Sample Complexity of Automated Mechanism Design

Maria-Florina Balcan, Tuomas Sandholm, Ellen Vitercik
School of Computer Science
Carnegie Mellon University
Pittsburgh, PA 15213
{ninamf,sandholm,vitercik}@cs.cmu.edu

Abstract

The design of revenue-maximizing combinatorial auctions, i.e. multi-item auctions over bundles of goods, is one of the most fundamental problems in computational economics, unsolved even for two bidders and two items for sale. In the traditional economic models, it is assumed that the bidders' valuations are drawn from an underlying distribution and that the auction designer has perfect knowledge of this distribution. Despite this strong and oftentimes unrealistic assumption, it is remarkable that the revenue-maximizing combinatorial auction remains unknown. In recent years, *automated mechanism design* has emerged as one of the most practical and promising approaches to designing high-revenue combinatorial auctions. The most scalable automated mechanism design algorithms take as input *samples* from the bidders' valuation distribution and then search for a high-revenue auction in a rich auction class. In this work, we provide the first sample complexity analysis for the standard hierarchy of deterministic combinatorial auction classes used in automated mechanism design. In particular, we provide tight sample complexity bounds on the number of samples needed to guarantee that the empirical revenue of the designed mechanism on the samples is close to its expected revenue on the underlying, unknown distribution over bidder valuations, for each of the auction classes in the hierarchy. In addition to helping set automated mechanism design on firm foundations, our results also push the boundaries of learning theory. In particular, the hypothesis functions used in our contexts are defined through multi-stage combinatorial optimization procedures, rather than simple decision boundaries, as are common in machine learning.

1 Introduction

Multi-item, multi-bidder auctions have been studied extensively in economics, operations research, and computer science. In a *combinatorial auction (CA)*, the bidders may submit bids on bundles of goods, rather than on individual items alone, and thereby they may fully express their complex valuation functions. Notably, these functions may be non-additive due to the presence of complementary or substitutable goods for sale. There are many important and practical applications of CAs, ranging from the US government's wireless spectrum license auctions to sourcing auctions, through which companies coordinate the procurement and distribution of equipment, materials and supplies.

One of the most important and tantalizing open questions in computational economics is the design of *optimal auctions*, that is, auctions that maximize the seller's expected. In the standard economic model, it is assumed that the bidders' valuations are drawn from an underlying distribution and that the mechanism designer has perfect information about this distribution. Astonishingly, even with this strong assumption, the optimal CA design problem is unsolved even for auctions with just two distinct items for sale and two bidders. A monumental advance in the study of optimal auction design was the characterization of the optimal 1-item auction [Myerson, 1981]. However, the problem becomes

significantly more challenging with multiple items for sale. In particular, Conitzer and Sandholm proved that the problem of finding a revenue-maximizing deterministic CA is NP-complete [Conitzer and Sandholm, 2004]. We note here that it is well-known that randomization can increase revenue in CAs, but we focus on deterministic CAs in this work because in many applications, randomization is not palatable and very few, if any, randomized CAs are used in practice.

In recent years, a novel approach known as *automated mechanism design* (AMD) has been adopted to attack the revenue-maximizing auction design problem [Conitzer and Sandholm, 2002, Sandholm, 2003]. In the most scalable strand of AMD, algorithms have been developed which take samples from the bidders' valuation distributions as input, optimize over a rich class of auctions, and return an auction which is high-performing over the sample [Likhodedov and Sandholm, 2004, 2005, Sandholm and Likhodedov, 2015]. AMD algorithms have yielded deterministic mechanisms with the highest known revenues in the contexts used for empirical evaluations [Sandholm and Likhodedov, 2015]. This approach relaxes the unrealistic assumption that the mechanism designer has perfect information about the bidders' valuation distribution.

However, until now, there was no formal characterization of the number of samples required to guarantee that the empirical revenue of the designed mechanism on the samples is close to its expected revenue on the underlying, unknown distribution over bidder valuations. In this paper, we provide that missing link. We present tight sample complexity guarantees over an extensive hierarchy of expressive CA families. These are the most commonly used auction families in AMD. The classes in the hierarchy are based on the classic VCG mechanism, which is a generalization of the well-known second-price, or Vickrey, single-item auction. The auctions we consider achieve significantly higher revenue than the VCG baseline by weighting bidders (multiplicatively increasing all of their bids) and boosting outcomes (additively increasing the liklihood that a particular outcome will be the result of the auction).

A major strength of our results is their applicability to any algorithm that determines the optimal auction over the sample, a nearly optimal approximation, or any other black box procedure. Therefore, they apply to any automated mechanism design algorithm, optimal or not. One of the key challenges in deriving these general sample complexity bounds is that to do so, we must develop deep insights into how changes to the auction parameters (the bidder weights and allocation boosts) effect the outcome of the auction (who wins which items and how much each bidder pays) and thereby the revenue of the auction. In our context, we show that the functions which determine the outcome of an auction are highly complex, consisting of multi-stage optimization procedures.

Therefore, the function classes we consider are much more challenging than those commonly found in machine learning contexts. Typically, for well-understood classes of functions used in machine learning, such as linear separators or other smooth curves in Euclidean spaces, there is a simple mapping from the parameters of a specific hypothesis to its prediction on a given example and a close connection between the distance in the parameter space between two parameter vectors and the distance in function space between their associated hypotheses. Roughly speaking, it is necessary to understand this connection in order to determine how many significantly different hypotheses there are over the full range of parameters. In our context, due to the inherent complexity of the classes we consider, connecting the parameter space to the space of revenue functions requires a much more delicate analysis. The key technical part of our work involves understanding this connection from a learning theoretic perspective. For the more general classes in the hierarchy, we use Rademacher complexity to derive our bounds, and for the auction classes with more combinatorial structure, we exploit that structure to prove pseudo-dimension bounds. This work is both of practical importance since we fill a fundamental gap in AMD, and of learning theoretical interest, as our sample complexity analysis requires a deep understanding of the structure of the revenue function classes we consider.

Related Work. In prior research, the sample complexity of revenue maximization has been studied primarily in the single-item or the more general single-dimensional settings [Elkind, 2007, Cole and Roughgarden, 2014, Huang et al., 2015, Medina and Mohri, 2014, Morgenstern and Roughgarden, 2015, Roughgarden and Schrijvers, 2016, Devanur et al., 2016], as well as some multi-dimensional settings which are reducible to the single-bidder setting [Morgenstern and Roughgarden, 2016]. In contrast, the combinatorial settings that we study are much more complex since the revenue functions consist of multi-stage optimization procedures that cannot be reduced to a single-bidder setting. The complexity intrinsic to the multi-item setting is explored in [Dughmi et al., 2014], who show that

for a single unit-demand bidder, when the bidder's values for the items may be correlated, $\Omega(2^m)$ samples are required to determine a constant-factor approximation to the optimal auction.

Learning theory tools such as pseudo-dimension and Rademacher complexity were used to prove strong guarantees in [Medina and Mohri, 2014, Morgenstern and Roughgarden, 2015, 2016], which analyze piecewise linear revenue functions and show that few samples are needed to learn over the revenue function classes in question. In a similar direction, bounds on the sample complexity of welfare-optimal item pricings have been developed [Feldman et al., 2015, Hsu et al., 2016]. Earlier work of Balcan et al. [2008] addressed sample complexity results for revenue maximization in unrestricted supply settings. In that context, the revenue function decomposes additively among bidders and does not apply to our combinatorial setting.

Despite the inherent complexity of designing high-revenue CAs, Morgenstern and Roughgarden use linear separability as a tool to prove that certain simple classes of multi-parameter auctions have small sample complexity. The auctions they study are sequential auctions with item and grand bundle pricings, as well as second-price item auctions with item reserve prices [Morgenstern and Roughgarden, 2016]. In the item pricing auctions, the bidders show up one at a time and the seller offers each item that remains at some price. Each buyer then chooses the subset of goods that maximizes her utility. In the grand bundle pricing auctions, the bidders are each offered the grand bundle in some fixed order, and the first bidder to have a value greater than the price buys it. They show that bounding the sample complexity of these sequential auctions can be reduced to the single-buyer setting.

In contrast, the auctions we study are more versatile than item pricing auctions, as they give the mechanism designer many more degrees of freedom than the number of items. This level of expressiveness allows the designer to increase competition between bidders, much like Myerson's optimal auction, and thus boost revenue. It is easy to construct examples where even simple AMAs achieve significantly greater revenue than sequential auctions with item and grand bundle prices. Moreover, even the simpler auction classes we consider pose a unique challenge because the parameters defining the auctions influence the multi-stage allocation procedure and resulting revenue in non-intuitive ways. This is unlike item and grand bundle pricing auctions, as well as second-price item auctions, which are simple by design. Our function classes therefore require us to understand the specific form of the weighted VCG payment rule and its interaction with the parameter space. Thus, our context and techniques diverge from those in [Morgenstern and Roughgarden, 2016].

Finally, there is a wealth of work on characterizing the optimal CA for restricted settings and designing mechanisms which achieve high, if not optimal revenue in specific contexts. Due to space constraints, in Section A of the supplementary materials, we describe these results as well as what is known theoretically about the classes in the hierarchy of deterministic CAs we study.

2 Preliminaries, notation, and the combinatorial auction hierarchy

In the following section, we explain the basic mechanism design problem, fix notation, and then describe the hierarchy of combinatorial auction families we study.

Mechanism Design Preliminaries. We consider the problem of selling m heterogeneous goods to n bidders. This means that there are 2^m different bundles of goods, $B = \{b_1, \ldots, b_{2^m}\}$. Each bidder $i \in [n]$ is associated with a set-wise valuation function over the bundles, $v_i : B \to \mathbb{R}$. We assume that the bidders' valuations are drawn from a distribution $\mathcal{D}$.

Every auction is defined by an *allocation function* and a *payment function*. The allocation function determines which bidders receive which items based on their bids and the payment function determines how much the bidders need to pay based on their bids and the allocation. It is up to the mechanism designer to determine which allocation and payment functions should be used. In our context, the two functions are fixed based on the samples from $\mathcal{D}$ before the bidders submit their bids.

Each auction family that we consider has a design based on the classic *Vickrey-Clarke-Groves mechanism (VCG)*. The VCG mechanism, which we describe below, is the canonical *strategy-proof* mechanism, which means that every bidder's dominant strategy is to bid truthfully. In other words, for every Bidder i, no matter the bids made by the other bidders, Bidder i maximizes her expected utility (her value for her allocation minus the price she pays) by bidding her true value. Therefore, we describe the VCG mechanism assuming that the bids equal the bidders' true valuations.

The VCG mechanism allocates the items such that the social welfare of the bidders, that is, the sum of each bidder's value for the items she wins, is maximized. Intuitively, each winning bidder then pays her bid minus a "rebate" equal to the increase in welfare attributable to Bidder i's presence in the auction. This form of the payment function is crucial to ensuring that the auction is strategy-proof. More concretely, the allocation of the VCG mechanism is the disjoint set of subsets $(b_1^*, \ldots, b_n^*) \subseteq B$ that maximizes $\sum v_i(b_i^*)$. Meanwhile, let $(b_1^{-i}, \ldots, b_n^{-i})$ be the disjoint set of subsets that maximizes $\sum_{j \neq i} v_j(b_j^{-i})$. Then Bidder i must pay $\sum_{j \neq i} \left[v_j(b_j^{-i}) - v_j(b_j^*) \right] = v_i(b_i^*) - \left[\sum v_j(b_j^*) - \sum_{j \neq i} v_j(b_j^{-i}) \right]$. In the special case where there is one item for sale, the VCG mechanism is known as the second price, or Vickrey, auction, where the highest bidder wins the item and pays the second highest bid. We note that every auction in the classes we study is strategy-proof, so we may assume that the bids equal the bidders' valuations.

Notation. We study auctions with n bidders and m items. We refer to the bundle of all m items as the *grand bundle*. In total, there are $(n+1)^m$ possible allocations, which we denote as the vectors $\mathcal{O} = \left\{ \vec{o}_1, \ldots, \vec{o}_{(n+1)^m} \right\}$. Each allocation vector $\vec{o}_i$ can be written as $(o_{i,1}, \ldots, o_{i,n})$, where $o_{i,j} = b_\ell \in B$ denotes the bundle of items allocated to Bidder j in allocation $\vec{o}_i$. We use the notation $\vec{v}_1 = (v_1(b_1), \ldots, v_1(b_{2^m}))$ and $\vec{v} = (\vec{v}_1, \ldots, \vec{v}_n)$ to denote a vector of bidder valuation functions. We say that $rev_A(\vec{v})$ is the revenue of an auction A on the valuation vector $\vec{v}$. Denoting the payment of any one bidder under auction A given valuation vector $\vec{v}$ as $p_{i,A}(\vec{v})$, we have that $rev_A(\vec{v}) = \sum_{i=1}^n p_{i,A}(\vec{v})$. Finally, U is an upper bound on the revenue achievable for any auction over the support of the bidders' valuation distribution.

Auction Classes. We now give formal definitions of the CA families in the hierarchy we study. See Figure 1 for the hierarchical organization of the auction classes, together with the papers which introduced each family.

Affine maximizer auctions (AMAs). An AMA A is defined by a set of weights per bidder $(w_1, \ldots, w_n) \subset \mathbb{R}_{>0}$ and boosts per allocation $\left(\lambda(\vec{o}_1), \ldots, \lambda(\vec{o}_{(n+1)^m}) \right) \subset \mathbb{R}$. An auction A uniquely corresponds to a set of these parameters, so we write $A = \left(w_1, \ldots, w_n, \lambda(\vec{o}_1), \ldots, \lambda(\vec{o}_{(n+1)^m}) \right)$. To simplify notation, we write $\lambda_i = \lambda(\vec{o}_i)$ interchangeably. These parameters allow the mechanism designer to multiplicatively boost any bidder's bids by their corresponding weight and to increase the likelihood that any one allocation is returned as the output of an auction. More concretely, the allocation $\vec{o}^*$ of an AMA A is the one which maximizes the weighted social welfare, i.e. $\vec{o}^* = \text{argmax}_{\vec{o}_i \in \mathcal{O}} \left\{ \sum_{j=1}^n w_j v_j(o_{i,j}) + \lambda(\vec{o}_i) \right\}$. The payment function of A has the same form as the VCG payment rule, with the parameters factored in to ensure that the auction remains strategy-proof. In particular, for all $j \in [n]$, the payments are $p_{j,A}(\vec{v}) = \frac{1}{w_j} \left[\sum_{\ell \neq j} w_\ell v_\ell(o_{-j,\ell}) + \lambda(\vec{o}_{-j}) - \sum_{\ell \neq j} w_\ell v_\ell(o_\ell^*) - \lambda(\vec{o}^*) \right]$, where $\vec{o}_{-j} = \text{argmax}_{\vec{o}_i \in \mathcal{O}} \left\{ \sum_{\ell \neq j} w_\ell v_\ell(o_{i,\ell}) + \lambda(\vec{o}_i) \right\}$.

We assume that $H_{\underline{w}} \leq w_i \leq H_{\overline{w}}$, $\lambda_i \leq H_\lambda$, and $v_i(b_\ell) \leq H_v$ for some $H_{\underline{w}}, H_{\overline{w}}, H_\lambda, H_v \in \mathbb{R}_{\geq 0}$. It is typical to assume an upper bound (here, H_v) on the bidders' valuation for any bundle. This is related to the fact that an upper bound on a target function's range is always assumed in standard machine learning sample complexity bounds. Intuitively, generalizability depends on how much any one sample can skew the empirical average of a hypothesis, or in this case, auction. The bounds on the AMA parameters are closely related to the bound on the bidders' valuations H_v. For example, it is a simple exercise to see that we need not search for a lambda value which is greater than H_v.

