15-780: Graduate AI

Lecture 1. Logic

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Logic

Why logic?

- *Search: can compactly write down, solve problems like Sudoku*
- *Reasoning: figure out consequences of the knowledge we've given our agent*
- *… and, logical inference is a special case of probabilistic inference*

Propositional logic

- *Constants: T or F*
- *Variables: x, y (values T or F)*
- *Connectives:* ∧, ∨, ¬
  - *Can get by w/ just NAND*
  - *Sometimes also add others:*

    ⊕, ⇒, ⇔, …

George Boole
1815–1864

Propositional logic

- *Build up expressions like* $\neg x \Rightarrow y$
- *Precedence:* $\neg$, $\wedge$, $\vee$, $\Rightarrow$
- *Terminology: variable or constant with or w/o negation =* ***literal***
- *Whole thing =* ***formula*** *or* ***sentence***

Expressive variable names

- *Rather than variable names like x, y, may use names like "rains" or "happy(John)"*
- *For now, "happy(John)" is just a string with no internal structure*
  - *there is no "John"*
  - *happy(John) $\Rightarrow \neg$ happy(Jack) means the same as $x \Rightarrow \neg y$*

But what does it mean?

- *A formula defines a mapping*

  $(assignment\ to\ variables) \mapsto \{T, F\}$

- *Assignment to variables =* ***model***
- *For example, formula* $\neg x$ *yields mapping:*

x	$\neg x$
T	F
F	T

More truth tables

x	y	$x \wedge y$
T	T	T
T	F	F
F	T	F
F	F	F

x	y	$x \vee y$
T	T	T
T	F	T
F	T	T
F	F	F

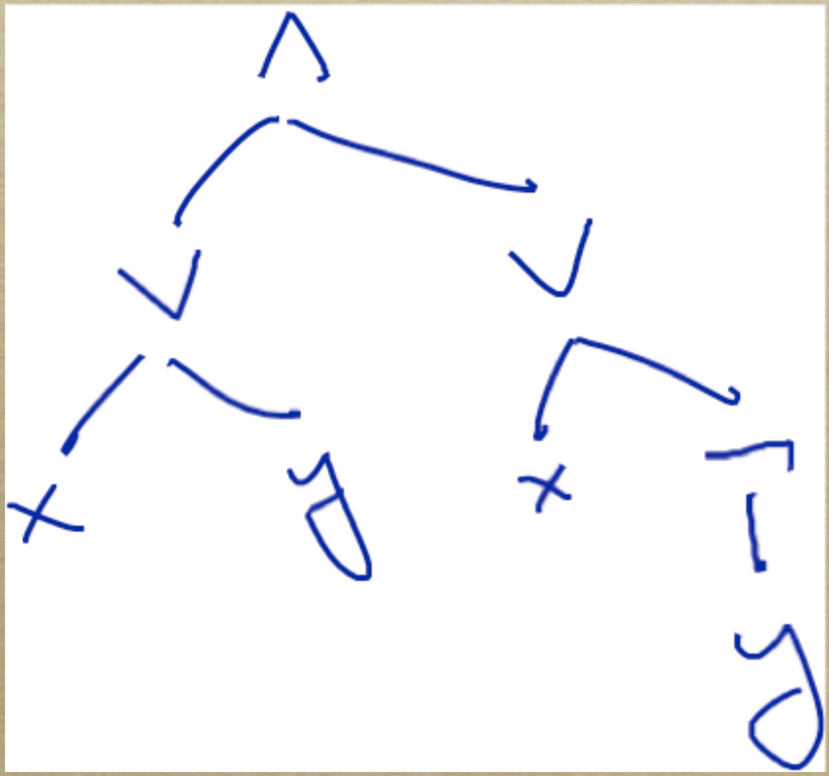
Truth table for implication

- *(a ⇒ b) is logically equivalent to (¬a ∨ b)*
- *If a is True, b must be True too*
- *If a False, no requirement on b*
- *E.g., "if I go to the movie I will have popcorn": if no movie, may or may not have popcorn*

a	b	$a \Rightarrow b$
T	T	T
T	F	F
F	T	T
F	F	T

Complex formulas

- *To evaluate a bigger formula*
  - $(x \lor y) \land (x \lor \neg y)$ *when* $x = F, y = F$
- *Build a parse tree*
- *Fill in variables at leaves using model*
- *Work upwards using truth tables for connectives*



