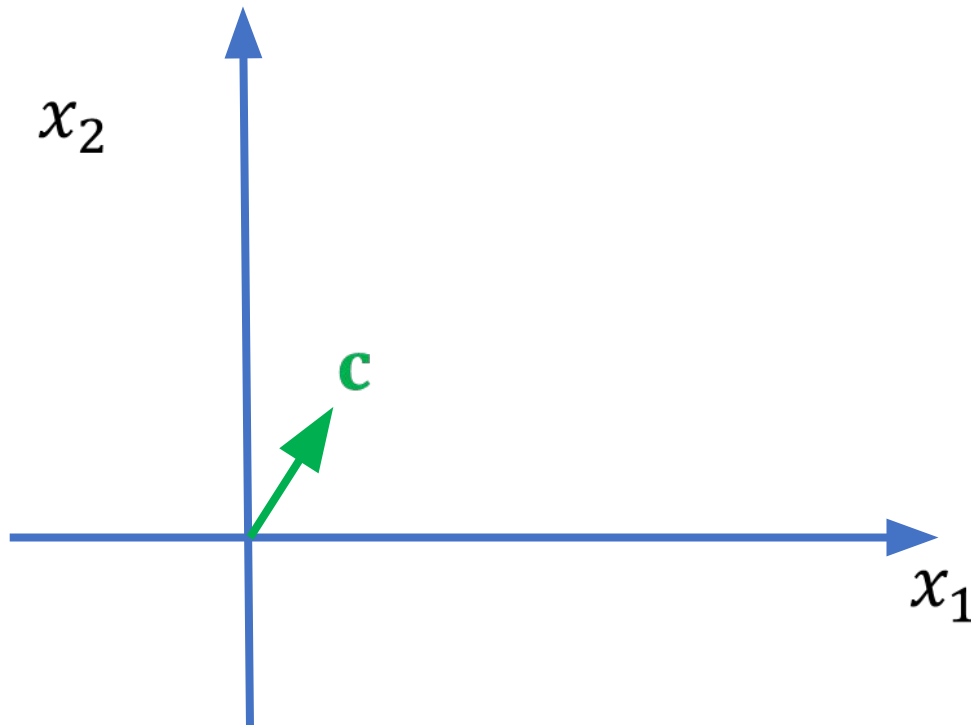
Cost Contours

Given the cost vector $[c_1, c_2]^T$ where will

$$\mathbf{c}^T \mathbf{x} = 0 ?$$

$$\mathbf{c}^T \mathbf{x} = 1 ?$$

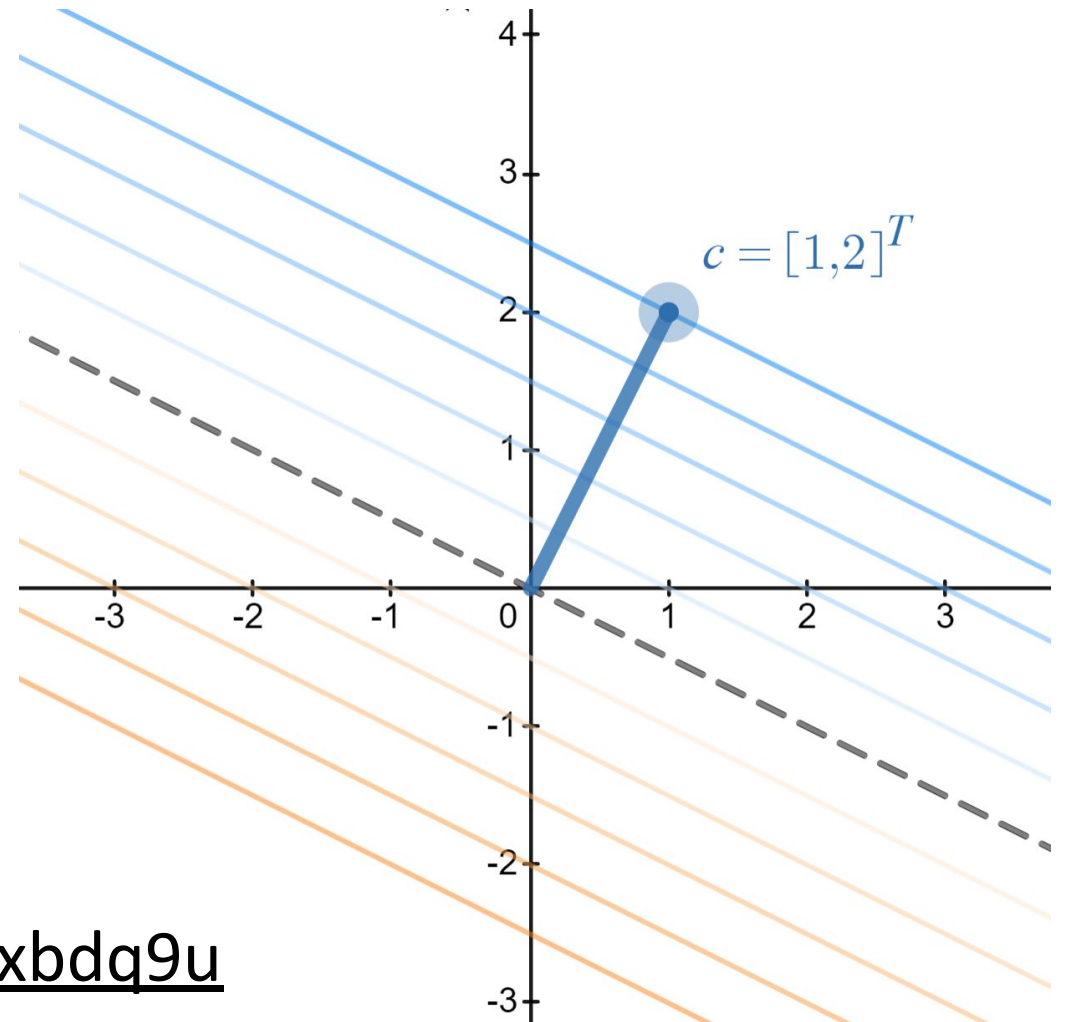
$$\mathbf{c}^T \mathbf{x} = 2 ?$$



Poll 5