Virtual valuation combinatorial auctions (VVCAs). VVCAs are a subset of AMAs. The defining characteristic of a VVCA is that each $\lambda(\vec{o}_j)$ is split into n terms such that $\lambda(\vec{o}_j) = \sum_{i=1}^n \lambda_i(\vec{o}_j)$ where $\lambda_i(\vec{o}_j) = c_{i,b}$ for all allocations $\vec{o}_j$ that give Bidder i exactly bundle $b \in B$.

λ-auctions. λ-auctions are the subclass of AMAs where $w_i = 1$ for all $i \in [n]$.

Mixed bundling auctions (MBAs). The class of MBAs is parameterized by a constant $c \geq 0$ which can be seen as a discount for any bidder who receives the grand bundle. Formally, the c-MBA is the λ-auction with $\lambda(\vec{o}) = c$ if some bidder receives the grand bundle in allocation $\vec{o}$ and 0 otherwise.

Mixed bundling auctions with reserve prices (MBARPs). MBARPs are identical to MBAs though with *reserve prices*. In a single-item VCG auction (i.e. second price auction) with a reserve price, the

item is only sold if the highest bidder's bid exceeds the reserve price, and the winner must pay the maximum of the second highest bid and the reserve price. We describe how this intuition generalizes to MBAs in Section 3.

Generalization bounds. In order to derive sample complexity bounds which apply to any algorithm that determines the optimal auction over the sample, a nearly optimal approximation, or any other black-box procedure, we derive uniform convergence sample complexity bounds with respect to the auction classes we examine. Formally, we define the sample complexity of uniform convergence over an auction class $\mathcal{A}$ as follows.

Definition 1 (Sample complexity of uniform convergence over $\mathcal{A}$). *We say that $N(\epsilon, \delta, \mathcal{A})$ is the sample complexity of uniform convergence over $\mathcal{A}$ if for any $\epsilon, \delta \in (0,1)$, if $\mathcal{S} = \left\{ \vec{v}^1, \ldots, \vec{v}^N \right\}$ is a sample of size $N \geq N(\epsilon, \delta, \mathcal{A})$ drawn at random from $\mathcal{D}$, with probability at least $1 - \delta$, for all auctions $A \in \mathcal{A}$, $\left| \frac{1}{N} \sum_{i=1}^{N} rev_A \left(\vec{v}^i \right) - \mathbb{E}_{\vec{v} \sim \mathcal{D}} \left[rev_A(\vec{v}) \right] \right| \leq \epsilon$.*

3 Sample complexity bounds over the hierarchy of auction classes

In this section, we provide an overview of our sample complexity guarantees over the hierarchy of auction classes we consider (Section 3.1 and 3.2). We show that more structured classes require drastically fewer samples to learn over. We conclude with a note about sample complexity guarantees for algorithms that find an approximately optimal mechanism over a sample, as opposed to the optimal mechanism. All omitted proofs are presented in full in the supplementary material.

3.1 The sample complexity of AMA, VVCA, and λ-auction revenue maximization

We begin by analyzing the most general families in the CA hierarchy — AMAs, VVCAs, and λ-auctions — proving a general upper bound and class-specific lower bounds.

Theorem 1. *The sample complexity of uniform convergence over the classes of n-bidder, m-item AMAs, VVCAs, and λ-Auctions is $N = \widetilde{O} \left(\left[U n^m \sqrt{m} \left(U + n^{m/2} \right) / \epsilon \right]^2 \right)$. Moreover, for λ-Auctions, $N = \Omega \left(n^m \right)$ and for VVCAs, $N = \Omega \left(2^m \right)$.*

We derive the upper bound by analyzing the Rademacher complexity of the class of n-bidder, m-item AMA revenue functions. For a family of functions G and a finite sample $S = \{x_1, \ldots, x_N\}$ of size N, the empirical Rademacher complexity is defined as $\widehat{\mathcal{R}}_S(G) = \mathbb{E}_\sigma [\sup_{g \in G} \frac{1}{N} \sum \sigma_i g(x_i)]$, where $\sigma = (\sigma_1, \ldots, \sigma_N)$, with σ_is independent uniform random variables taking values in $\{-1, 1\}$. The Rademacher complexity of G is defined as $\mathcal{R}_N(G) = \mathbb{E}_{S \sim \mathcal{D}^N} [\widehat{\mathcal{R}}_S(G)]$.

The AMA revenue function, defined in Section 2, can be summarized as a multi-stage optimization procedure: determine the weighted-optimal allocation and then compute the n different payments, each of which requires a separate optimization procedure. Luckily, we are able to decompose the revenue functions into small components, each of which is easier to analyze on its own, and then combine our results to prove the following theorem about this class of revenue functions as a whole.

Theorem 2. *Let $\mathcal{F}$ be the set of n-bidder, m-item AMA revenue functions rev_A such that $A = \left(w_1, \ldots, w_n, \lambda_1, \ldots, \lambda_{(n+1)^m} \right), H_{\underline{w}} \leq |w_i| \leq H_{\overline{w}}, |\lambda_i| \leq H_\lambda$. Then*

$$\mathcal{R}_N(\mathcal{F}) = O \left(\frac{n^{m+2} \left(H_{\overline{w}} H_v + H_\lambda \right)}{H_{\underline{w}}} \sqrt{\frac{m \log n}{N}} \left(\frac{n \hat{H}_v \left(n H_{\overline{w}} + H_\lambda \right)}{H_{\underline{w}}} + \sqrt{n^m \log N} \right) \right),$$

where $\hat{H}_v = \max \left\{ H_v, 1 \right\}$.

Proof sketch. First, we describe how we split each revenue function into smaller, easier to analyze atoms, which together allow us to bound the Rademacher complexity of the class of AMA revenue functions. To this end, it is well-known (e.g. [Mohri et al., 2012]) that if every function f in a class $\mathcal{F}$ can be written as the summation of two functions g and h from classes $\mathcal{G}$ and $\mathcal{H}$, respectively, then $\mathcal{R}_N(\mathcal{F}) \leq \mathcal{R}_N(\mathcal{G}) + \mathcal{R}_N(\mathcal{H})$. Therefore, we split each revenue function into $n + 1$ components such that the sum of these components equals the revenue function.
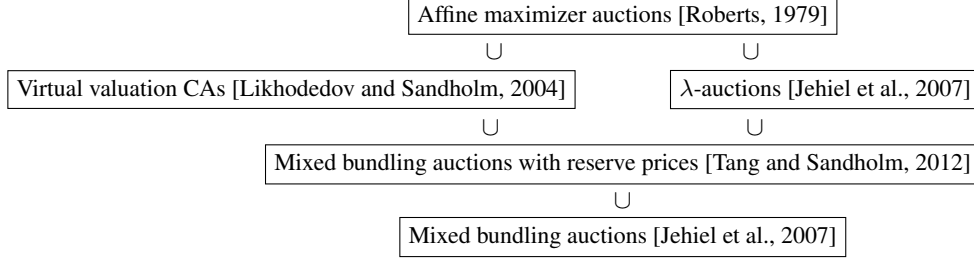
Figure 1: The hierarchy of deterministic CA families. Generality increases upward in the hierarchy.

With this objective in mind, let $\vec{o}_A^*(\vec{v})$ be the outcome of the AMA A on the bidding instance $\vec{v}$, i.e. $\vec{o}_A^* = \operatorname{argmax}_{\vec{o}_i \in \mathcal{O}} \left\{ \sum_{j=1}^n w_j v_j(o_{i,j}) + \lambda_i \right\}$ and let $\phi_{A,-j}(\vec{v})$ be the weighted social welfare of the welfare-maximizing outcome without Bidder j's participation. In other words, $\phi_{A,-j}(\vec{v}) = \max_{\vec{o}_i \in \mathcal{O}} \left\{ \sum_{\ell \neq j} w_\ell v_\ell(o_{i,\ell}) + \lambda_i \right\}$. Then we can write

$$rev_A(\vec{v}) = \sum_{j=1}^n \frac{1}{w_j} \phi_{A,-j}(\vec{v}) - \sum_{i=1}^{(n+1)^m} \left(\sum_{j=1}^n \frac{1}{w_j} \sum_{\ell \neq j} w_\ell v_\ell(o_{i,\ell}) + \lambda_i \right) \mathbf{1}_{\vec{o}_i = \vec{o}_A^*(\vec{v})}.$$

We can now split rev_A into $n+1$ simpler functions: $rev_{A,j}(\vec{v}) = \frac{1}{w_j} \phi_{A,-j}(\vec{v})$ for $j \in [n]$ and

$$rev_{A,n+1}(\vec{v}) = - \sum_{i=1}^{(n+1)^m} \left(\sum_{j=1}^n \frac{1}{w_j} \sum_{\ell \neq j} w_\ell v_\ell(o_{i,\ell}) + \lambda_i \right) \mathbf{1}_{\vec{o}_i = \vec{o}_A^*(\vec{v})},$$

so $rev_A(\vec{v}) = \sum_{j=1}^{n+1} rev_{A,j}(\vec{v})$. Intuitively, for $j \in [n]$, $rev_{A,j}$ is a weighted version of what the social welfare would be if Bidder j had not participated in the auction, whereas $rev_{A,n+1}(\vec{v})$ measures the amount of revenue subtracted to ensure that the resulting auction is strategy-proof.

As to be expected, bounding the Rademacher complexity of each smaller class of functions $\mathcal{L}_j = \left\{ rev_{A,j} \mid (w_1, \ldots, w_n, \lambda_1, \ldots, \lambda_{(n+1)^m}), H_{\underline{w}} \leq |w_i| \leq H_{\overline{w}}, |\lambda_i| \leq H_\lambda \right\}$ for $j \in [n+1]$ is simpler than bounding the Rademacher complexity the class of revenue functions itself, and if $\mathcal{F}$ is the set of all n-bidder, m-item AMA revenue functions, then $\mathcal{R}_N(\mathcal{F}) \leq \sum_{j=1}^{n+1} \mathcal{R}_N(\mathcal{L}_j)$. In Lemma 2 and Lemma 3 of Section B.1 in the supplementary materials, we obtain bounds on $\mathcal{R}_N(\mathcal{L}_j)$ for $j \in [n+1]$ which lead us to our bound on $\mathcal{R}_N(\mathcal{F})$. $\square$

3.2 The sample complexity of MBA revenue maximization

Fortunately, these negative sample complexity results are not the end of the story. We do achieve polynomial sample complexity upper bounds for the important classes of mixed bundling auctions (MBAs) and mixed bundling auctions with reserve prices (MBARPs). We derive these sample complexity bounds by analyzing the pseudo-dimensions of these classes of auctions. In this section, we present our results in increasing complexity, beginning with the class of n-bidder, m-item MBAs, which we show has a pseudo-dimension of 2. We build on the proof of this result to show that the class of n-bidder, m-item MBARPs has a pseudo-dimension of $O\left(m^3 \log n\right)$.

We note that when we analyze the class of MBARPs, we assume additive reserve prices, rather than bundle reserve prices. In other words, each item has its own reserve price, and the reserve price of a bundle is the sum of its components' reserve prices, as opposed to each bundle having its own reserve price. We have good reason to make this restriction; in Section C.1, we prove that an exponential number of samples are required to learn over the class of MBARPs with bundle reserve prices.

Before we prove our sample complexity results, we fix some notation. For any c-MBA, let $rev_c(\vec{v})$ be its revenue on $\vec{v}$, which is determined in the exact same way as the general AMA revenue function with the λ terms set as described in Section 2. Similarly, let $rev_{\vec{v}^\ell}(c)$ be the revenue of the c-MBA on $\vec{v}^\ell$ *as a function of c.* We will use the following result regarding the structure of $rev_c(\vec{v})$ in order to derive our pseudo-dimension results. The proof is in Section C of the supplementary materials.

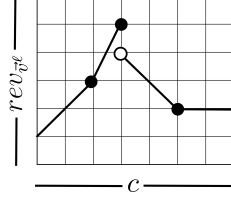Figure 2: Example of $rev_{\vec{v}^\ell}(c)$.

Lemma 1. *There exists $c^* \in [0, \infty)$ such that $rev_{\vec{v}}(c)$ is non-decreasing on the interval $[0, c^*]$ and non-increasing on the interval (c^*, ∞).*

The form of $rev_{\vec{v}}(c)$ as described in Lemma 1 is depicted in Figure 2. The full proof of the following pseudo-dimension bound can be found in Section C of the supplementary materials.

Theorem 3. *The pseudo-dimension of the class of n-bidder, m-item MBAs is 2.*

Proof sketch. First, we recall what we must show in order to prove that the pseudo-dimension of this class is 2 (for more on pseudo-dimension, see, for example, [Mohri et al., 2012]). The proof structure is similar to those involved in VC dimension derivations. To begin with, we must provide a set of two valuation vectors $\mathcal{S} = \{\vec{v}^1, \vec{v}^2\}$ that can be *shattered* by the class of MBA revenue functions. This means that there exist two *targets* $z^1, z^2 \in \mathbb{R}$ with the property that for any $T \subseteq \mathcal{S}$, there exists a $c_T \in C$ such that if $\vec{v}^i \in T$, then $rev_{c_T}(\vec{v}^i) \leq z^i$ and if $\vec{v}^i \notin T$, then $rev_{c_T}(\vec{v}^i) > z^i$. In other words, $\mathcal{S}$ can be labeled in every possible way by MBA revenue functions (whether or not $rev_c(\vec{v}^j)$ is greater than its target z^j). We must also prove that no set of three valuation vectors is shatterable.

Our construction of the set $\mathcal{S} = \{\vec{v}^1, \vec{v}^2\}$ that can be shattered by the set of MBAs can be found in the full proof of this theorem in Section C of the supplementary materials. We now show that no set of size $N \geq 3$ can be shattered by the class of MBAs. Fix one sample $\vec{v}^i \in \mathcal{S}$ and consider $rev_{\vec{v}^i}(c)$. From Lemma 1, we know that there exists $c_i^* \in [0, \infty)$, such that $rev_{\vec{v}^i}(c)$ is non-decreasing on the interval $[0, c_i^*]$ and non-increasing on the interval (c_i^*, ∞). Therefore, there exist two thresholds $t_i^1 \in [0, c_i^*]$ and $t_i^2 \in (c_i^*, \infty) \cup \{\infty\}$ such that $rev_{\vec{v}^i}(c)$ is below its threshold for $c \in [0, t_i^1)$, above its threshold for $c \in (t_i^1, t_i^2)$, and below its threshold for $c \in (t_i^2, \infty)$. Now, merge these thresholds for all N samples on the real line and consider the interval (t_1, t_2) between two adjacent thresholds. The binary labeling of the samples in $\mathcal{S}$ on this interval is fixed. In other words, for any sample $\vec{v}^j \in \mathcal{S}$, $rev_{\vec{v}^j}(c)$ is either at least z^j or strictly less than z^j for all $c \in (t_1, t_2)$. There are at most $2N + 1$ intervals between adjacent thresholds, so at most $2N + 1$ different binary labelings of $\mathcal{S}$. Since we assumed $\mathcal{S}$ is shatterable, it must be that $2^N \leq 2N + 1$, so $N \leq 2$. $\qquad\square$

This result allows us to prove the following sample complexity guarantee.