Another example

$$(x \lor y) \Rightarrow z$$ x = F, y = T, z = F

Questions about models and sentences

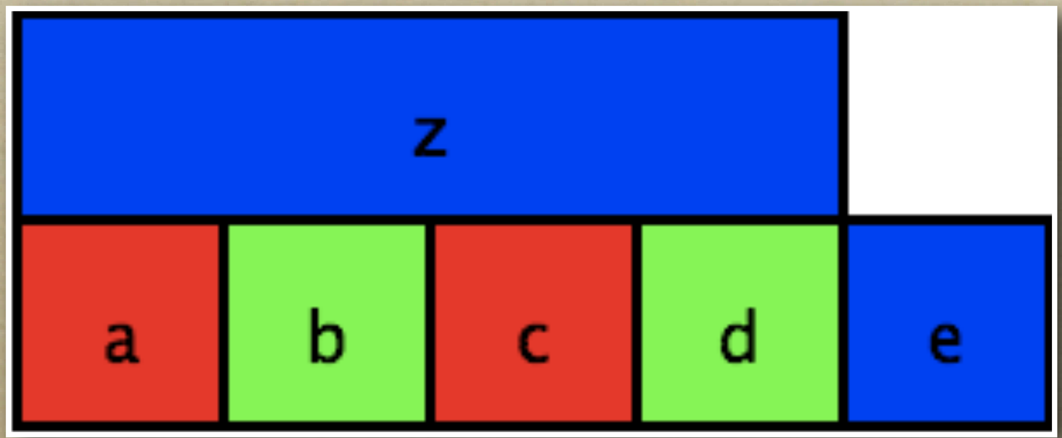
- *How many models make a sentence true?*
  - *Sentence is **satisfiable** if true in some model (famous NP-complete problem)*
  - *If not satisfiable, it is a **contradiction** (false in every model)*
  - *A sentence is **valid** if it is true in every model (called a **tautology**)*

Questions about models and sentences

- *How is the variable X set in {some, all} satisfying models?*
- *This is the most frequent question an agent would ask: given my assumptions, can I conclude X? Can I rule X out?*
- *SAT answers all the above questions*

Bigger Examples

3-coloring



Sudoku

SuDoku Puzzle

		6	3			4	7	
		5	8		7			
1							2	3
	6		1	9				
4	9							
						1	9	8
6					3	5		
		8		5				2
	7	4			6		8	

http://www.cs.qub.ac.uk/~I.Spence/SuDoku/SuDoku.html

Constraint satisfaction problems

$$(aR \vee aB \vee aG) \wedge (bR \vee bB \vee bG) \wedge \ldots \wedge (\overline{aR} \vee \overline{bR}) \wedge (\overline{aB} \vee \overline{bB}) \wedge (\overline{aG} \vee \overline{bG}) \wedge (\overline{aR} \vee \overline{zR}) \ldots$$

- *Like SAT, but:*
  - *variable domains are arbitrary (vs. TF)*
  - *complex constraints (vs. $a \vee b \vee \neg c$)*
- *Sudoku: "at most one 3 in row 5"*
- *Can translate SAT ⇔ CSP*
  - *often CSP more compact*

Minesweeper

0	0	1	v1		
0	0	1	v2		
0	0	1	v3		
1	1	2	v4		
v8	v7	v6	v5		

V = { v1 , v2 , v3 , v4 , v5 , v6 , v7 , v8 }, D = { B (bomb) , S (space) }

C = { (v1,v2) : { (B , S) , (S,B) } ,(v1,v2,v3) : { (B,S,S) , (S,B,S) , (S,S,B)},...}

