The Profit Maximizing Cutoff in Observable Queues with State Dependent Pricing

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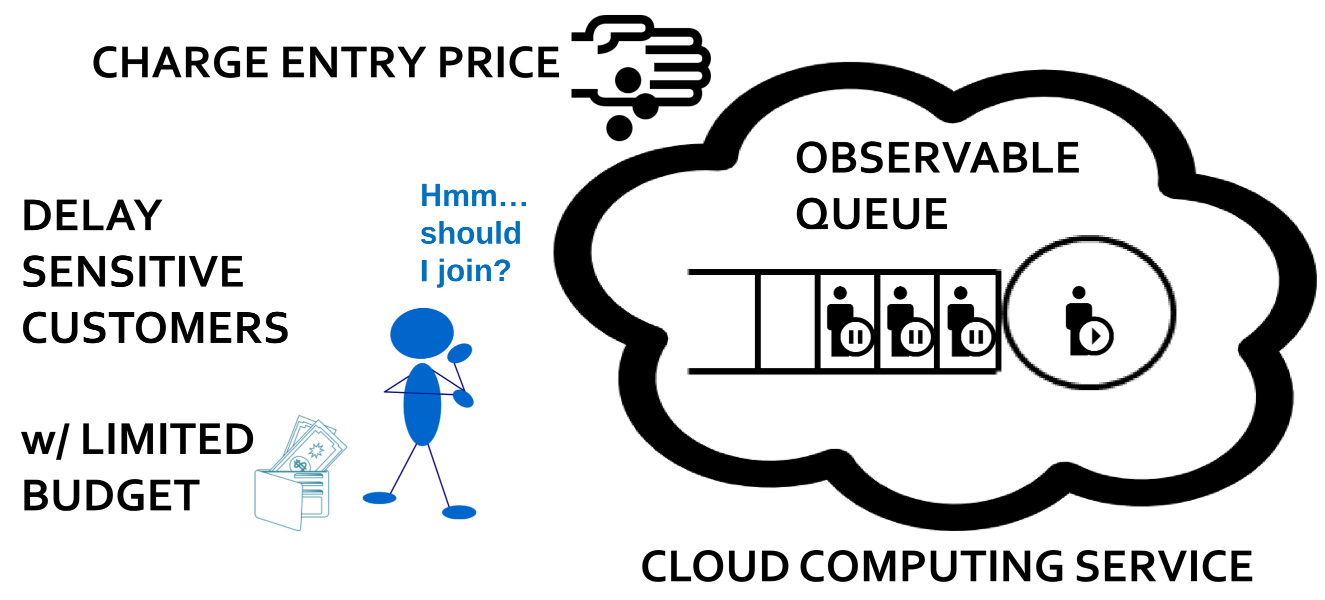
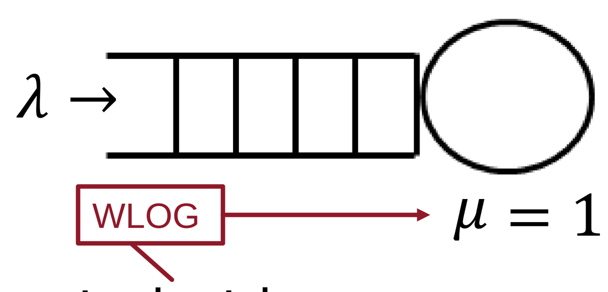
1. Background & Motivation



GOAL: Choose prices and admissions policy to maximize profits

2. The Queueing Model

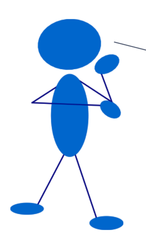
- $M/M/1$ queue



- All customers are identical with parameters:

V — Value for receiving service

$v = 1$ — Waiting cost per unit time



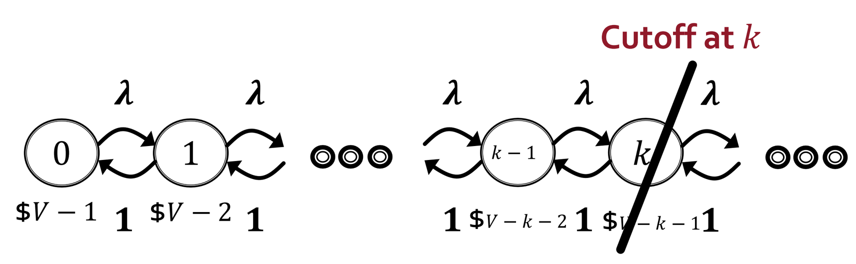
I wait only if: $(n+1) + p \leq V$

p: price customers must pay

n : # of customers in (state of) system

3. How to Price?

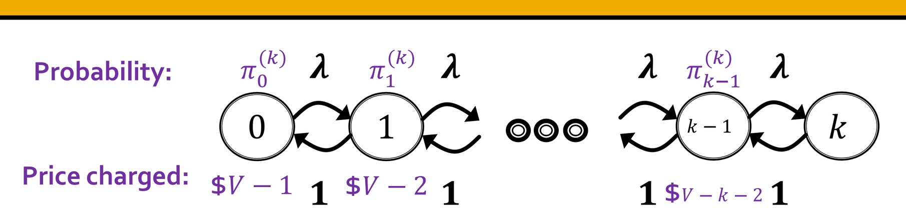
Charge as much as customers are willing to pay in each state: $p(n) = V - (n+1)$



But… we use an explicit early cutoff at state k. *Why?*

Answer: Cutoff ➔ shorter queue ➔ high prices more often

So how much do we make?



Earning rate = (arrival rate) x (average price charged)

$$R(k,\lambda,V) = \lambda \cdot \sum_{n=0}^{k-1} \{\pi_n^{(k)} \cdot p(n)\}$$

$\pi_n^{(k)}$: Probability that a new arrival sees state n, given a cutoff of k

$p(n) = V - (n+1)$

$$R(k,\lambda,V) = \left(\frac{\lambda}{1-\lambda^{k+1}} \cdot \frac{1}{1-\lambda}\right)\left((1-\lambda^k)(1-\lambda)V + \lambda^k(1+k-k\lambda) - 1\right)$$

4. Problem Statement & Approach

- **INPUTS:** fixed λ and V
- **OBJECTIVE:** Find cutoff k^* to maximize the earning rate $R(k,\lambda,V)$
- **TECHNIQUE:** use a "discrete derivative"
  - Define $\Delta_x f(x) \equiv f(x+1) - f(x)$
  - Δ_x is the *forward difference operator*
  - Solve $\Delta_k[R(k,\lambda,V)] = 0$ for $k \in \mathbf{R}$
  - Obtain optimal cutoff $k^* = [k]$

5. The Analysis

- Solve: $R(k+1,\lambda,V) - R(k,\lambda,V) = 0$
- Define: $\tilde{k} \equiv k+2$ and $G(\lambda,V) \equiv (1-\lambda)V + \frac{1}{1-\lambda}$
- **Question**: How do we solve this for $\tilde{k}$?

$$\ln(\lambda)\left(G(\lambda,V) - \tilde{k}\right)e^{\ln(\lambda)(G(\lambda,V)-\tilde{k})} = \frac{\ln(\lambda)\lambda^{G(V,\lambda)}}{1-\lambda}$$

Call this constant C

Call both of these x

- **Now solve:** $xe^x = C$ for x
- **Answer**: Lambert W (product log) function

$W(x)e^{W(x)} = x$ Or equivalently ⇔ $W(xe^x) = x$

6. Conclusions

$$k^* = \text{ceil}\left((1-\lambda)V + \frac{1}{1-\lambda} - \frac{1}{\ln\lambda} \cdot W\left(\frac{\ln\lambda \cdot \lambda^{(1-\lambda)V + \frac{1}{1-\lambda}}}{1-\lambda}\right) - 2\right)$$

So what can the closed form do for us?

Asymptotic Results

For fixed $\lambda < 1$, as $V \to \infty$: $k^* \sim (1-\lambda)V$

When $\lambda = 1$, as $V \to \infty$: $k^* \sim \sqrt{2V}$

For fixed $\lambda > 1$, as $V \to \infty$: $k^* \sim \log_\lambda(V)$

Is the cutoff of practical use?

When $\lambda > 1$: high cutoff ➔ huge losses!