Theorem 4. *The sample complexity of uniform convergence over the class of n-bidder, m-item MBAs is $N = O\left((U/\epsilon)^2 \left(\log(U/\epsilon) + \log(1/\delta)\right)\right)$.*

Mixed bundling auctions with reserve prices (MBARPs). MBARPs are a variation on MBAs, with the addition of reserve prices. Reserve prices in the single-item case, as described in Section 2, can be generalized to the multi-item case as follows. We enlarge the set of agents to include the seller, who is now Bidder 0 and whose valuation for a set of items is the set's reserve price. Working in this expanded set of agents, the bidder weights are all 1 and the λ terms are the same as in the standard MBA setup. Importantly, the seller makes no payments, no matter her allocation. More formally, given a vector of valuation functions $\vec{v}$, the MBARP allocation is $\vec{o}^* = \text{argmax}_{\vec{o} \in \mathcal{O}} \sum_{i=0}^{n} v_i(o_i) + \lambda(\vec{o})$. For each $i \in \{1, \ldots, n\}$, Bidder i's payment is

$$p_{A,i}(\vec{v}) = \sum_{j \in \{0, \ldots, n\} \setminus \{i\}} v_j(o_{-i,j}) + \lambda(\vec{o}_{-i}) - \sum_{j \in \{0, \ldots, n\} \setminus \{i\}} v_j(o_j^*) - \lambda(\vec{o}^*),$$

where

$$\vec{o}_{-i} = \text{argmax}_{\vec{o} \in \mathcal{O}} \sum_{j \in \{0, \ldots, n\} \setminus \{i\}} v_j(o_j) + \lambda(\vec{o}).$$

As mentioned, we restrict our attention to item-specific reserve prices. In this case, the the reserve price of a bundle is the sum of the reserve prices of the items in the bundle.

Each MBARP is therefore parameterized by $m + 1$ values $(c, r_1, \ldots, r_m)$, where r_i is the reserve price for the ith good. For a fixed valuation function vector $\vec{v} = (v_1(b_1), \ldots, v_1(b_{2^m}), \ldots, v_n(b_1), \ldots, v_n(b_{2^m}))$, we can analyze the MBARP revenue function on $\vec{v}$ as a mapping $rev_{\vec{v}} : \mathbb{R}^{m+1} \to \mathbb{R}$, where $rev_{\vec{v}}(c, r_1, \ldots, r_m)$ is the revenue of the MBARP parameterized by $(c, r_1, \ldots, r_m)$ on $\vec{v}$.

Theorem 5. *The psuedo-dimension of the class of n-bidder, m-item MBARPs with item-specific reserve prices is $O\left(m^3 \log n\right)$.*

Proof sketch. Let $\mathcal{S} = \left\{\vec{v}^1, \ldots, \vec{v}^N\right\}$ of size N be a set of n-bidder valuation function samples that can be shattered by a set C of 2^N MBARPs. This means that there exist N targets $z^1, \ldots, z^N$ such that each MBARP in C induces a binary labeling of the samples $\vec{v}^j$ in $\mathcal{S}$ (whether the revenue of the MBARP on $\vec{v}^j$ is greater than or less than z^j). Since $\mathcal{S}$ is shatterable, we can thus label $\mathcal{S}$ in every possible way using MBARPs in C.

This proof is similar to the proof of Theorem 3, where we split the real line into a set of intervals $\mathcal{I}$ such that for any $I \in \mathcal{I}$, the binary labeling of $\mathcal{S}$ by the c-MBA revenue function was fixed for all $c \in I$. In the case of MBARPs, however, the domain is $\mathbb{R}^{m+1}$, so we cannot split the domain into intervals in the same way. Instead, we show that we can split the domain into cells such that the binary labeling of $\mathcal{S}$ by the MBARP revenue function is a fixed linear function as we range over parameters in a single cell. In this way, we show that $N = O\left(m^3 \log n\right)$. $\qquad\square$

This is enough to prove the following guarantee.

Theorem 6. *The sample complexity of uniform convergence over the class of n-bidder, m-item MBARPs with item-specific reserve prices is $N = O\left((U/\epsilon)^2 \left(m^3 \log n \log (U/\epsilon) + \log (1/\delta)\right)\right)$.*

3.3 Sample complexity bounds for approximation algorithms

It may not always be computationally feasible to solve for the best auction over $\mathcal{S}$ for the given auction family. Rather, we may only be able to determine an auction A that has average revenue over $\mathcal{S}$ that is within a $(1 + \alpha)$ multiplicative factor of the revenue-maximizing auction over $\mathcal{S}$ within the family. Nonetheless, in Theorem 11 of the supplementary materials, we prove that with slightly more samples, we can ensure that the expected revenue of A is close to being with a $(1 + \alpha)$ multiplicative factor of the expected revenue of the optimal auction within the family with respect to the real distribution $\mathcal{D}$. We prove a similar bound for an additive factor approximation as well.

4 Conclusion

In this paper, we proved strong bounds on the sample complexity of uniform convergence for the well-studied and standard auction families that constitute the hierarchy of deterministic combinatorial auctions. We thereby answered a crucial question in the study of (automated) mechanism design: how to relate the performance of the mechanisms in the search space over the input samples to their expectation over the underlying—unknown—distribution. Specifically, for a fixed class of auctions, we determine the sample complexity necessary to ensure that with high probability, for any auction in that class, the average revenue over the sample is close to the expected revenue with respect to the underlying, unknown distribution over bidders' valuations. Our bounds apply to any algorithm that finds an optimal or approximately optimal auction over an input sample, and therefore to any automated mechanism design algorithm. Moreover, our results and analyses are of interest from a learning theoretic perspective because the function classes which make up the hierarchy of deterministic combinatorial auctions diverge significantly from well-understood hypothesis classes typically found in machine learning.

A Additional Related Work

In this section, we discuss additional related work regarding multi-dimensional auction design. The simplicity of Myerson's optimal *single-item* auction might lead one to hope that the optimal multi-item auction could be so elegantly characterizable [Myerson, 1981]. Recent work has made considerable progress toward this end (e.g. [Alaei et al., 2013, Bhalgat et al., 2013, Bhattacharya et al., 2010, Cai et al., 2012b,a, 2013, Daskalakis et al., 2014, Kleinberg and Weinberg, 2012]) but there is still relatively little known about optimal multi-item auction design. The problem has also garnered significant interest from a more applied perspective, resulting in significant advances from the artificial intelligence and machine learning communities (e.g. [Parkes and Ungar, 2000, Lahaie, 2011, Wurman and Wellman, 2000, Parkes et al., 2004, Amin et al., 2013, Mohri and Munoz, 2014, 2015]).

Revenue-maximizing mechanism design complements an active research area in theoretical computer science which strives to answer the question: can *simple* mechanisms achieve near-optimal revenue? This question was posed by Hartline and Roughgarden [2009], who left the precise definition of a simple mechanism open for interpretation. Recently, Morgenstern and Roughgarden [2015, 2016] proposed an auction class's pseudo-dimension as a formal means of defining simplicity. In particular, Morgenstern and Roughgarden [2016] complemented pseudo-dimension bounds with known approximation guarantees for the corresponding simple auction classes. See Morgenstern and Roughgarden [2016] and references therein for descriptions of these guarantees.

In contrast, most analyses of the revenue achieved by the classes that make up the hierarchy of deterministic CAs have been empirical [Sandholm, 2003, Likhodedov and Sandholm, 2004, 2005, Tang and Sandholm, 2012, Sandholm and Likhodedov, 2015]. However, from a theoretical standpoint, Roberts, when introducing the class of AMAs [Roberts, 1979], proved that they are the only *ex post* strategy-proof mechanisms over unrestricted domains of valuations[1]. Lavi et al. [2003] went on to prove that under certain natural assumptions, every incentive compatible CA is almost[2] an affine maximizer.

B Proofs from Section 3.1

B.1 AMA Upper Bound Proofs

Proof of Theorem 1. This bound follows from Theorem 2 and standard Rademacher complexity bounds (e.g. [Mohri et al., 2012]). The lower bounds are proved in Theorems 8 and 9 $\qquad\square$

Proof of Theorem 2. First, we describe how we split each revenue function into smaller, easier to analyze atoms, which together allow us to bound the Rademacher complexity of the class of AMA revenue functions. To this end, it is well-known (e.g. [Mohri et al., 2012]) that if every function f in a class $\mathcal{F}$ can be written as the summation of two functions g and h from classes $\mathcal{G}$ and $\mathcal{H}$, respectively, then $\mathcal{R}_N(\mathcal{F}) \leq \mathcal{R}_N(\mathcal{G}) + \mathcal{R}_N(\mathcal{H})$. Therefore, we split each revenue function into $n+1$ components such that the sum of these components equals the revenue function.

With this objective in mind, let $\vec{o}_A^*(\vec{v})$ be the outcome of the AMA A on the bidding instance $\vec{v}$, i.e. $\vec{o}_A^* = \operatorname{argmax}_{\vec{o}_i \in \mathcal{O}} \left\{ \sum_{j=1}^n w_j v_j (o_{i,j}) + \lambda_i \right\}$ and let $\phi_{A,-j}(\vec{v})$ be the weighted social welfare of the welfare-maximizing outcome without Bidder j's participation. In other words, $\phi_{A,-j}(\vec{v}) = \max_{\vec{o}_i \in \mathcal{O}} \left\{ \sum_{\ell \neq j} w_\ell v_\ell (o_{i,\ell}) + \lambda_i \right\}$. Then we can write

$$rev_A(\vec{v}) = \sum_{j=1}^n \frac{1}{w_j} \phi_{A,-j}(\vec{v}) - \sum_{i=1}^{(n+1)^m} \left(\sum_{j=1}^n \frac{1}{w_j} \sum_{\ell \neq j} w_\ell v_\ell(o_{i,\ell}) + \lambda_i \right) \mathbf{1}_{\vec{o}_i = \vec{o}_A^*(\vec{v})}.$$

---

[1] A mechanism is *ex post* strategy-proof if truthful bidding is an *ex post Nash equilibrium* in which all bidders always receive nonnegative utility. By *ex post Nash equilibrium*, we mean that for each player, no matter the valuations of the other players but given that they are bidding truthfully, that player will maximize her utility if she bids truthfully as well.

[2] A mechanism is an *almost affine maximizer* if it is an affine maximizer for sufficiently high valuations. Lavi et al. [2003] conjecture that the "almost" qualifier is merely technical, and can be removed in future research.

We can now split rev_A into $n+1$ simpler functions: $rev_{A,j}(\vec{v}) = \frac{1}{w_j}\phi_{A,-j}(\vec{v})$ for $j \in [n]$ and

$$rev_{A,n+1}(\vec{v}) = -\sum_{i=1}^{(n+1)^m}\left(\sum_{j=1}^{n}\frac{1}{w_j}\sum_{\ell \neq j}w_\ell v_\ell\left(o_{i,\ell}\right) + \lambda_i\right)\mathbf{1}_{\vec{o}_i = \vec{\sigma}_A^*(\vec{v})},$$

so $rev_A(\vec{v}) = \sum_{j=1}^{n+1}rev_{A,j}(\vec{v})$. Intuitively, for $j \in [n]$, $rev_{A,j}$ is a weighted version of what the social welfare would be if Bidder j had not participated in the auction, whereas $rev_{A,n+1}(\vec{v})$ measures the amount of revenue subtracted to ensure that the resulting auction is strategy-proof.

As to be expected, bounding the Rademacher complexity of each smaller class of functions $\mathcal{L}_j = \left\{rev_{A,j} \mid \left(w_1, \ldots, w_n, \lambda_1, \ldots, \lambda_{(n+1)^m}\right), H_{\underline{w}} \leq |w_i| \leq H_{\overline{w}}, |\lambda_i| \leq H_\lambda\right\}$ for $j \in [n+1]$ is simpler than bounding the Rademacher complexity the class of revenue functions itself, and if $\mathcal{F}$ is the set of all n-bidder, m-item AMA revenue functions, then $\mathcal{R}_N(\mathcal{F}) \leq \sum_{j=1}^{n+1}\mathcal{R}_N(\mathcal{L}_j)$. In Lemma 2 and Lemma 3 of Section B.1 in the supplementary materials, we obtain bounds on $\mathcal{R}_N(\mathcal{L}_j)$ for $j \in [n+1]$. In fact, we have that $\sum_{j=1}^{n}\mathcal{R}_N(\mathcal{L}_j) = O\left(\mathcal{R}_N(\mathcal{L}_{n+1})\right)$. Therefore,

$$\mathcal{R}_N(\mathcal{F}) \leq O\left(\mathcal{R}_N(\mathcal{L}_j)\right)$$
$$= O\left(\frac{n^{m+2}\left(H_{\overline{w}}H_v + H_\lambda\right)}{H_{\underline{w}}}\sqrt{\frac{m\log n}{N}}\left(\frac{n\hat{H}_v\left(nH_{\overline{w}} + H_\lambda\right)}{H_{\underline{w}}} + \sqrt{n^m\log N}\right)\right),$$

as claimed. $\qquad\square$

Lemma 2. *For $j \in [n]$,*

$$\mathcal{R}_N(\mathcal{L}_j) = O\left(\frac{n^m\hat{H}_v(nH_{\overline{w}} + H_\lambda)}{H_{\underline{w}}}\sqrt{\frac{m\log n}{N}}\right).$$

Proof. At a high level, we will show that every function $\phi_{A,-j}$, which is the weighted version of what the social welfare would be under the AMA A if Bidder j had not participated in the auction, can be written as the maximum over $O(n^m)$ linear functions. Each linear function maps from the high dimensional space $\mathbb{R}^{n2^m+1}$, but its corresponding weight vector has a small ℓ_1 norm, which is a good sign in Rademacher complexity analyses, as we describe below. We take advantage of these structural properties in our Rademacher analysis, as follows.

First, we fix the notation that we will need for this proof. Let Φ_j be the set of functions $\phi_{A,-j}$ for all AMAs A. In other words, $\Phi_j = \{\phi_{A,-j} \mid A = (w_1, \ldots, w_n, \lambda_1, \ldots, \lambda_{(n+1)^m}), H_{\underline{w}} \leq |w_i| \leq H_{\overline{w}}, |\lambda_i| \leq H_\lambda\}$. Recall that the vector $\vec{v}$ has the form and $\vec{v} = (\vec{v}_1, \ldots, \vec{v}_n)$ where $\vec{v}_i = (v_i(b_1), \ldots, v_i(b_{2^m}))$ for some ordering of all 2^m bundles $b_1, \ldots, b_{2^m}$. In this proof, we will make use of the function $\sigma : [\#(\text{allocations})] \times [\#(\text{bidders})] \to [\#(\text{bundles})]$, where $\sigma(i,t)$ is the index of the bundle that Bidder t receives in allocation $\vec{o}_i$. For example, if Bidder 7 receives the bundle b_2 in outcome $\vec{o}_i$, then $\sigma(i,7) = 2$.