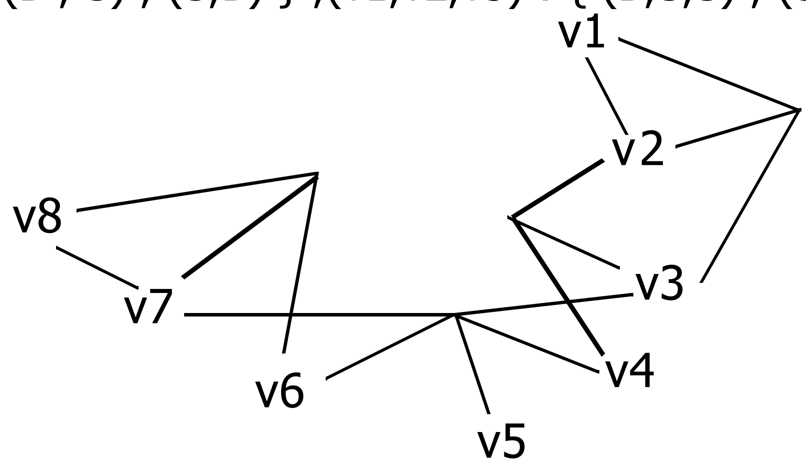


image courtesy Andrew Moore

Propositional planning

init: have(cake)

goal: have(cake), eaten(cake)

eat(cake):

  pre: have(cake)

  eff: -have(cake), eaten(cake)

bake(cake):

  pre: -have(cake)

  eff: have(cake)

Other important logic problems

- *Scheduling (e.g., of factory production)*
- *Facility location*
- *Circuit layout*
- *Multi-robot planning*

Handling uncertainty

- *Minesweeper: what if no safe move?*
- *Say each mine initially present w/ prob p*
- *Common situation: independent "Nature" choices, deterministic rules thereafter*
- *Logic represents deterministic rules ⇒ use logical reasoning as subroutine*

		1			1			1	
1	1	1	1	1	1	1	1	1	

Handling uncertainty

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	$\otimes$	1		$\otimes$	1		$\otimes$	1	
1	1	1	1	1	1	1	1	1	

Handling uncertainty

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- *Say each mine initially present w/ prob p*
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⊗		1	⊗		1	⊗		1	⊗
1	1	1	1	1	1	1	1	1	

Handling uncertainty

- *Minesweeper: what if no safe move?*
- *Say each mine initially present w/ prob p*
- *Common situation: independent "Nature" choices, deterministic rules thereafter*
- *Logic represents deterministic rules ⇒ use logical reasoning as subroutine*

⊗		1	⊗		1	⊗		1	
1	1	1	1	1	1	1	1	1	⊗

Working with formulas

Truth tables get big fast

x	y	z	$(x \vee y) \Rightarrow z$
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

Truth tables get big fast

x	y	z	a		$(x \vee y \vee a) \Rightarrow z$
T	T	T	T		
T	T	F	T		
T	F	T	T		
T	F	F	T		
F	T	T	T		
F	T	F	T		
F	F	T	T		
F	F	F	T		
T	T	T	F		
T	T	F	F		
T	F	T	F		
T	F	F	F		
F	T	T	F		
F	T	F	F		
F	F	T	F		
F	F	F	F		

Definitions

- *Two sentences are* ***equivalent****,* $A \equiv B$*, if they have same truth value in every model*
  - $(rains \Rightarrow pours) \equiv (\neg rains \lor pours)$
  - *reflexive, transitive, symmetric*
- ***Simplifying*** *= transforming a formula into a simpler, equivalent formula*

Transformation rules

$$
\begin{aligned}
(\alpha \land \beta) &\equiv (\beta \land \alpha) \quad \text{commutativity of } \land \\
(\alpha \lor \beta) &\equiv (\beta \lor \alpha) \quad \text{commutativity of } \lor \\
((\alpha \land \beta) \land \gamma) &\equiv (\alpha \land (\beta \land \gamma)) \quad \text{associativity of } \land \\
((\alpha \lor \beta) \lor \gamma) &\equiv (\alpha \lor (\beta \lor \gamma)) \quad \text{associativity of } \lor \\
\neg(\neg\alpha) &\equiv \alpha \quad \text{double-negation elimination}
\end{aligned}
$$

$$
\begin{aligned}
(\alpha \land (\beta \lor \gamma)) &\equiv ((\alpha \land \beta) \lor (\alpha \land \gamma)) \quad \text{distributivity of } \land \text{ over } \lor \\
(\alpha \lor (\beta \land \gamma)) &\equiv ((\alpha \lor \beta) \land (\alpha \lor \gamma)) \quad \text{distributivity of } \lor \text{ over } \land
\end{aligned}
$$

