We can now establish a remarkable result, known as the *revenue equivalence theorem.*[35]

Proposition 23.D.3: (*The Revenue Equivalence Theorem*) Consider an auction setting with I risk-neutral buyers, in which buyer i's valuation is drawn from an interval $[\underline{\theta}_i, \bar{\theta}_i]$ with $\underline{\theta}_i \neq \bar{\theta}_i$ and a strictly positive density $\phi_i(\cdot) > 0$, and in which buyers' types are statistically independent. Suppose that a given pair of Bayesian Nash equilibria of two different auction procedures are such that for every buyer i: (i) For each possible realization of $(\theta_1, \ldots, \theta_I)$, buyer i has an identical probability of getting the good in the two auctions; and (ii) Buyer i has the same expected utility level in the two auctions when his valuation for the object is at its lowest possible level. Then these equilibria of the two auctions generate the same expected revenue for the seller.

Proof: By the revelation principle, we know that the social choice function that is (indirectly) implemented by the equilibrium of any auction procedure must be Bayesian incentive compatible. Thus, we can establish the result by showing that if two Bayesian incentive compatible social choice functions in this auction setting have the same functions $(y_1(\theta), \ldots, y_I(\theta))$ and the same values of $(U_1(\underline{\theta}_1), \ldots, U_I(\underline{\theta}_I))$ then they generate the same expected revenue for the seller.

To show this, we derive an expression for the seller's expected revenue from an arbitrary Bayesian incentive compatible mechanism. Note, first, that the seller's expected revenue is equal to $\sum_{i=1}^{I} E[-t_i(\theta)]$. Now,

$$\begin{aligned} E[-t_i(\theta)] &= E_{\theta_i}[-\bar{t}_i(\theta_i)] \\ &= \int_{\underline{\theta}_i}^{\bar{\theta}_i} [\bar{y}_i(\theta_i)\theta_i - U_i(\theta_i)]\phi_i(\theta_i)\, d\theta_i \\ &= \int_{\underline{\theta}_i}^{\bar{\theta}_i} \left(\bar{y}_i(\theta_i)\theta_i - U_i(\underline{\theta}_i) - \int_{\underline{\theta}_i}^{\theta_i} \bar{y}_i(s)\, ds\right)\phi_i(\theta_i)\, d\theta_i \\ &= \left[\int_{\underline{\theta}_i}^{\bar{\theta}_i} \left(\bar{y}_i(\theta_i)\theta_i - \int_{\underline{\theta}_i}^{\theta_i} \bar{y}_i(s)\, ds\right) \phi_i(\theta_i)\, d\theta_i\right] - U_i(\underline{\theta}_i). \end{aligned}$$

Moreover, integration by parts implies that

$$\begin{aligned} \int_{\underline{\theta}_i}^{\bar{\theta}_i} \left(\int_{\underline{\theta}_i}^{\theta_i} \bar{y}_i(s)\, ds\right)\phi_i(\theta_i)\, d\theta_i &= \left(\int_{\underline{\theta}_i}^{\bar{\theta}_i} \bar{y}_i(\theta_i)\, d\theta_i\right) - \left(\int_{\underline{\theta}_i}^{\bar{\theta}_i} \bar{y}_i(\theta_i)\Phi_i(\theta_i)\, d\theta_i\right) \\ &= \int_{\underline{\theta}_i}^{\bar{\theta}_i} \bar{y}_i(\theta_i)(1 - \Phi_i(\theta_i))\, d\theta_i. \end{aligned}$$

Substituting, we see that

$$E[-\bar{t}_i(\theta_i)] = \left[\int_{\underline{\theta}_i}^{\bar{\theta}_i} \bar{y}_i(\theta_i)\left(\theta_i - \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)}\right)\phi_i(\theta_i)\, d\theta_i\right] - U_i(\underline{\theta}_i), \qquad (23.D.14)$$

35. Versions of the revenue equivalence theorem have been derived by many authors; see McAfee and McMillan (1987) and Milgrom (1987) for references as well as for a further discussion of the result.

or, equivalently,

$$E[-\bar{t}_i(\theta_i)] = \left[\int_{\underline{\theta}_1}^{\bar{\theta}_1} \cdots \int_{\underline{\theta}_I}^{\bar{\theta}_I} y_i(\theta_1, \ldots, \theta_I)\left(\theta_i - \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)}\right)\left(\prod_{j=1}^{I} \phi_j(\theta_j)\right) d\theta_I \cdots d\theta_1\right] - U_i(\underline{\theta}_i). \tag{23.D.15}$$

Thus, the seller's expected revenue is equal to

$$\left[\int_{\underline{\theta}_1}^{\bar{\theta}_1} \cdots \int_{\underline{\theta}_I}^{\bar{\theta}_I} \left[\sum_{i=1}^{I} y_i(\theta_1, \ldots, \theta_I)\left(\theta_i - \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)}\right)\right]\left(\prod_{j=1}^{I} \phi_j(\theta_j)\right) d\theta_I \cdots d\theta_1\right] - \sum_{i=1}^{I} U_i(\underline{\theta}_i). \tag{23.D.16}$$

By inspection of (23.D.16), we see that any two Bayesian incentive compatible social choice functions that generate the same functions $(y_1(\theta), \ldots, y_I(\theta))$ and the same values of $(U_1(\underline{\theta}_1), \ldots, U_I(\underline{\theta}_I))$ generate the same expected revenue for the seller. ■

As an example of the application of Proposition 23.D.3, consider the equilibria of the first-price and second-price sealed-bid auctions that we identified in Examples 23.B.5 and 23.B.6 (where the buyers' valuations were independently drawn from the uniform distribution on [0, 1]). For these equilibria, the conditions of the revenue equivalence theorem are satisfied: in both auctions the buyer with the highest valuation always gets the good and a buyer with a zero valuation has an expected utility of zero. Thus, the revenue equivalence theorem tells us that the seller receives exactly the same level of expected revenue in these equilibria of the two auctions (you can confirm this fact in Exercise 23.D.3). More generally, it can be shown that in any *symmetric auction setting* (i.e., one where the buyers' valuations are independently drawn from identical distributions), the conditions of the revenue equivalence theorem will be met for *any* Bayesian Nash equilibrium of the first-price sealed-bid auction and the (dominant strategy) equilibrium of the second-price sealed-bid auction (see Exercise 23.D.4 for a consideration of symmetric equilibria in these settings). We can conclude from Proposition 23.D.3, therefore, that in any such setting the first-price and second-price sealed-bid auctions generate exactly the same revenue for the seller.

23.E Participation Constraints

In Sections 23.B to 23.D, we have studied the constraints that the presence of private information puts on the set of implementable social choice functions. Our analysis up to this point, however, has assumed implicitly that each agent i has no choice but to participate in any mechanism chosen by the mechanism designer. That is, agent i's discretion was limited to choosing his optimal actions within those allowed by the mechanism.

In many applications, however, agents' participation in the mechanism is *voluntary*. As a result, the social choice function that is to be implemented by a mechanism must not only be incentive compatible but must also satisfy certain *participation* (or *individual rationality*) *constraints* if it is to be successfully implemented. In this section, we provide a brief discussion of these additional