Recall that $\phi_{A,-j}(\vec{v}) = \max_{\vec{o}_i}\left\{\sum_{\ell \neq j}w_\ell v_\ell(o_{i,\ell}) + \lambda_i\right\}$. We will now show that for every allocation $\vec{o}_i$, we can write $\sum_{\ell \neq j}w_\ell v_\ell(o_{i,\ell}) + \lambda_i$ as a linear function. In particular, $\sum_{\ell \neq j}w_\ell v_\ell(o_{i,\ell}) + \lambda_i$ is a linear function $h^i_{A,j}$ from $\mathbb{R}^{n2^m+1}$ to $\mathbb{R}$, since we can write $h^i_{A,j}(\vec{v}, 1) = (\vec{v}, 1) \cdot \vec{a}^i_{A,j}$, where

$$\vec{a}^i_{A,j}[\ell] = \begin{cases} w_t & \text{if } \ell = 2^m(t-1) + \sigma(i,t) \text{ and } t \neq j \\ \lambda_i & \text{if } \ell = n2^m + 1 \\ 0 & \text{otherwise}, \end{cases}.$$

To see why $\vec{a}^i_{A,j}$ has this form, notice that if $\ell = 2^m(t-1) + \sigma(i,t)$, then ℓ is the index in $\vec{v}$ of Bidder t's valuation for bundle it receives in allocation $\vec{o}_i$. In other words, $\vec{v}[\ell] = v_t(o_{i,t})$. We set $\vec{a}^i_{A,j}[\ell]$ to w_t so that when we dot $\vec{a}^i_{A,j}$ with $(\vec{v}, 1)$, we obtain the summand $w_t v_t(o_{i,t})$. We repeat this process for all $t \neq j$. If we also set the final index of $\vec{a}^i_{A,j}$ to λ_i, i.e. $\vec{a}^i_{A,j}[n2^m + 1] = \lambda_i$, then $(\vec{v}, 1) \cdot \vec{a}^i_{A,j} = \sum_{\ell \neq j}w_\ell v_\ell(o_{i,\ell}) + \lambda_i$.

Notice that each vector $\vec{a}^i_{A,j}$ is high-dimensional but very sparse. In fact, its ℓ_1-norm is relatively small: $\|\vec{a}^i_{A,j}\|_1 \leq nH_{\overline{w}} + H_\lambda$. Moreover, the set of all "input" vectors $(\vec{v}, 1)$ has a bounded ℓ_∞

norm: $||(\vec{v}, 1)||_\infty \leq \max\{H_v, 1\} = \hat{H}_v$. Therefore, we may use special bounds for sets of linear functions with small ℓ_1-norm. Specifically, it is well known that if $\mathcal{K}$ is the set of linear functions in $\mathbb{R}^d$ of the form $\mathcal{K} = \{\vec{x} \mapsto \vec{w} \cdot \vec{x} \mid ||\vec{w}||_1 \leq B\}$ and if with probability one, $||\vec{x}||_\infty \leq X_\infty$, then $\mathcal{R}_N(\mathcal{K}) \leq X_\infty B \sqrt{2 \log d / N}$. Therefore, if $\mathcal{H}_j^i$ is the class of linear functions $h_{A,j}^i(\vec{v}) = \vec{a}_{A,j}^i \cdot (\vec{v}, 1)$ for all AMAs A, i.e.

$$\mathcal{H}_j^i = \left\{ h_{A,j}^i \mid A = (w_1, \ldots, w_n, \lambda_1, \ldots, \lambda_{(n+1)^m}), H_{\underline{w}} \leq |w_i| \leq H_{\overline{w}}, |\lambda_i| \leq H_\lambda \right\},$$

then

$$\mathcal{R}_N(\mathcal{H}_j^i) \leq \hat{H}_v(nH_{\overline{w}} + H_\lambda) \sqrt{\frac{2 \log(n2^m + 1)}{N}}.$$

Now, for two hypothesis sets H and H' of functions mapping from X to $\mathbb{R}$,

$$\mathcal{R}_N(\{\max(h, h') \mid h \in H, h' \in H'\}) \leq \mathcal{R}_N(H) + \mathcal{R}_N(H'), \tag{1}$$

where $\max(h, h')$ denotes the function $x \mapsto \max(h(x), h'(x))$ [Mohri et al., 2012]. Therefore,

$$\mathcal{R}_N(\Phi_j) \leq (n+1)^m \mathcal{R}_N(\mathcal{H}_j^1) \leq (n+1)^m \hat{H}_v(nH_{\overline{w}} + H_\lambda) \sqrt{\frac{2 \log(n2^m + 1)}{N}},$$

which means that,

$$\mathcal{R}_N(\mathcal{L}_j) = \frac{1}{H_{\underline{w}}} \mathcal{R}_N(\Phi_j) \leq \frac{(n+1)^m \hat{H}_v(nH_{\overline{w}} + H_\lambda)}{H_{\underline{w}}} \sqrt{\frac{2 \log(n2^m + 1)}{N}}$$

$$= O\left(\frac{n^m \hat{H}_v(nH_{\overline{w}} + H_\lambda)}{H_{\underline{w}}} \sqrt{\frac{m \log n}{N}} \right).$$

$\square$

Lemma 3.

$$\mathcal{R}_N(\mathcal{L}_{n+1}) = O\left(\frac{n^{m+2}(H_{\overline{w}}H_v + H_\lambda)}{H_{\underline{w}}} \sqrt{\frac{m \log n}{N}} \left(\frac{n\hat{H}_v(nH_{\overline{w}} + H_\lambda)}{H_{\underline{w}}} + \sqrt{n^m \log N} \right) \right).$$

Proof. Recall that

$$rev_{A,n+1}(\vec{v}) = - \sum_{i=1}^{(n+1)^m} \left(\sum_{j=1}^{n} \frac{1}{w_j} \sum_{\ell \neq j} w_\ell v_\ell(o_{i,\ell}) + \lambda_i \right) \mathbf{1}_{\vec{o}_i = \vec{o}_A^*(\vec{v})}$$

and $\mathcal{L}_{n+1} = \{rev_{A,j} \mid (w_1, \ldots, w_n, \lambda_1, \ldots, \lambda_{(n+1)^m}), H_{\underline{w}} \leq |w_i| \leq H_{\overline{w}}, |\lambda_i| \leq H_\lambda\}$. In essence, for a fixed AMA A, $rev_{A,n+1}$ is the sum of $(n+1)^m$ linear functions multiplied by a binary function. In this proof, we will analyze the complexity of the linear and binary functions separately. Then, we can combine our analyses by making use of the following helpful lemma, which is similar to Lemma 3 in DeSalvo et al. [2015].

Lemma 4. *Let $\mathcal{F}$ be a family of functions mapping $\mathcal{X}$ to $[-c, c]$, let $\mathcal{G}$ be a a family of functions mapping $\mathcal{X}$ to $\{0, 1\}$, and let $\mathcal{H} = \{fg \mid f \in \mathcal{F}, g \in \mathcal{G}\}$. Then*

$$\mathcal{R}_N(\mathcal{H}) \leq (c+1)(\mathcal{R}_N(\mathcal{F}) + \mathcal{R}_N(\mathcal{G})).$$

Proof of Lemma 4. Notice that for any $f \in \mathcal{F}, g \in \mathcal{G}$, we have that $fg = \frac{1}{4}[(f+g)^2 - (f-g)^2]$. For $x \in [-c, c+1]$, the function $x \mapsto \frac{1}{4}x^2$ is $\frac{1}{2}(c+1)$-Lipschitz. The same holds for $x \in [-c-1, c]$. Therefore, by Talagrand's lemma (see Mohri et al. [2012], for example), we have that

$$\widehat{\mathcal{R}}_S(\mathcal{H}) \leq \frac{1}{2}(c+1)[\widehat{\mathcal{R}}_S(\mathcal{F}+\mathcal{G}) + \widehat{\mathcal{R}}_S(\mathcal{F}-\mathcal{G})] \leq (c+1)(\widehat{\mathcal{R}}_S(\mathcal{F}) + \widehat{\mathcal{R}}_S(\mathcal{G})).$$

Therefore, $\mathcal{R}_N(\mathcal{H}) \leq (c+1)(\mathcal{R}_N(\mathcal{F}) + \mathcal{R}_N(\mathcal{G}))$. $\square$

To use Lemma 4, we first define a set of functions for each $i \in [(n+1)^m]$ which will function as the set of real-valued functions in the lemma statement. In particular, let

$$\mathcal{F}_i = \left\{ f_{A,i} \mid A = (w_1, \ldots, w_n, \lambda_1, \ldots, \lambda_{(n+1)^m}), H_{\underline{w}} \le |w_j| \le H_{\overline{w}}, |\lambda_j| \le H_\lambda \right\},$$

where

$$f_{A,i}(\vec{v}) = \sum_{j=1}^n \frac{1}{w_j} \sum_{\ell \ne j} w_\ell v_\ell(o_{i,\ell}) + \lambda_i.$$

As in the proof of Lemma 2, we can write each $f_{A,i}$ as a linear function from $\mathbb{R}^{n2^m+1}$ to $\mathbb{R}$ as follows. Let $h_{A,i}(\vec{v}, 1) = (\vec{v}, 1) \cdot \vec{a}_{A,i}$, where

$$\vec{a}_{A,i}[\ell] = \begin{cases} w_t \sum_{s \ne t} \frac{1}{w_s} & \text{if } \ell = 2^m(t-1) + \sigma(i,t) \\ \lambda_i \sum_{i=1}^n \frac{1}{w_i} & \text{if } \ell = n2^m + 1 \\ 0 & \text{otherwise} \end{cases}.$$

Then $f_{A,i}(\vec{v}) = h_{A,i}(\vec{v}, 1)$. As before, we have that $||(\vec{v}, 1)||_\infty \le \max\{H_v, 1\} = \hat{H}_v$. Moreover, $||\vec{a}_{A,i}||_1 \le \frac{n}{H_{\underline{w}}}(nH_{\overline{w}} + H_\lambda)$. Again, using the same ℓ_1-norm Rademacher complexity bound for linear functions as in Lemma 2, we have that

$$\mathcal{R}_N(\mathcal{F}_i) \le \frac{n\hat{H}_v}{H_{\underline{w}}}(nH_{\overline{w}} + H_\lambda)\sqrt{\frac{2\log(n2^m+1)}{N}}. \tag{2}$$

Now, we define a set of functions $\mathcal{G}_i$ for each $i \in [(n+1)^m]$ which will function as the set of binary-valued functions in the statement of Lemma 4. Let

$$\mathcal{G}_i = \left\{ g_{A,i} \mid A = (w_1, \ldots, w_n, \lambda_1, \ldots, \lambda_{(n+1)^m}), H_{\underline{w}} \le |w_j| \le H_{\overline{w}}, |\lambda_j| \le H_\lambda \right\},$$

where $g_{A,i}(\vec{v}) = 1$ if and only if $i = \vec{o}_A^*(\vec{v})$, i.e.

$$g_{A,i}(\vec{v}) = \begin{cases} 1 & \text{if } i = \underset{i \in [(n+1)^m]}{\operatorname{argmax}} \left\{ \sum_{j=1}^n w_j v_j(o_{i,j}) + \lambda_i \right\} \\ 0 & \text{otherwise} \end{cases}.$$

If we multiply $f_{A,i}(\vec{v})$ and $g_{A,i}(\vec{v})$, then we get $\left(\sum_{j=1}^n \frac{1}{w_j} \sum_{\ell \ne j} w_\ell v_\ell(o_{i,\ell}) + \lambda_i \right) \mathbf{1}_{\vec{o}_i = \vec{o}_A^*(\vec{v})}$, which is one of the summands in $rev_{A,n+1}(\vec{v})$.

We now bound the Rademacher complexity of the class $\mathcal{G}_i$ so that we can apply Lemma 4. Notice that we can also write each function $g_{A,i}(\vec{v})$ as an intersection of $(n+1)^m - 1$ binary-valued functions $\{c_\ell\}$, where $\ell \in [(n+1)^m] \setminus \{i\}$, as follows.

$$c_\ell(\vec{v}) = \begin{cases} 1 & \text{if } \sum_{j=1}^n w_j v_j(o_{i,j}) + \lambda_i \ge \sum_{j=1}^n w_j v_j(o_{\ell,j}) + \lambda_\ell \\ 0 & \text{otherwise} \end{cases}. \tag{3}$$

Indeed, $c_\ell(\vec{v}) = 1$ for all $\ell \ne i$ if and only if $g_{A,i}(\vec{v}) = 1$, i.e.

$$i = \underset{i \in [(n+1)^m]}{\operatorname{argmax}} \left\{ \sum_{j=1}^n w_j v_j(o_{i,j}) + \lambda_i \right\}.$$

Each function c_ℓ can be written as a linear separator over $\mathbb{R}^{n2^m}$, so the VC dimension of $\{c_\ell\}$ is $n2^m + 1$. This allows us to use Lemma 3.2.3 from Blumer et al. [1989] to bound the VC dimension of $\mathcal{G}_i$.

Lemma 5 (Lemma 3.2.3 from Blumer et al. [1989]). *Let C be a concept class of finite VC dimension $d \ge 1$. For all $s \ge 1$, let $C_s\{\cap_{i=1}^s c_i \mid c_i \in C, 1 \le i \le s\}$. Then for all $s \ge 1$, the VC dimension of C_s is less than $2ds \log(3s)$.*

Therefore, the VC dimension of $\mathcal{G}_i$ is less than $2(n2^m + 1)(n+1)^m \log(3 \cdot (n+1)^m) = O(mn^m \log n)$. By Corollary 3.1 in Mohri et al. [2012], we have that

$$\mathcal{R}_N(\mathcal{G}_i) = O\left(\sqrt{\frac{mn^m \log n \log N}{N}}\right). \tag{4}$$

Putting Equations (2) and (4) together with Lemma 4, we conclude that if $\mathcal{H}_i = \{f_{A,i} g_{A,i} \mid f_{A,i} \in \mathcal{F}_i, g_{A,i} \in \mathcal{G}_i\}$, then

$$\mathcal{R}_N(\mathcal{H}_i) = O\left(\left(\frac{n^2(H_{\overline{w}}H_v + H_\lambda)}{H_{\underline{w}}}\right)\left(\frac{n\hat{H}_v}{H_{\underline{w}}}(nH_{\overline{w}} + H_\lambda)\sqrt{\frac{m \log n}{N}} + \sqrt{\frac{mn^m \log n \log N}{N}}\right)\right).$$

This follows from Lemma 4, since the range of any function in $\mathcal{F}_i$ is $[0, n(n-1)(H_{\overline{w}}H_v + H_\lambda)/H_{\underline{w}}]$. Finally, since

$$rev_{A,n+1}(\vec{v}) = -\sum_{i=1}^{(n+1)^m} f_{A,i}(\vec{v})g_{A,i}(\vec{v}),$$

we have that

$$\mathcal{R}_N(\mathcal{L}_{n+1})$$
$$= O\left(n^m\left(\frac{n^2(H_{\overline{w}}H_v + H_\lambda)}{H_{\underline{w}}}\right)\left(\frac{n\hat{H}_v}{H_{\underline{w}}}(nH_{\overline{w}} + H_\lambda)\sqrt{\frac{m \log n}{N}} + \sqrt{\frac{mn^m \log n \log N}{N}}\right)\right).$$

By rearranging terms, we get the desired result. $\square$

B.2 Lower Bound on Sample Complexity for λ-Auctions

In this section, we show that it is not possible to learn over the set of λ-auction revenue functions under an arbitrary distribution with subexponential sample complexity. Since λ-auctions are a subset of AMAs, this lower bound applies to AMAs as well. In particular, we prove Theorem 8, which states that no algorithm can learn over the class of n-bidder, m-item λ-auction revenue functions with sample complexity $o(n^m)$. This holds even when the bidders' valuation functions are additive.