α, β, γ are arbitrary formulas

More rules

$$(\alpha \;\Rightarrow\; \beta) \;\equiv\; (\neg\beta \;\Rightarrow\; \neg\alpha) \quad \text{contraposition}$$
$$(\alpha \;\Rightarrow\; \beta) \;\equiv\; (\neg\alpha \vee \beta) \quad \text{implication elimination}$$
$$(\alpha \;\Leftrightarrow\; \beta) \;\equiv\; ((\alpha \;\Rightarrow\; \beta) \wedge (\beta \;\Rightarrow\; \alpha)) \quad \text{biconditional elimination}$$
$$\neg(\alpha \wedge \beta) \;\equiv\; (\neg\alpha \vee \neg\beta) \quad \text{de Morgan}$$
$$\neg(\alpha \vee \beta) \;\equiv\; (\neg\alpha \wedge \neg\beta) \quad \text{de Morgan}$$

α, β are arbitrary formulas

Still more rules…

- *…can be derived from truth tables*
- *For example:*
  - $(a \vee \neg a) \equiv True$
  - $(True \vee a) \equiv True$ *(T elim)*
  - $(False \wedge a) \equiv False$ *(F elim)*

Example

$$(a \vee \neg b) \wedge (a \vee \neg c) \wedge (\neg(b \vee c) \vee \neg a)$$

Normal Forms

Normal forms

- *A normal form is a standard way of writing a formula*
- *E.g.,* ***conjunctive normal form*** *(CNF)*
  - *conjunction of disjunctions of literals*
  - $(x \lor y \lor \neg z) \land (x \lor \neg y) \land (z)$
  - *Each disjunct called a* ***clause***
- *Any formula can be transformed into CNF w/o changing meaning*

CNF cont'd

happy(John) ∧
(¬*happy(Bill)* ∨ *happy(Sue)*) ∧
man(Socrates) ∧
(¬*man(Socrates)* ∨ *mortal(Socrates)*)

- *Often used for storage of knowledge database*
  - *called **knowledge base** or KB*
- *Can add new clauses as we find them out*
- *Each clause in KB is separately true (if KB is)*

Another normal form: DNF

- *DNF = disjunctive normal form = disjunction of conjunctions of literals*
- *Doesn't compose the way CNF does: can't just add new conjuncts w/o changing meaning of KB*

$$(rains \lor pours) \land (\neg pours \Rightarrow fishing)$$

Transforming to CNF or DNF

- *Naive algorithm:*
  - *replace all connectives with* ∧∨¬
  - *move negations inward using De Morgan's laws and double-negation*
  - *repeatedly distribute over* ∧ *over* ∨ *for DNF (*∨ *over* ∧ *for CNF)*

Example

- *Put in CNF:*

  $(a \vee \neg c) \wedge \neg(a \wedge b \wedge d \wedge \neg e)$

Discussion

- *Problem with naive algorithm: it's exponential! (Space, time, size of result.)*
- *Each use of distributivity can almost double the size of a subformula*

A smarter transformation

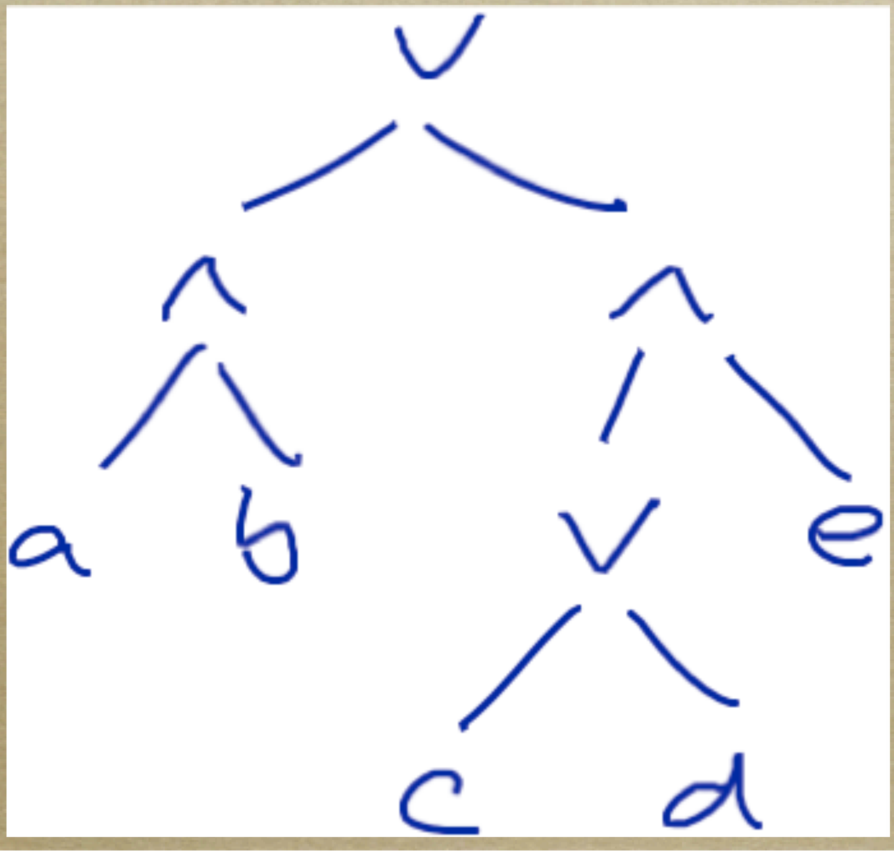
- *Can we avoid exponential blowup in CNF?*
- *Yes, if we're willing to introduce new variables*
- *G. Tseitin. On the complexity of derivation in propositional calculus. Studies in Constrained Mathematics and Mathematical Logic, 1968.*