As the magnitude of c increases, the distance between

- A) Increases
- B) Decreases



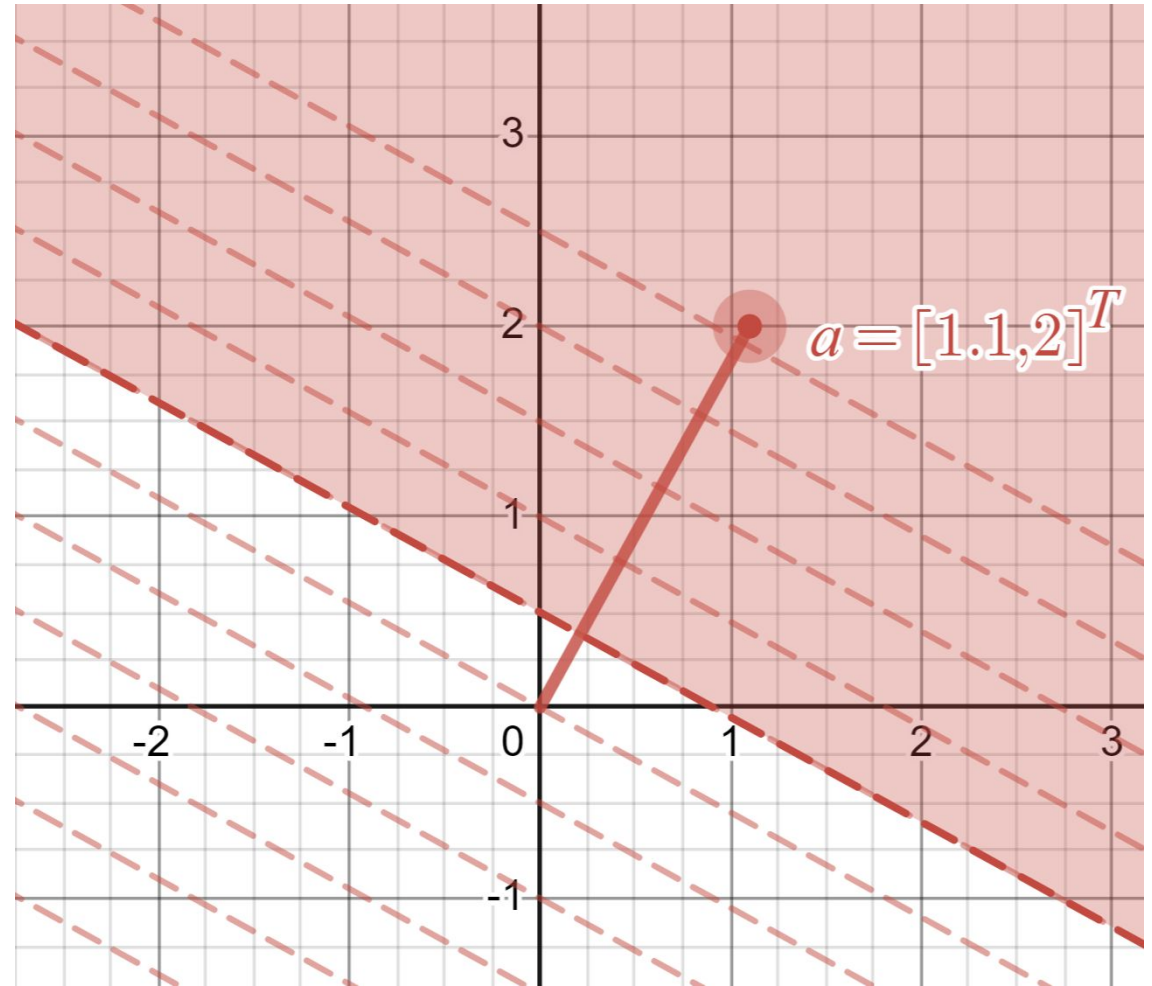
<https://www.desmos.com/calculator/8d9kxbdq9u>

Graphics Representation

Geometry / Algebra I Quiz

What shape do these represent?