To prove Theorem 8, we construct a set V of n-bidder, m-item valuation functions taking values in $\{0, 1\}$ where, under each valuation function, each bidder is interested in a specific subset of items, and these subsets are all pairwise disjoint. Moreover, $|V| = n^m - n$. The high level idea is to show that for any subset H of V, there exists a λ-auction that has high revenue over valuation functions in H, but low revenue on the valuation functions in $V \setminus H$. Theorem 7 describes V in more detail. Now suppose that the distribution over the bidders' valuation functions is the uniform distribution over V. This means that if a learning algorithm's input samples consist of only a small subset of V, then we cannot guarantee that any output revenue function will achieve average revenue over the sample which is close to its expected revenue over the distribution, as we require. This immediately implies hardness for learning over the uniform distribution on V. See Theorem 8 for the formal proof.

We now present Theorem 7, wherein we describe the set V of valuation functions which we will use to prove Theorem 8.

Theorem 7. *For any $n, m \geq 2$ and any $\gamma \in (0, 1)$, there exists a set of $N = n^m - n$ n-bidder, m-item additive valuation functions $V = \{\vec{v}^1, \ldots, \vec{v}^N\}$ such that for any $H \subseteq V$, there exists a λ-auction A_H with revenue 0 on $\vec{v}^i$ if $\vec{v}^i \notin H$ and revenue at least $2 - 2\gamma$ on $\vec{v}^i$ otherwise.*

Proof. We define the set $V = \{\vec{v}^1, \ldots, \vec{v}^N\}$ of n-bidder, m-item additive valuation functions, where $\vec{v}^j = \left(v_1^j(\{1\}), \ldots, v_1^j(\{m\}), \ldots, v_n^j(\{1\}) \ldots, v_n^j(\{m\})\right)$, with $N = n^m - n$. Recall that every allocation vector $\vec{o}_j$ is written as $(o_{j,1}, \ldots, o_{j,n})$ where $o_{j,1}, \ldots, o_{j,n}$ are disjoint subsets of the m items being auctioned. First, let $\hat{o}_j$ be the allocation where Bidder j receives all m items. Next, let $\tilde{o}_1, \ldots, \tilde{o}_N$ be a fixed ordering of the $n^m - n$ allocations where all m goods are allocated except $\{\hat{o}_1, \ldots, \hat{o}_n\}$. Let the bundles allocated to the n bidders in $\tilde{o}_\ell$ be $(\tilde{o}_{\ell,1}, \ldots, \tilde{o}_{\ell,n})$ and let N_ℓ be the

13

set of bidders who are allocated some item in allocation $\tilde{o}_\ell$. In other words, $N_\ell = \{j \mid \tilde{o}_{\ell,j} \neq \emptyset\}$. For a sanity check, notice that $\bigcup_{i=1}^n \tilde{o}_{\ell,i}$ is the grand bundle.

We will now define the valuation vectors $\{\vec{v}^1, \dots, \vec{v}^N\}$ in terms of this set of special allocations $\{\tilde{o}_1, \dots, \tilde{o}_N\}$. Specifically, we define $\vec{v}^\ell$ for $\ell \in [N]$ as follows.

If $i \notin N_\ell$ (i.e. $\tilde{o}_{\ell,j} = \emptyset$), set $v_i^\ell(\{j\}) = 0$ for all $j \in [m]$. Otherwise, set

$$v_i^\ell(\{j\}) = \begin{cases} 0 & \text{if } j \notin \tilde{o}_{\ell,i} \\ 1 & \text{if } j \in \tilde{o}_{\ell,i} \end{cases}.$$

We proceed to prove that for any subset $H \subseteq V$, there exists a λ-auction with 0 revenue on all valuation functions in $V \setminus H$ and at least $2 - 2\gamma$ revenue on all valuation functions in H. To define this λ-auction, we set the λ terms such that

$$\lambda(\vec{o_j}) = \begin{cases} 0 & \text{if } \vec{o_j} = \tilde{o}_\ell \text{ for some } \vec{v}^\ell \in H \\ 1 - \gamma & \text{otherwise} \end{cases}.$$

Lemma 6. *If $\vec{v}^\ell \in H$, then the revenue on $\vec{v}^\ell$ is at least $2 - 2\gamma$.*

Proof of Lemma 6. First, note that $\sum_{i=1}^n v_i^\ell(\tilde{o}_{\ell,i}) + \lambda(\tilde{o}_\ell) = m$, and for all allocations $\vec{o}_j \neq \tilde{o}_\ell$, $\sum_{i=1}^n v_i^\ell(o_{j,i}) + \lambda(\vec{o}_j) \leq m - 1 + 1 - \gamma < m$. Therefore, the λ-auction allocation is $\tilde{o}_\ell$.

In order to analyze the revenue of this λ-auction, we must understand the payments of each bidder, which means that we must investigate what the outcome of this λ-auction would be without any one bidder's participation. To this end, suppose $i \in N_\ell$, so Bidder i is allocated some item in $\tilde{o}_\ell$, i.e. $\tilde{o}_{\ell,i} \neq \emptyset$. Then $\sum_{j \neq i} v_j^\ell(\tilde{o}_{\ell,j}) + \lambda(\tilde{o}_\ell) = m - |\tilde{o}_{\ell,i}|$ because Bidder i's valuation for the bundle $\tilde{o}_{\ell,i}$ is exactly $|\tilde{o}_{\ell,i}|$.

By construction, no bidder receives all m items in $\tilde{o}_\ell$, so we know that there exists some $i' \in N_\ell, i' \neq i$. With this fact in mind, let $\vec{o}_{-i}$ be the allocation where all bidders in N_ℓ are allocated the same items as they are in $\tilde{o}_\ell$ and Bidder i receives the empty set. This is one possible allocation of the λ-auction without Bidder i's participation, and therefore the social welfare of the other bidders will be at least as high under this allocation as it would be in the true allocation of the λ-auction without Bidder i's participation. By construction, $\lambda(\vec{o}_{-i}) = 1 - \gamma$. Therefore, $\sum_{\ell \neq i} v_j^\ell(o_{-i,j}) + \lambda(\vec{o}_{-i}) = m - |\tilde{o}_{\ell,i}| + 1 - \gamma$ which means that Bidder i must pay at least $(m - |\tilde{o}_{\ell,i}| + 1 - \gamma) - (m - |\tilde{o}_{\ell,i}|) = 1 - \gamma$. We know that $|N_\ell| \geq 2$, i.e. there are at least 2 bidders who receive a non-empty bundle and therefore must pay at least $1 - \gamma$, so the revenue of this λ-auction is at least $2 - 2\gamma$. $\square$

Lemma 7. *If $\vec{v}^\ell \notin H$, then the revenue on $\vec{v}^\ell$ is 0.*

Proof of Lemma 7. First, note that $\sum_{i=1}^n v_i^\ell(\tilde{o}_{\ell,i}) + \lambda(\tilde{o}_\ell) = m + 1 - \gamma$, and for all allocations $\vec{o}_j \neq \tilde{o}_\ell$, $\sum_{i=1}^n v_i^\ell(\vec{o}_{j,i}) + \lambda(\vec{o}_j) \leq m - 1 + 1 - \gamma < m$, so the λ-auction allocation is $\tilde{o}_\ell$. Now, suppose $i \in N_\ell$. Then $\sum_{j \neq i} v_j^\ell(\tilde{o}_{\ell,j}) + \lambda(\tilde{o}_\ell) = m - |\tilde{o}_{\ell,i}| + 1 - \gamma$. Since Bidder i is the only bidder with nonzero valuations for the items in $\tilde{o}_{\ell,i}$ under $\vec{v}^\ell$, any allocation $\vec{o}_{-i}$ without his participation will have social welfare at most $\sum_{j \neq i} v_j^\ell(o_{-i,j}) + \lambda(\vec{o}_{-i}) \leq m - |\tilde{o}_{\ell,i}| + 1 - \gamma$. Therefore, Bidder i pays nothing.

Of course, for any Bidder $i \notin N_\ell$, her presence in the auction makes no difference on the resulting allocation because her valuation function under $\vec{v}^\ell$ is 0 on all items, so she pays nothing as well. Therefore, the revenue on $\vec{v}^\ell$ is 0. $\square$

Putting Lemmas 6 and 7 together, we have the desired result. $\square$

We now use Theorem 7 to prove Theorem 8.

Theorem 8. *Let $\mathcal{ALG}$ be an arbitrary learning algorithm that uses only a polynomial number of training samples drawn i.i.d. from the underlying distribution and produces a λ-auction. For any $\epsilon \in (0, 1)$, there exists a distribution $\mathcal{D}$ and a λ-auction A^* such that, with probability 1 (over the draw of the set of training samples $\mathcal{S}$),*

$$\frac{1}{|\mathcal{S}|} \sum_{\vec{v} \in \mathcal{S}} rev_{A^*}(\vec{v}) - \underset{\vec{v} \sim \mathcal{D}}{\mathbb{E}}[rev_{A^*}(\vec{v})] > \epsilon.$$

14

Proof. Let $\gamma = 1 - \epsilon$ and let V be the set of valuation functions proven to exist in Theorem 7 corresponding to γ (i.e. for any $H \subseteq V$, there exists a λ-auction A_H with revenue 0 on $\vec{v}$ if $\vec{v} \in H$ and revenue at least $2 - 2\gamma$ on $\vec{v}$ otherwise). Let $\mathcal{D}$ be the uniform distribution on V.

Suppose that $\mathcal{ALG}$ uses a set $\mathcal{S}$ of $\ell \leq n^c$ samples, where c is a constant. Of course, $\mathcal{S} \subseteq V$, so let A^* be the λ-auction with 0 revenue on every valuation function not in the sample and revenue at least $2 - 2\gamma$ on every valuation function in the sample. We know that A^* exists due to Theorem 7.

Notice that the average empirical revenue of A^* on $\mathcal{S}$ is at least $2 - 2\gamma$. Meanwhile, the probability, on a random draw $\vec{v} \sim \mathcal{D}$ that $rev_{A^*}(\vec{v})$ is 0 is exactly the probability that $\vec{v} \notin \mathcal{S}$. Given that the set of training examples has measure $\frac{n^c}{n^m - n} < \frac{1}{2}$, we have that

$$\frac{1}{|\mathcal{S}|} \sum_{\vec{v} \in \mathcal{S}} rev_{A^*}(\vec{v}) - \mathop{\mathbb{E}}_{\vec{v} \sim \mathcal{D}} [rev_{A^*}(\vec{v})] \geq 2 - 2\gamma - (2 - 2\gamma) \mathop{\mathbb{P}}_{\vec{v} \sim \mathcal{D}} [\vec{v} \in \mathcal{S}]$$

$$> 2 - 2\gamma - (1 - \gamma)$$
$$= 1 - \gamma$$
$$= \epsilon,$$

as desired. $\qquad \square$

B.3 Lower Bound on Sample Complexity for VVCAs

In this section, we prove that it is not possible to learn over the set of VVCA revenue function under and arbitrary distribution with subexponential sample complexity. In particular, we prove that no algorithm can learn over the class of n-bidder, m-item VVCA revenue functions with sample complexity $o(2^m)$. This holds even when the bidders' valuation functions are additive.

The format of this proof similar to that of Theorem 8. Namely, we construct a set V of n-bidder, m-item valuation functions such that $|V| = 2^m - 2$. We then show that for any subset H of V, there exists a VVCA that has high revenue over valuation functions in H, but low revenue on the valuation functions in $V \setminus H$. The set V is described in more detail in Theorem 9. As described in Theorem 8, this immediately implies hardness for learning over the uniform distribution on V. Given the parallel proof structure, we present Theorem 9 and refer the reader to Theorem 8 to see how it implies hardness for learning.

Theorem 9. *For any $m \geq 2$ and any $\gamma \in (0, 1)$, there exists a set of $N = 2^m - 2$ 2-bidder additive valuation functions $V = \{\vec{v}^1, \ldots, \vec{v}^N\}$ such that for any $H \subseteq V$, there exists a VVCA with revenue 0 on $\vec{v}^i$ if $\vec{v}^i \in V$ and revenue $1 - \gamma$ on $\vec{v}^i$ if $\vec{v}^i \notin V$.*

Proof. We define the set $V = \{\vec{v}^1, \ldots, \vec{v}^N\}$ of 2-bidder valuation functions, where $\vec{v}^j = (v_1^j(\{1\}), \ldots, v_1^j(\{m\}), v_2^j(\{1\}) \ldots, v_2^j(\{m\}))$, with $N = 2^m - 2$. Recall that every allocation vector $\vec{o}_j$ can be written as $(o_{j,1}, o_{j,2})$ where $o_{j,1}$ and $o_{j,2}$ are disjoint subsets of the m items being auctioned. In order to define the valuation functions in V, we define $\tilde{b}_1, \ldots, \tilde{b}_N$ to be a arbitrary, fixed ordering of all subsets of $[m]$ except the empty set and the grand bundle. In other words, $\tilde{b}_1, \ldots, \tilde{b}_N$ is an ordering of $2^{[m]} \setminus \{\emptyset, [m]\}$. We will define each valuation function in V in terms of this ordering. In particular, let $\tilde{o}_\ell = (\tilde{b}_\ell^c, \tilde{b}_\ell)$ be the allocation where Bidder 1 receives $\tilde{b}_\ell^c$ and Bidder 2 receives $\tilde{b}_\ell$. Finally, let $\vec{v}^\ell$ for $\ell \in [N]$ be defined as follows.

$$v_1^\ell(\{i\}) = \begin{cases} 1 & \text{if } i \in \tilde{b}_\ell^c \\ 0 & \text{otherwise} \end{cases}$$

and

$$v_2^\ell(\{i\}) = \begin{cases} 1 & \text{if } i \in \tilde{b}_\ell \\ 0 & \text{otherwise} \end{cases}.$$

Clearly, if $w_1 = w_2 = 1$ and $\lambda_1(\vec{o}) = \lambda_2(\vec{o}) = 0$ for all $\vec{o} \in \mathcal{O}$, then the VVCA allocation on any $\vec{v}^\ell \in S$ is the one in which Bidder 2 receives $\tilde{b}_\ell^c$ and Bidder 1 receives $\tilde{b}_\ell$. This has a social welfare of m, whereas any other allocation has a social welfare at most $m - 1$.

We claim that for any $H \subseteq V$, there exists a VVCA with revenue 0 on $\vec{v}^i$ if $\vec{v}^i \in H$ and revenue $1 - \gamma$ on $\vec{v}^i$ if $\vec{v}^i \notin H$. The VVCA has bidder weights $w_1 = w_2 = 1$, and for all $\vec{v}^\ell \in H$, we set $\lambda_1(\tilde{o}_\ell) = c_{1,\tilde{b}_\ell^c} = c_{2,\tilde{b}_\ell} = \lambda_2(\tilde{o}_\ell) = 0$. Otherwise, we set $\lambda_i(\vec{o}) = (1-\gamma)/2$ for each $i \in \{1,2\}$.