Tseitin example

- *Put the following formula in CNF:*

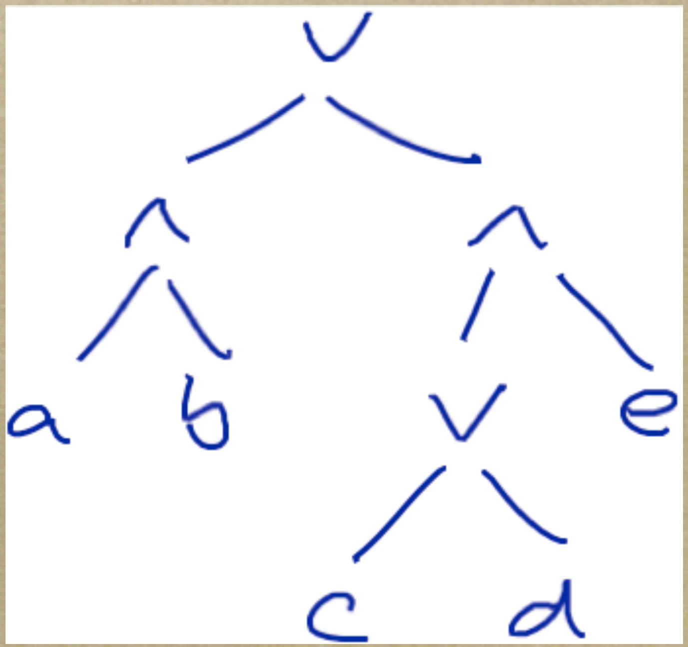
  $(a \wedge b) \vee ((c \vee d) \wedge e)$

- *Parse tree:*



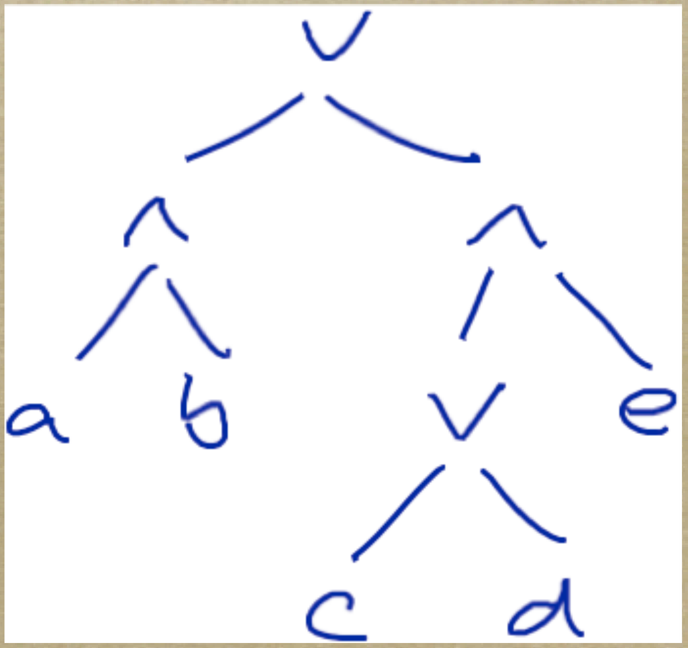
Tseitin transformation

- *Introduce temporary variables*
  - $x = (a \wedge b)$
  - $y = (c \vee d)$
  - $z = (y \wedge e)$