1. $a_1 x_1 + a_2 x_2 = b_1$
2. $a_1 x_1 + a_2 x_2 \leq b_1$
- 3.



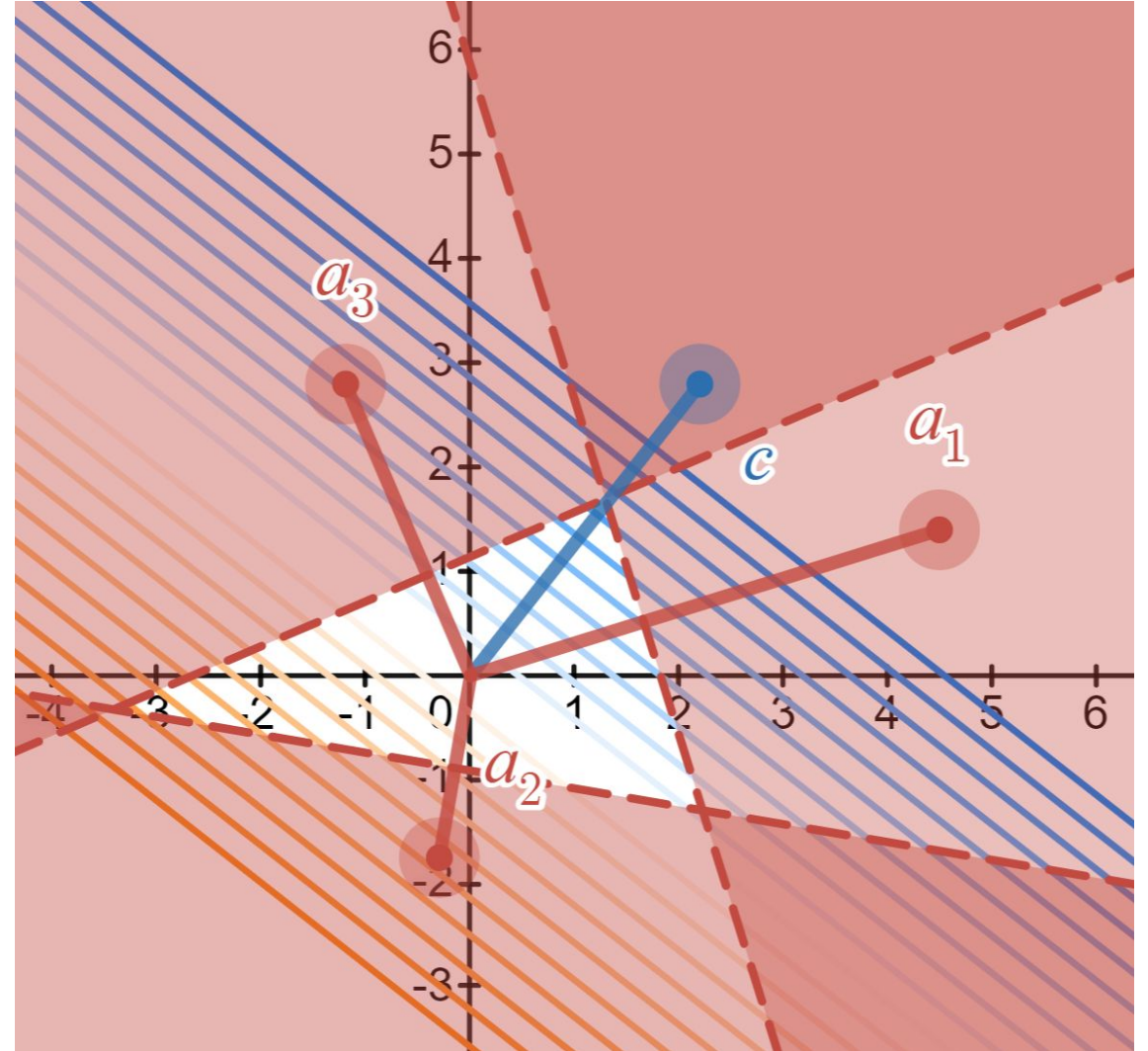
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Graphics Representation

Geometry / Algebra I Quiz

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3. $a_{1,1} x_1 + a_{1,2} x_2 \leq b_1$
 $a_{2,1} x_1 + a_{2,2} x_2 \leq b_2$
 $a_{3,1} x_1 + a_{3,2} x_2 \leq b_3$
 $a_{4,1} x_1 + a_{4,2} x_2 \leq b_4$



<https://www.desmos.com/calculator/plp1thgsbh>