Lemma 8. *If $\vec{v}^\ell \in H$, then the revenue on $\vec{v}^\ell$ is $1 - \gamma$.*

Proof of Lemma 8. First, note that $v_1(\tilde{b}_\ell^c) + v_2(\tilde{b}_\ell) + \lambda_1(\tilde{o}_\ell) + \lambda_2(\tilde{o}_\ell) = m$, and for all allocations $\vec{o}_j \neq \tilde{o}_\ell$, $v_1(o_{j,1}) + v_2(o_{j,2}) + \lambda_1(\vec{o}_j) + \lambda_2(\vec{o}_j) \leq m - 1 + 1 - \gamma$. Therefore, the VVCA allocation is $\tilde{o}_\ell$. However, this is neither Bidder 1 nor Bidder 2's favorite weighted allocation, since $v_1(\tilde{b}_\ell^c) + \lambda_1(\tilde{o}_\ell) = |\tilde{b}_\ell^c| < v_1([m]) + c_{1,[m]} = |\tilde{b}_\ell^c| + (1-\gamma)/2$ and $v_2(\tilde{b}_\ell) + \lambda_2(\tilde{o}_\ell) = |\tilde{b}_\ell| < v_2([m]) + c_{2,[m]} = |\tilde{b}_\ell| + (1-\gamma)/2$. This follows from the fact that $\tilde{b}_\ell \neq [m]$ and $\tilde{b}_\ell^c \neq [m]$ for all $\ell \in [N]$, it must be that $\lambda_1([m]) = \lambda_2([m]) = (1-\gamma)/2$.

Since $|\tilde{b}_\ell^c|$ and $|\tilde{b}_\ell|$ are Bidder 1 and 2's highest valuations for any allocation, respectively, and because $(1-\gamma)/2$ is the highest value of any λ term, $v_1([m]) + c_{1,[m]}$ and $v_2([m]) + c_{2,[m]}$ are the maximum weighted valuation that either bidder has for any allocation under this VVCA. Therefore, the revenue of this VVCA on $\vec{v}_\ell$ is $|\tilde{b}_\ell| + |\tilde{b}_\ell^c| + 1 - \gamma - |\tilde{b}_\ell| - |\tilde{b}_\ell^c| = 1 - \gamma$. $\qquad\square$

Lemma 9. *If $\vec{v}^\ell \notin H$, then the revenue on that valuation function pair is 0.*

Proof of Lemma 9. First, note that $v_1(\tilde{b}_\ell^c) + v_2(\tilde{b}_\ell) + \lambda_1(\tilde{o}_\ell) + \lambda_2(\tilde{o}_\ell) = m + 1 - \gamma$, and for all allocations $\vec{o}_j \neq \tilde{o}_\ell$, $v_1(o_{j,1}) + v_2(o_{j,2}) + \lambda_1(\vec{o}_j) + \lambda_2(\vec{o}_j) \leq m - 1 + 1 - \gamma < m + 1 - \gamma$, so the AMA allocation is $\tilde{o}_\ell$. Moreover, $v_1(\tilde{b}_\ell^c) + \lambda_1(\tilde{o}_\ell) = |\tilde{b}_\ell^c| + (1-\gamma)/2 \geq v_1(o_{j,1}) + \lambda_1(\vec{o}_j)$ and $v_2(\tilde{b}_\ell) + \lambda_2(\tilde{o}_\ell) = |\tilde{b}_\ell| + (1-\gamma)/2 \geq v_2(o_{j,2}) + \lambda_2(\vec{o}_j)$ for all allocations $\vec{o}_j \in \mathcal{O}$. Therefore, both bidders receive one of their favorite weighted allocations, so the revenue is 0. $\qquad\square$

$\qquad\square$

C Proofs from Section 3.2

Proof of Lemma 1. We will show that $rev_{\vec{v}}$ can be decomposed into simple components, each of which can be easily analyzed on its own, and by combining these analyses, we prove the lemma statement. To this end, recall that under the VCG mechanism, each winning bidder pays her bid minus a "rebate" equal to the increase in welfare attributable to her presence in the auction. In a c-MBA, each winning bidder pays the boosted version of this amount. In other words, suppose $\vec{o}^*$ is the resulting allocation of a certain c-MBA A and $\vec{o}_{-i}$ is the boosted social-welfare maximizing allocation without Bidder i's participation. More explicitly, $\vec{o}^* = \max_{\vec{o}_j} \left\{ \sum_{i=1}^n v_i(o_{j,i}) + \lambda(\vec{o}_j) \right\}$ and $\vec{o}_{-i} = \max_{\vec{o}_j} \left\{ \sum_{k \neq i} v_k(o_{j,k}) + \lambda(\vec{o}_j) \right\}$, where $\lambda(\vec{o}_j)$ is set according to the MBA allocation boosting rule for all $\vec{o}_j$. Then Bidder i pays

$$p_{i,\vec{v}}(c) = v_i(o_i^*) - \left[\sum_{j=1}^n v_j(o_j^*) + \lambda(\vec{o}^*) - \left(\sum_{j \neq i} v_j(o_{-i,j}) + \lambda(\vec{o}_{-i}) \right) \right],$$

where c is the parameter of the c-MBA, factored into the λ terms. This means that

$$rev_{\vec{v}}(c) = \sum_{i=1}^n p_{i,\vec{v}}(c) = (1-n) \sum_{i=1}^n v_i(o_i^*) - n\lambda(\vec{o}^*) + \sum_{i=1}^n \sum_{j \neq i} v_j(o_{-i,j}) + \lambda(\vec{o}_{-i}).$$

The revenue function can be split into $n + 1$ functions:

$$f_{i,\vec{v}}(c) = \sum_{j \neq i} v_j(o_{-i,j}) + \lambda(\vec{o}_{-i}) \text{ for } i \in \{1, \ldots, n\}$$

and

$$g_{\vec{v}}(c) = (1-n) \sum_{i=1}^n v_i(o_i^*) - n\lambda(\vec{v}^*).$$

16

We claim that $f_{i,\vec{v}}(c)$ is continuous for all i, whereas $g_{\vec{v}}(c)$ has at most one discontinuity. This means that $rev_{\vec{v}}(c) = \sum_{i=1}^{n} f_{i,\vec{v}}(c) + g_{\vec{v}}(c)$ has at most one discontinuity as well. Moreover, the slope of $\sum_{i=1}^{n} f_{i,\vec{v}}(c)$ is between zero and n, whereas the slope of $g_{\vec{v}}(c)$ is zero until its discontinuity, and then is $-n$. Therefore, the slope of $rev_{\vec{v}}(c)$ is at least zero before its discontinuity and at most zero after its discontinuity. This is enough to prove the lemma statement.

To see why these properties are true for the functions $f_{i,\vec{v}}(c)$, first let $\vec{o}^1_{-i}$ be the VCG allocation without Bidder i's participation. In other words, $\vec{o}^1_{-i} = \max_{\vec{o}_j}\left\{\sum_{k \neq i} v_k(o_{j,k})\right\}$. If one bidder is allocated the grand bundle in outcome $\vec{o}^1_{-i}$, then this allocation will only be more valuable as c grows, so $\vec{o}^1_{-i} = \max_{\vec{o}_j}\left\{\sum_{k \neq i} v_k(o_{j,k}) + \lambda(\vec{o}_j)\right\}$ for all values of c, which means that $f_{i,\vec{v}}(c) = \sum_{j \neq i} v_j\left(o^1_{-i,j}\right) + \lambda\left(\vec{o}^1_{-i}\right) = \sum_{j \neq i} v_j\left(o^1_{-i,j}\right) + c$ for all values of c as well. Clearly, in this case, $f_{i,\vec{v}}(c)$ is increasing and continuous. Otherwise, there exists some value c_i such that

$$\sum_{j \neq i} v_j\left(o^1_{-i,j}\right) + \lambda\left(\vec{o}^1_{-i}\right) = \sum_{j \neq i} v_j\left(o^1_{-i,j}\right) \geq \max_{k \neq i}\{v_k([m])\} + c \qquad \text{if } c \leq c_i$$

$$\sum_{j \neq i} v_j\left(o^1_{-i,j}\right) < \max_{k \neq i}\{v_k([m])\} + c \qquad \text{if } c > c_i.$$

This means that $\vec{o}^1_{-i}$ is the allocation of the c-MBA without Bidder i's participation for $c \leq c_i$, and the allocation of the c-MBA without Bidder i's participation for $c > c_i$ is the one where the highest bidder for the grand bundle (excluding Bidder i) wins the grand bundle. Therefore,

$$f_{i,\vec{v}}(c) = \begin{cases} \sum_{j \neq i} v_j\left(o^1_{-i,j}\right) & \text{if } c \leq c_i \\ \max_{k \neq i}\{v_k([m])\} + c & \text{if } c > c_i. \end{cases}$$

Notice that $\sum_{j \neq i} v_j\left(o^1_{-i,j}\right) = \max_{k \neq i}\{v_k([m])\} + c_i$, so $f_{i,\vec{v}}(c)$ is continuous. Finally, it is clear that the slope of each $f_{i,\vec{v}}(c)$ is between 0 and 1, so the slope of $\sum_{i=1}^{n} f_{i,\vec{v}}(c)$ is between 0 and n.

Similarly, let $\vec{o}^1$ be the allocation of the VCG mechanism run on $\vec{v}$. Then there exists some c^* such that $\vec{o}^1$ is the allocation of the c-MBA for $c \leq c^*$ and the allocation of the c-MBA for $c > c_i$ is the one where the highest bidder for the grand bundle wins the grand bundle. More explicitly,

$$\sum_{i=1}^{n} v_i\left(o^1_i\right) + \lambda\left(\vec{o}^1\right) = \sum_{i=1}^{n} v_i\left(o^1_i\right) \geq \max\{v_k([m])\} + c \qquad \text{if } c \leq c^*$$

$$\sum_{i=1}^{n} v_i\left(o^1_i\right) < \max\{v_k([m])\} + c \qquad \text{if } c > c^*.$$

Therefore,

$$g_{\vec{v}}(c) = \begin{cases} (1-n)\sum_{i=1}^{n} v_i\left(o^1_i\right) & \text{if } c \leq c^* \\ (1-n)\max\{v_k([m])\} - nc & \text{if } c > c^*. \end{cases}$$

Therefore, $g_{\vec{v}}(c)$ has at most one discontinuity, which falls at c^*. Moreover, the slope of $g_{\vec{v}}(c)$ is 0 for $c < c^*$ and $-n$ for $c > c^*$. As described, these properties of $f_{i,\vec{v}}(c)$ and $g_{\vec{v}}(c)$ are enough to show that the lemma statement holds. $\square$

Proof of Theorem 3. First, we show that the pseudo-dimension of the class of n-bidder, m-item MBAs is at most 2. Let $\mathcal{S} = \left\{\vec{v}^1, \ldots, \vec{v}^N\right\}$ of size N be a set of n-bidder valuation functions that can be shattered by a set C of 2^N MBAs. This means that there exist N witnesses $z^1, \ldots, z^N$ such that each MBA in C induces a binary labeling of the samples $\vec{v}^j$ of $\mathcal{S}$ (whether the revenue of the MBA on $\vec{v}^j$ is at least z^j or strictly less than z^j). Since $\mathcal{S}$ is shatterable, we can thus label $\mathcal{S}$ in every possible way using MBAs in C.

Now, fix one sample $\vec{v}^i \in \mathcal{S}$ and consider $rev_{\vec{v}^i}(c)$. From Lemma 1, we know that there exists $c_i^* \in [0, \infty)$, such that $rev_{\vec{v}^i}(c)$ is non-decreasing on the interval $[0, c_i^*]$ and non-increasing on the interval (c_i^*, ∞). Therefore, there exist two thresholds $t_i^1 \in [0, c_i^*]$ and $t_i^2 \in (c_i^*, \infty) \cup \{\infty\}$ such that $rev_{\vec{v}^i}(c)$ is below its threshold for $c \in [0, t_i^1)$, above its threshold for $c \in (t_i^1, t_i^2)$, and below its threshold for $c \in (t_i^2, \infty)$. Now, merge these thresholds for all N samples on the real line and

c value	Revenue on $\vec{v}^1$	Revenue on $\vec{v}^2$
0	$0 \leq z^1$	$2 \leq z^2$
1.5	$3 \leq z^1$	$5 > z^2$
2.5	$5 > z^1$	$4 \leq z^2$
2	$4 > z^1$	$6 > z^2$

Table 1: Example of a shattered set of size 2

consider the interval (t_1, t_2) between two adjacent thresholds. The binary labeling of the samples in $\mathcal{S}$ on this interval is fixed. In other words, for any sample $\vec{v}^j \in \mathcal{S}$, $rev_{\vec{v}^j}(c)$ is either at least z^j or strictly less than z^j for all $c \in (t_1, t_2)$. There are at most $2N + 1$ intervals between adjacent thresholds, so at most $2N + 1$ different binary labelings of $\mathcal{S}$. Since we assumed $\mathcal{S}$ is shatterable, it must be that $2^N \leq 2N + 1$, so $N \leq 2$.

Finally, we show that the pseudo-dimension of the class of n-bidder, m-item MBAs is at least 2 by constructing a set $\mathcal{S} = \{\vec{v}^1, \vec{v}^2\}$ that can be shattered by the set of MBAs. To construct this sample $\mathcal{S}$, let

$$v_1^1(b_i) = v_2^1(b_i) = \begin{cases} 0 & \text{if } |b_i| < \lfloor m/2 \rfloor \\ 3 & \text{if } \lfloor m/2 \rfloor \leq |b_i| \end{cases} \quad \text{and } v_1^2(b_i) = v_2^2(b_i) = \begin{cases} 0 & \text{if } |b_i| < \lfloor m/2 \rfloor \\ 3 & \text{if } \lfloor m/2 \rfloor \leq |b_i| < m \\ 4 & \text{if } |b_i| = m. \end{cases}$$

Finally, let Bidders 3 through n have all-zero valuations in both $\vec{v}^1$ and $\vec{v}^2$.

Now, let $z^1 = 3$ and $z^2 = 4$. We define four MBAs parameterized by the coefficients $c_1 = 0, c_2 = 1.5, c_3 = 2.5, c_4 = 2$. It is easy to check that this set of MBAs shatters $\mathcal{S}$, witnessed by z^1 and z^2. For example, see Table 1. $\qquad \square$

Proof of Theorem 4. This follows from standard pseudo-dimension sample complexity bounds (e.g. Theorem 19.2 of Anthony and Bartlett [2009]) and Theorem 3. $\qquad \square$

Proof of Theorem 5. Let $\mathcal{S} = \{\vec{v}^1, \dots, \vec{v}^N\}$ of size N be a set of n-bidder valuation function samples that can be shattered by a set C of 2^N MBARPs. This means that there exist N witnesses $z^1, \dots, z^N$ such that each MBARP in C induces a binary labeling of the samples $\vec{v}^j$ in $\mathcal{S}$ (whether the revenue of the MBARP on $\vec{v}^j$ is greater than z^j or at most z^j). Since $\mathcal{S}$ is shatterable, we can thus label $\mathcal{S}$ in every possible way using MBARPs in C.

This proof is similar to the proof of Theorem 3, where we split the real line into a set of intervals $\mathcal{I}$ such that for any $I \in \mathcal{I}$, the binary labeling of $\mathcal{S}$ by the c-MBA revenue function was fixed for all $c \in I$. In the case of MBARPs, however, the domain is $\mathbb{R}^{m+1}$, so we cannot split the domain into intervals in the same way. Instead, we show that we can split the domain into cells such that the binary labeling of $\mathcal{S}$ by the MBARP revenue function is fixed as we range over parameters in a single cell. In this way, we show that $N = O(m^3 \log n)$.