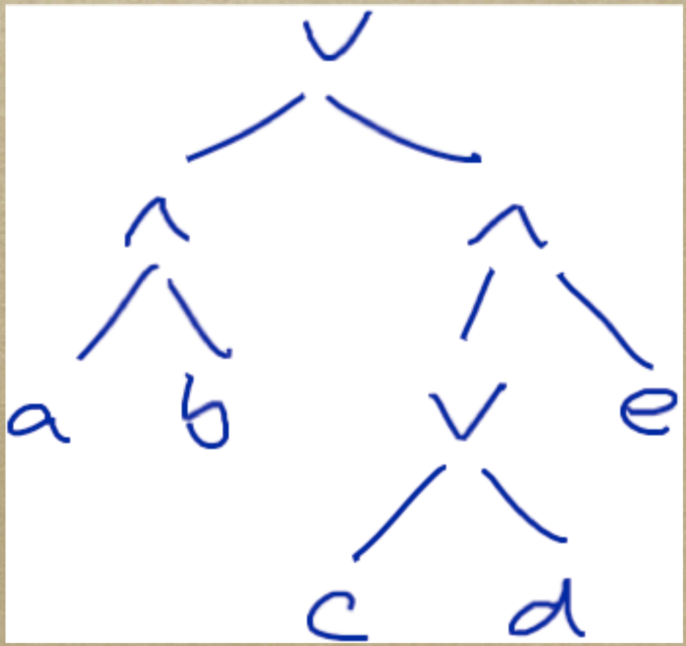
Tseitin transformation

- *To ensure $x = (a \wedge b)$, want*
  - $x \Rightarrow (a \wedge b)$
  - $(a \wedge b) \Rightarrow x$



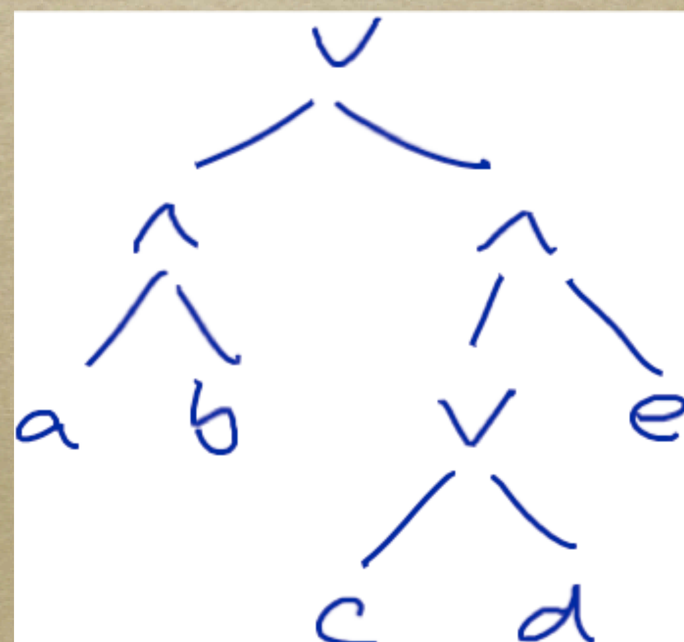
Tseitin transformation

- $x \Rightarrow (a \wedge b)$
- $(\neg x \vee (a \wedge b))$
- $(\neg x \vee a) \wedge (\neg x \vee b)$