Now, fix $\vec{v}^t \in \mathcal{S}$. First, for each $T \subseteq [m]$, let $\mathcal{O}_T$ be the set of allocations where exactly the elements of T are allocated, and let

$$\vec{o}^T = \underset{\vec{o} \in \mathcal{O}_T}{\arg\max} \left\{ \sum_{i=1}^n v_i^t(o_i) \right\}.$$

Notice that regardless of the reserve prices, if T comprises of the items allocated in the allocation of an MBARP, then $\vec{o}^T$ will be the allocation. After all, if $(r_1, \dots, r_m)$ are the reserve prices of an arbitrary MBARP, then it will always be the case that

$$\sum_{i=1}^n v_i^t(o_i^T) + \sum_{j \notin T} r_j \geq \sum_{i=1}^n v_i^t(o_i') + \sum_{j \notin T} r_j$$

for any allocation $\vec{o}' \in \mathcal{O}_T$ by definition of $\vec{o}^T$.

18

Now, consider an MBARP parameterized by $(c, r_1, \ldots, r_m)$. The allocation will be

$$\vec{o}^T = \operatorname{argmax}\left\{\sum_{i=1}^n v_i^t\left(o_i^{[m]}\right) + c, \left\{\sum_{i=1}^n v_i^t\left(o_i^T\right) + \sum_{j \notin T} r_j\right\}_{T \neq [m]}\right\}.$$

For any $T \subseteq [m]$, let $R_T^{\vec{v}^t}$ be the subset of $\mathbb{R}^{m+1}$ such that if an MBARP is parameterized by $(c, r_1, \ldots, r_m) \in R_T^{\vec{v}^t}$, then the allocation of the MBARP on $\vec{v}^t$ is $\vec{o}^T$. This means that if $T \neq [m]$

$$\sum_{i=1}^n v_i^t\left(o_i^T\right) + \sum_{j \notin T} r_j \geq \sum_{i=1}^n v_i^t\left(o_i^{T'}\right) + \sum_{j \notin T'} r_j \qquad \forall T' \notin \{T, [m]\} \text{ and}$$

$$\sum_{i=1}^n v_i^t\left(o_i^T\right) + \sum_{j \notin T} r_j \geq \sum_{i=1}^n v_i^t\left(o_i^{[m]}\right) + c.$$

In other words, $(c, r_1, \ldots, r_m) \in R_T^{\vec{v}^t}$ if and only if it falls in the intersection of $2^m - 1$ halfspaces:

$$\sum_{j \notin T} r_j - \sum_{j \notin T'} r_j \geq \sum_{i=1}^n v_i^t\left(o_i^{T'}\right) - v_i^t\left(o_i^T\right) \qquad \forall T' \notin \{T, [m]\}$$

$$\sum_{j \notin T} r_j - c \geq \sum_{i=1}^n v_i^t\left(o_i^{[m]}\right) - v_i^t\left(o_i^T\right).$$

Similarly, if $T = [m]$, it is not hard to see that we can write $R_T^{\vec{v}^t}$ as the intersection of $2^m - 1$ halfspaces.

We can also analyze the allocation of an MBARP parameterized by $(c, r_1, \ldots, r_m)$ without Bidder i's participation for any $i \in [n]$, which we need to do in order to analyze the revenue function. To this end, for all $T \subseteq [m]$, let $\mathcal{O}_{T_{-i}}$ be the set of all allocations where exactly the elements of T are allocated to all of the bidders except i, and let

$$\vec{o}^{T-i} = \operatorname*{argmax}_{\vec{o} \in \mathcal{O}_{T_{-i}}}\left\{\sum_{j \neq i} v_j^t\left(o_j\right)\right\}.$$

Again, regardless of the reserve prices, if T consists of the items allocated by an MBARP without Bidder i's participation, then $\vec{o}^{T-i}$ will be the allocation. Now, for an MBARP parameterized by $(c, r_1, \ldots, r_m)$ without Bidder i's participation, the allocation will be

$$\vec{o}^{T-i} = \operatorname{argmax}\left\{\sum_{j \neq i} v_j^t\left(o_j^{[m]-i}\right) + c, \left\{\sum_{j \neq i} v_j^t\left(o_j^{T-i}\right) + \sum_{\ell \notin T} r_\ell\right\}_{T \neq [m]}\right\}.$$

For any $T \subseteq [m]$, let $R_{T-i}^{\vec{v}^t}$ be the subset of $\mathbb{R}^{m+1}$ such that if an MBARP is parameterized by $(c, r_1, \ldots, r_m) \in R_{T-i}^{\vec{v}^t}$, then the allocation of the MBARP without Bidder i's partitipation on $\vec{v}$ is $\vec{o}^{T-i}$. This means that if $T \neq [m]$, then just as before, $(c, r_1, \ldots, r_m) \in R_{T-i}^{\vec{v}^t}$ if and only if it falls in the intersection of $2^m - 1$ halfspaces:

$$\sum_{\ell \notin T} r_\ell - \sum_{\ell \notin T'} r_\ell \geq \sum_{j \neq i} v_j\left(o_j^{T'-i}\right) - \sum_{j \neq i} v_j\left(o_j^{T-i}\right) \qquad \forall T' \notin \{T, [m]\}$$

$$\sum_{\ell \notin T} r_\ell - c \geq \sum_{j \neq i} v_j\left(o_j^{[m]-i}\right) - \sum_{j \neq i} v_j\left(o_j^{T-i}\right).$$

Similarly, if $T = [m]$, we can write $R_{T-i}^{\vec{v}}$ as the intersection of $2^m - 1$ halfspaces.

19

Clearly, $\left\{R_T^{\vec{v}^t}\right\}_{T\subseteq[m]}$ partition $\mathbb{R}^{m+1}$, since there will always be some allocation of an MBARP parameterized by an arbitrary point in $\mathbb{R}^{m+1}$. Similarly, $\left\{R_{T-i}^{\vec{v}^t}\right\}_{T\subseteq[m]}$ partition $\mathbb{R}^{m+1}$ for every $i \in [n]$.

Now, suppose

$$(c, r_1, \ldots, r_m) \in R_{T^0}^{\vec{v}^t} \bigcap R_{T_{-1}^1}^{\vec{v}^t} \bigcap \cdots \bigcap R_{T_{-n}^n}^{\vec{v}^t} = R$$

for some $T^0, T^1, \ldots, T^n \subseteq [m]$. We show that $rev_{\vec{v}^t}(c, r_1, \ldots, r_m)$ is linear on R by splitting the analysis into four cases.

1. If $T^0, T^1, \ldots, T^n \neq [m]$ we can write

$$rev_{\vec{v}^t}(c, r_1, \ldots, r_m) = \sum_{i=1}^{n}\left[\sum_{j\neq i}\left(v_j^t\left(o_j^{T_{-i}^i}\right) - v_j^t\left(o_j^{T^0}\right)\right) + \sum_{\ell\notin T^i} r_\ell - \sum_{\ell\notin T^0} r_\ell\right]. \quad (5)$$

2. If $T^0 \neq [m]$ and $T^i = [m]$ for some $i \in \{1, \ldots, n\}$, then we replace the summand of Equation 5 indexed by i with

$$\sum_{j\neq i}\left(v_j^t\left(o_j^{T_{-i}^i}\right) - v_j^t\left(o_j^{T^0}\right)\right) + c - \sum_{\ell\notin T^0} r_\ell.$$

3. If $T^0 = [m]$ and $T_1, \ldots, T_n \neq [m]$, then

$$rev_{\vec{v}^t}(c, r_1, \ldots, r_m) = \sum_{i=1}^{n}\left[\sum_{j\neq i}\left(v_j^t\left(o_j^{T_{-i}^i}\right) - v_j^t\left(o_j^{T^0}\right)\right) + \sum_{\ell\notin T^i} r_\ell - c\right]. \quad (6)$$

4. If $T^0 = [m]$ and $T^i = [m]$ for some $i \in \{1, \ldots, n\}$, then we replace the summand of Equation 6 indexed by i with

$$\sum_{j\neq i}\left(v_j^t\left(o_j^{T_{-i}^i}\right) - v_j^t\left(o_j^{T^0}\right)\right).$$

In all of these cases, $rev_{\vec{v}^t}(c, r_1, \ldots, r_m)$ is a linear function over

$$(c, r_1, \ldots, r_m) \in R_{T^0}^{\vec{v}^t} \bigcap R_{T_{-1}^1}^{\vec{v}^t} \bigcap \cdots \bigcap R_{T_{-n}^n}^{\vec{v}^t} = R.$$

To summarize, we fixed $\vec{v}^t \in \mathcal{S}$ and introduced $n+1$ partitions of $\mathbb{R}^{m+1}$. Each partition is made up of 2^m cells and each cell is defined as the intersection of $2^m - 1$ halfspaces. If we restrict the domain of the revenue function to the intersection of any $n+1$ cells, one from each of the $n+1$ partitions, then the revenue function on that restricted domain is linear and therefore, there is one subregion where $rev_{\vec{v}^t}(c, r_1, \ldots, r_m)$ exceeds its target revenue and one subregion where it does not.

One generous upper bound on the number of different regions induced by taking the intersection of any $n+1$ cells, one from each of the $n+1$ partitions, is the number of different regions induced by the $(n+1)2^m (2^m - 1)$ total hyperplanes. This is at most $(m+1)\left((n+1)2^m (2^m - 1)\right)^{m+1} \leq (m+1)\left((n+1)4^m\right)^{m+1}$ because the number of regions induced by k hyperplanes in $\mathbb{R}^d$ is at most $\sum_{i=1}^{d}\binom{k}{i} \leq dk^d$. Again, if we restrict the domain of the revenue function to any of these induced regions, the revenue function will be linear. As we saw in cases (1)-(4) of our case analysis, depending on the region, the revenue function will take a specific linear form. For a given region R, denote this specific linear form of the revenue function on by R as $rev_{\vec{v}^j}|_R$. With this in mind, we define one more hyperplane per region: $rev_{\vec{v}^j}|_R > z^j$. In total, this contributes at most $(m+1)\left((n+1)4^m\right)^{m+1}$ more hyperplanes, since this is the maximum number of induced regions on $\mathbb{R}^{m+1}$. We are therefore left with at most $\alpha = (m+1)\left((n+1)4^m\right)^{m+1} + (n+1)2^m (2^m - 1) = O\left(mn^m 8^{m^2}\right)$ total hyperplanes per valuation vector function $\vec{v}^j \in \mathcal{S}$.

If we merge all of the N sets of α hyperplanes, $\mathbb{R}^{m+1}$ will be split into at most $(m+1)(N\alpha)^{m+1}$ regions, each of which induces a specific binary labeling of $\mathcal{S}$ (whether or not $\vec{v}^j$ exceeds its target revenue). Therefore, it must be that $2^N \leq (m+1)(N\alpha)^{m+1}$, so $N = O(m \log \alpha) = O\left(m^3 \log n\right)$. $\qquad\square$

C.1 Bundle Reserve Prices Lower Bound

In this section, we justify our choice to concentrate on MBARPs with item-specific reserve prices. In particular, we prove that no algorithm can learn over the class of n-bidder, m-item MBARP revenue functions with bundle-specific reserve prices using sample complexity $o\left(4^m/\sqrt{m}\right)$.

As in the proof of Theorem 8, we construct a special set V of valuation functions. In this case, V is a set of single-bidder, m-item valuation functions and $|V| = \Omega\left(4^m/\sqrt{m}\right)$. We then show that for any subset H of V, there exists a setting of the bundle reserve prices that has high revenue over valuation functions in H, but low revenue on the valuation functions in $V \setminus H$. We describe V more formally in Theorem 10. As we show in Remark 1, this construction can trivially be extended to a set of n-bidder, m-item valuation functions of the same size. Then, as in Theorem 8, this immediately implies hardness for learning over the uniform distribution on V. We provide the construction of V and, given the parallel proof structure, we refer the reader to Theorem 8 to see how this implies hardness of learning.

We now present the construction of the set of valuation functions V.

Theorem 10. *For any $m \geq 2$, there exists a set of $N = \Omega\left(\frac{4^m}{\sqrt{m}}\right)$ single-bidder, m-item valuation function vectors $V = \left\{\vec{v}^1, \ldots, \vec{v}^N\right\}$ such that for any $H \subseteq V$, there exists a set of monotone bundle reserve prices such that the resulting auction has revenue 0 on $\vec{v}^i$ if $\vec{v}^i \in H$ and revenue $1 - \gamma$ on $\vec{v}^i$ if $\vec{v}^i \notin H$, for any $\gamma \in (0,1)$.*

Proof. We define the set $V = \left\{\vec{v}^1, \ldots, \vec{v}^N\right\}$ of single-bidder valuation functions, where $\vec{v}^j = \left(v_1^j(b_1), \ldots, v_1^j(b_{2^m})\right)$. Assume for now that m is even, and let $\tilde{b}_1, \ldots, \tilde{b}_N$ be a fixed ordering of the subsets of $2^{[m]}$ of size $m/2$, so $N = \Omega\left(\frac{4^m}{\sqrt{m}}\right)$. Let $\vec{v}^\ell$ for $\ell \in [N]$ be defined as follows.

$$v_1^\ell(b_i) = \begin{cases} 0 & \text{if } |b_i| < m/2 \\ 1 & \text{if } b_i = \tilde{b}_\ell \\ 0 & \text{if } |b_i| = m/2 \text{ and } b_i \neq \tilde{b}_\ell \\ 1 & \text{if } |b_i| > m/2 \end{cases}.$$

We claim that for any $H \subseteq V$, there exists a set of monotone bundle reserve prices $\{r(b_1), \ldots, r(b_{2^m})\}$ such that the resulting auction has 0 revenue on all valuation functions $\vec{v}^i$ such that $\vec{v}^i \notin H$ and $1 - \gamma$ revenue on all valuation functions $\vec{v}^i \in H$. The reserve prices are defined as follows:

$$v_0(b_i) = r(b_i) = \begin{cases} 0 & \text{if } |b_i| < m/2 \\ 1 - \gamma & \text{if } b_i = \tilde{b}_\ell^c \text{ and } \vec{v}^\ell \notin H \\ 0 & \text{if } b_i = \tilde{b}_\ell^c \text{ and } \vec{v}^\ell \in H \\ 1 - \gamma & |b_i| > m/2 \end{cases}.$$

Regardless of whether or not $\vec{v}^\ell$ is in H, $\left(\tilde{b}_\ell, \tilde{b}_\ell^c\right) = \mathrm{argmax}_{\vec{o} \in \mathcal{O}}\left\{v_0(o_0) + v_1^\ell(o_1)\right\}$ and $1 - \gamma = \max_{b_i \in 2^{[m]}}\left\{v_0(b_i)\right\}$. Therefore, Bidder 1 pays

$$1 - \gamma - v_0\left(\tilde{b}_\ell^c\right) = \begin{cases} 1 - \gamma & \text{if } \vec{v}^\ell \in H \\ 0 & \text{if } \vec{v}^\ell \notin H \end{cases}.$$

$\qquad\square$

Remark 1. *For any $m \geq 2, n \geq 1$, there exists a set of $N = \Omega\left(4^m/\sqrt{m}\right)$ n-bidder valuation function vectors $V = \left\{\vec{v}^1, \ldots, \vec{v}^N\right\}$ such that for any $H \subseteq V$, there exists a set of monotone bundle reserve prices such that the resulting auction has revenue 0 on $\vec{v}^i$ if $\vec{v}^i \notin H$ and revenue $1 - \epsilon$ on $\vec{v}^i$ if $\vec{v}^i \in H$.*

This follows simply by setting v_1^ℓ as in the proof of Theorem 10 and setting $v_j^\ell(b_i) = 0$ for all $\ell \in [N], i \in [2^m], j \neq 1$.

D   Sample complexity bounds for approximation algorithms

Theorem 11. *Let $\mathcal{S} = \{x_1, \dots, x_N\}$ be a sample drawn from $\mathcal{D}$ and $\epsilon, \delta \in (0,1)$ be given. Suppose that N is sufficiently large to ensure that with probability at least $1 - \delta/2$, for any $h \in \mathcal{H}$, $\mathbb{E}_{x \sim \mathcal{D}}[\ell(h,x)] - \frac{1}{N}\sum_{i=1}^N \ell(h, x_i) < \epsilon$.*

Suppose $h^* \in \mathcal{H}$ is a function that minimizes expected loss with respect to the distribution, $\hat{h}$ is a function that minimizes average loss over the sample $\mathcal{S}$, and $\tilde{h} \in \mathcal{H}$ is a function such that the average loss of $\hat{h}$ over $\mathcal{S}$ is within an additive ρ factor of the average loss of $\tilde{h}$ over $\mathcal{S}$. In other words, $\frac{1}{N}\sum_{i=1}^N \ell(\tilde{h}, x_i) - \frac{1}{N}\sum_{i=1}^N \ell(\hat{h}, x_i) \leq \rho$ for some $\rho > 0$. Then with probability at least $1 - \delta$,

$$\mathbb{E}_{x \sim \mathcal{D}}\left[\ell\left(\tilde{h}, x\right)\right] - \mathbb{E}_{x \sim \mathcal{D}}[\ell(h^*, x)] \leq \epsilon + c\sqrt{\frac{\ln(4/\delta)}{2N}} + \rho.$$

Meanwhile, if $\frac{1}{N}\sum_{i=1}^N \ell\left(\tilde{h}, x_i\right) \leq (1+\alpha)\frac{1}{N}\sum_{i=1}^N \ell\left(\hat{h}, x_i\right)$ for some $\alpha \in [0,1)$, then

$$\mathbb{E}_{x \sim \mathcal{D}}\left[\ell\left(\tilde{h}, x\right)\right] - \mathbb{E}_{x \sim \mathcal{D}}[\ell(h^*, x)] \leq \epsilon + (1+\alpha)\left(c\sqrt{\frac{\ln(4/\delta)}{2N}}\right) + \alpha \mathbb{E}_{x \sim \mathcal{D}}[\ell(h^*, x)].$$

Moreover, both bounds are tight in the worst case.

Proof. First, let $\epsilon = 2R_N(\mathcal{H}) + c\sqrt{\frac{2\ln(4/\delta)}{N}}$. For ease of notation, for any $h \in \mathcal{H}$, let $L_{\mathcal{S}}(h) = \frac{1}{N}\sum_{i=1}^N \ell(h, x_i)$ and $L_{\mathcal{D}}(h) = \mathbb{E}_{x \sim \mathcal{D}}[\ell(h, x)]$. Suppose that h^* is the optimal hypothesis in $\mathcal{H}$ (i.e. it minimizes $L_{\mathcal{D}}(h)$, the expected loss over the distribution $\mathcal{D}$), $\hat{h}$ is the empirical risk minimizer (i.e. it minimizes $L_{\mathcal{S}}(h)$, the average loss over the sample $\mathcal{S}$), and $\tilde{h}$ is a hypothesis such that $L_{\mathcal{S}}\left(\tilde{h}\right) - L_{\mathcal{S}}\left(\hat{h}\right) \leq \rho$ for some $\rho > 0$. Then with probability at least $1 - \delta$,

$$L_{\mathcal{D}}\left(\tilde{h}\right) - \epsilon \leq L_{\mathcal{S}}\left(\tilde{h}\right) \tag{7}$$

$$\leq L_{\mathcal{S}}\left(\hat{h}\right) + \rho \tag{8}$$

$$\leq L_{\mathcal{S}}(h^*) + \rho \tag{9}$$

$$\leq L_{\mathcal{D}}(h^*) + c\sqrt{\frac{2\ln(4/\delta)}{2N}} + \rho. \tag{10}$$

Inequality 7 follows from Equation standard Rademacher complexity uniform convergence bounds: with probability at least $1 - \delta/2$, $L_{\mathcal{D}}\left(\tilde{h}\right) - L_{\mathcal{S}}\left(\tilde{h}\right) \leq \epsilon$ (see, for example, Shalev-Shwartz and Ben-David [2014]). Inequality 8 follows from the fact that $L_{\mathcal{S}}\left(\tilde{h}\right) \leq L_{\mathcal{S}}\left(\hat{h}\right) + \rho$. Inequality 9 follows because $\hat{h}$ is the empirical risk minimizer (i.e. it minimizes $L_{\mathcal{S}}(h)$). Finally, inequality 10 is a result, again, of Hoeffding's inequality, which guarantees that with probability at least $1 - \delta/2$, $L_{\mathcal{S}}(h^*) \leq L_{\mathcal{D}}(h^*) + c\sqrt{\frac{2\ln(4/\delta)}{2N}}$.

Rearranging, we get that

$$L_{\mathcal{D}}\left(\tilde{h}\right) - L_{\mathcal{D}}(h^*) \leq \epsilon + c\sqrt{\frac{2\ln(4/\delta)}{2N}} + \rho,$$

as claimed.

Next, suppose that $\tilde{h}$ is a hypothesis such that $L_{\mathcal{S}}\left(\tilde{h}\right) \leq (1+\alpha)L_{\mathcal{S}}\left(\hat{h}\right)$. We similarly can deduce that with probability at least $1 - \delta$,

$$L_{\mathcal{D}}\left(\tilde{h}\right) - \epsilon \leq L_{\mathcal{S}}\left(\tilde{h}\right) \tag{11}$$

$$\leq (1+\alpha)L_{\mathcal{S}}\left(\hat{h}\right) \tag{12}$$

$$\leq (1+\alpha)L_{\mathcal{S}}\left(h^*\right) \tag{13}$$

$$\leq (1+\alpha)\left(L_{\mathcal{D}}\left(h^*\right) + c\sqrt{\frac{2\ln(4/\delta)}{2N}}\right). \tag{14}$$

Rearranging, we get that

$$L_{\mathcal{D}}\left(\tilde{h}\right) \leq \epsilon + (1+\alpha)L_{\mathcal{D}}\left(h^*\right) + (1+\alpha)c\sqrt{\frac{2\ln(4/\delta)}{2N}},$$

which means that

$$L_{\mathcal{D}}\left(\tilde{h}\right) - L_{\mathcal{D}}\left(h^*\right) \leq \epsilon + (1+\alpha)c\sqrt{\frac{2\ln(4/\delta)}{2N}} + \alpha L_{\mathcal{D}}\left(h^*\right),$$

as desired.

We remark that inequalities 7-14 could be tight in the worst case, so both bounds are tight.

$$\square$$

References

Saeed Alaei, Hu Fu, Nima Haghpanah, and Jason Hartline. The simple economics of approximately optimal auctions. In *Proceedings of the Annual Symposium on Foundations of Computer Science (FOCS)*, 2013.

Kareem Amin, Afshin Rostamizadeh, and Umar Syed. Learning prices for repeated auctions with strategic buyers. In *Proceedings of the Annual Conference on Neural Information Processing Systems (NIPS)*, pages 1169–1177, 2013.

Martin Anthony and Peter Bartlett. *Neural Network Learning: Theoretical Foundations*. Cambridge University Press, 2009.

Maria-Florina Balcan, Avrim Blum, Jason Hartline, and Yishay Mansour. Reducing mechanism design to algorithm design via machine learning. *Journal of Computer and System Sciences*, 74:78–89, 2008.

Anand Bhalgat, Sreenivas Gollapudi, and Kamesh Munagala. Optimal auctions via the multiplicative weight method. In *Proceedings of the ACM Conference on Economics and Computation (EC)*, pages 73–90, 2013.

Sayan Bhattacharya, Gagan Goel, Sreenivas Gollapudi, and Kamesh Munagala. Budget constrained auctions with heterogeneous items. In *Proceedings of the Annual Symposium on Theory of Computing (STOC)*, pages 379–388, 2010.

Anselm Blumer, Andrzej Ehrenfeucht, David Haussler, and Manfred K. Warmuth. Learnability and the Vapnik-Chervonenkis dimension. *Journal of the ACM*, 36(4):865–929, 1989.

Yang Cai, Constantinos Daskalakis, and Matt Weinberg. Optimal multi-dimensional mechanism design: Reducing revenue to welfare maximization. In *Proceedings of the Annual Symposium on Foundations of Computer Science (FOCS)*, 2012a.

Yang Cai, Constantinos Daskalakis, and S Matthew Weinberg. An algorithmic characterization of multi-dimensional mechanisms. In *Proceedings of the Annual Symposium on Theory of Computing (STOC)*, pages 459–478, 2012b.

Yang Cai, Constantinos Daskalakis, and S Matthew Weinberg. Reducing revenue to welfare maximization: Approximation algorithms and other generalizations. In *Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 578–595, 2013.

Richard Cole and Tim Roughgarden. The sample complexity of revenue maximization. In *Proceedings of the Annual Symposium on Theory of Computing (STOC)*, pages 243–252, 2014.

Vincent Conitzer and Tuomas Sandholm. Complexity of mechanism design. In *Proceedings of the 18th Annual Conference on Uncertainty in Artificial Intelligence (UAI)*, pages 103–110, 2002.

Vincent Conitzer and Tuomas Sandholm. Self-interested automated mechanism design and implications for optimal combinatorial auctions. In *Proceedings of the ACM Conference on Electronic Commerce (ACM-EC)*, pages 132–141, New York, NY, 2004.

Constantinos Daskalakis, Alan Deckelbaum, and Christos Tzamos. The complexity of optimal mechanism design. In *Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1302–1318, 2014.

Giulia DeSalvo, Mehryar Mohri, and Umar Syed. Learning with deep cascades. In *Proceedings of the Annual Conference on Learning Theory (ALT)*, 2015.

Nikhil R Devanur, Zhiyi Huang, and Christos-Alexandros Psomas. The sample complexity of auctions with side information. In *Proceedings of the Annual Symposium on Theory of Computing (STOC)*, 2016.

Shaddin Dughmi, Li Han, and Noam Nisam. Sampling and representation complexity of revenue maximization. In *International Workshop On Internet And Network Economics (WINE)*, pages 277–291, 2014.

Edith Elkind. Designing and learning optimal finite support auctions. In *Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 736–745, 2007.

Michal Feldman, Nick Gravin, and Brendan Lucier. Combinatorial auctions via posted prices. In *Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, 2015.

Jason D Hartline and Tim Roughgarden. Simple versus optimal mechanisms. In *Proceedings of the ACM Conference on Economics and Computation (EC)*, pages 225–234, 2009.

Justin Hsu, Jamie Morgenstern, Ryan Rogers, Aaron Roth, and Rakesh Vohra. Do prices coordinate markets? *Proceedings of the Annual Symposium on Theory of Computing (STOC)*, 2016.

Zhiyi Huang, Yishay Mansour, and Tim Roughgarden. Making the most of your samples. In *Proceedings of the ACM Conference on Economics and Computation (EC)*, pages 45–60, 2015.

Philippe Jehiel, Moritz Meyer-Ter-Vehn, and Benny Moldovanu. Mixed bundling auctions. *Journal of Economic Theory*, 127(1):494–512, 2007.

Robert Kleinberg and Seth Matthew Weinberg. Matroid prophet inequalities. In *Proceedings of the Annual Symposium on Theory of Computing (STOC)*, pages 123–136, 2012.

Sébastien Lahaie. A kernel-based iterative combinatorial auction. In *AAAI Conference on Artificial Intelligence (AAAI)*, 2011.

Ron Lavi, Ahuva Mu'Alem, and Noam Nisan. Towards a characterization of truthful combinatorial auctions. In *Proceedings of the Annual Symposium on Foundations of Computer Science (FOCS)*, pages 574–583, 2003.

Anton Likhodedov and Tuomas Sandholm. Methods for boosting revenue in combinatorial auctions. In *Proceedings of the National Conference on Artificial Intelligence (AAAI)*, pages 232–237, 2004.

Anton Likhodedov and Tuomas Sandholm. Approximating revenue-maximizing combinatorial auctions. In *Proceedings of the National Conference on Artificial Intelligence (AAAI)*, 2005.

Andres Munoz Medina and Mehryar Mohri. Learning theory and algorithms for revenue optimization in second price auctions with reserve. In *International Conference on Machine Learning (ICML)*, pages 262–270, 2014.

Mehryar Mohri and Andres Munoz. Optimal regret minimization in posted-price auctions with strategic buyers. In *Proceedings of the Annual Conference on Neural Information Processing Systems (NIPS)*, pages 1871–1879, 2014.

Mehryar Mohri and Andres Munoz. Revenue optimization against strategic buyers. In *Proceedings of the Annual Conference on Neural Information Processing Systems (NIPS)*, pages 2521–2529, 2015.

Mehryar Mohri, Afshin Rostamizadeh, and Ameet Talwalkar. *Foundations of Machine Learning*. MIT Press, 2012.

Jamie Morgenstern and Tim Roughgarden. On the pseudo-dimension of nearly optimal auctions. In *Proceedings of the Annual Conference on Neural Information Processing Systems (NIPS)*, pages 136–144, 2015.

Jamie Morgenstern and Tim Roughgarden. Learning simple auctions. In *Conference on Learning Theory (COLT)*, 2016.

Roger Myerson. Optimal auction design. *Mathematics of Operation Research*, 6:58–73, 1981.

David Parkes and Lyle Ungar. Iterative combinatorial auctions: Theory and practice. In *Proceedings of the National Conference on Artificial Intelligence (AAAI)*, pages 74–81, 2000.

David C Parkes, Dimah Yanovsky, and Satinder P Singh. Approximately efficient online mechanism design. In *Proceedings of the Annual Conference on Neural Information Processing Systems (NIPS)*, pages 1049–1056, 2004.

Kevin Roberts. The characterization of implementable social choice rules. In J-J Laffont, editor, *Aggregation and Revelation of Preferences*. North-Holland Publishing Company, 1979.

Tim Roughgarden and Okke Schrijvers. Ironing in the dark. *Proceedings of the ACM Conference on Economics and Computation (EC)*, 2016.

Tuomas Sandholm. Automated mechanism design: A new application area for search algorithms. In *International Conference on Principles and Practice of Constraint Programming (CP)*, pages 19–36, 2003.

Tuomas Sandholm and Anton Likhodedov. Automated design of revenue-maximizing combinatorial auctions. *Operations Research*, 63(5):1000–1025, 2015.

Shai Shalev-Shwartz and Shai Ben-David. *Understanding Machine Learning: From Theory to Algorithms*. Cambridge University Press, 2014.

Pingzhong Tang and Tuomas Sandholm. Mixed-bundling auctions with reserve prices. In *International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, 2012.

Peter Wurman and Michael Wellman. AkBA: A progressive, anonymous-price combinatorial auction. In *Proceedings of the ACM Conference on Electronic Commerce (ACM-EC)*, pages 21–29, 2000.