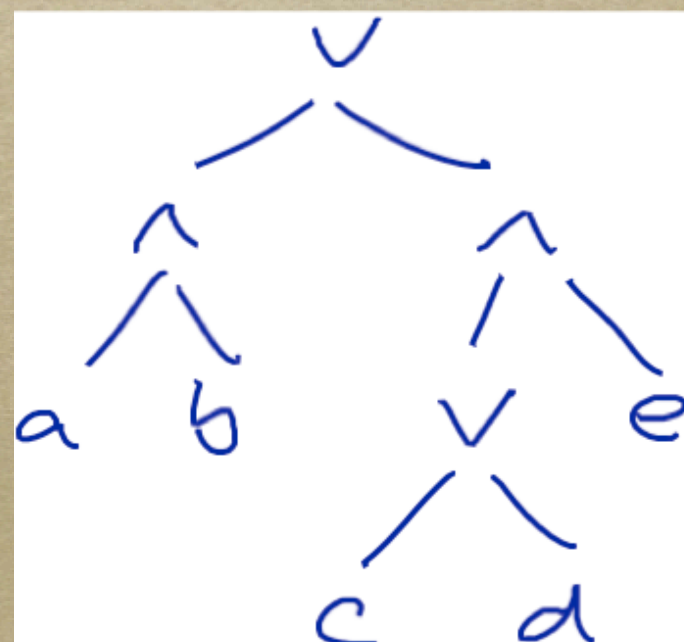
Tseitin transformation

- $(a \wedge b) \Rightarrow x$
- $(\neg(a \wedge b) \vee x)$
- $(\neg a \vee \neg b \vee x)$



Tseitin transformation

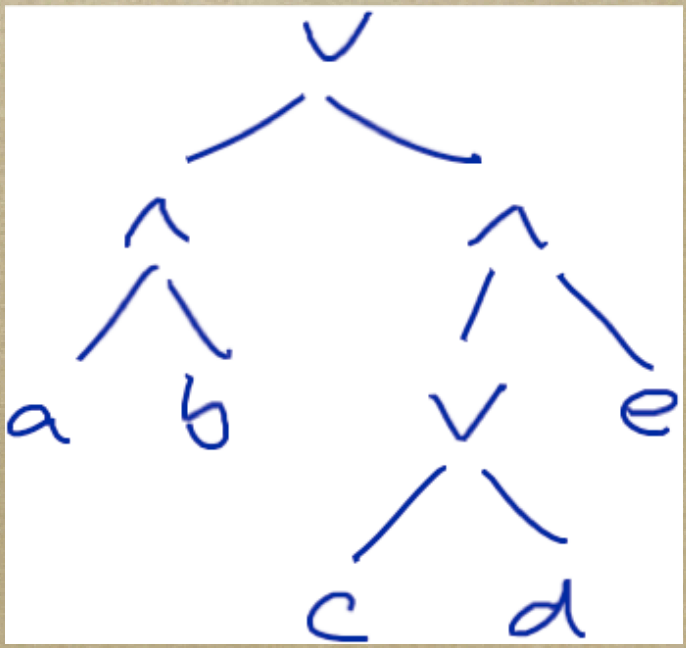
- *To ensure* $y = (c \vee d)$, *want*
  - $y \Rightarrow (c \vee d)$
  - $(c \vee d) \Rightarrow y$



Tseitin transformation

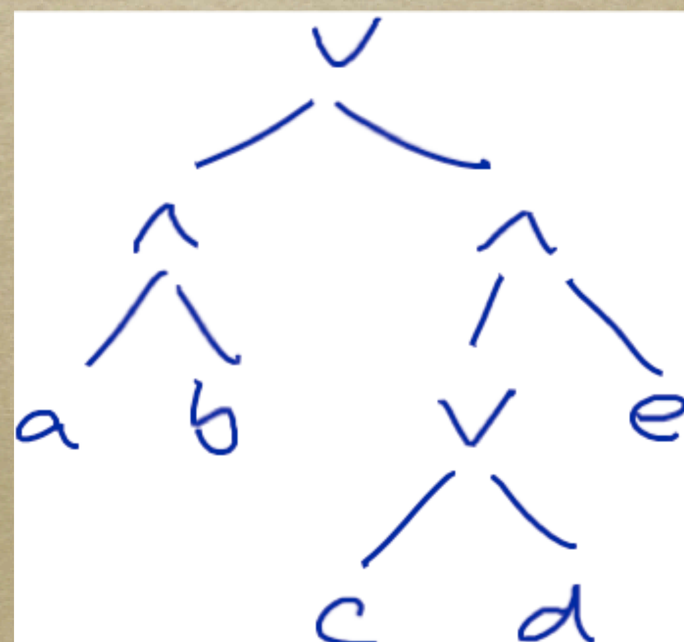
- $y \Rightarrow (c \vee d)$
- $(\neg y \vee c \vee d)$

- $(c \vee d) \Rightarrow y$
- $((\neg c \wedge \neg d) \vee y)$
- $(\neg c \vee y) \wedge (\neg d \vee y)$

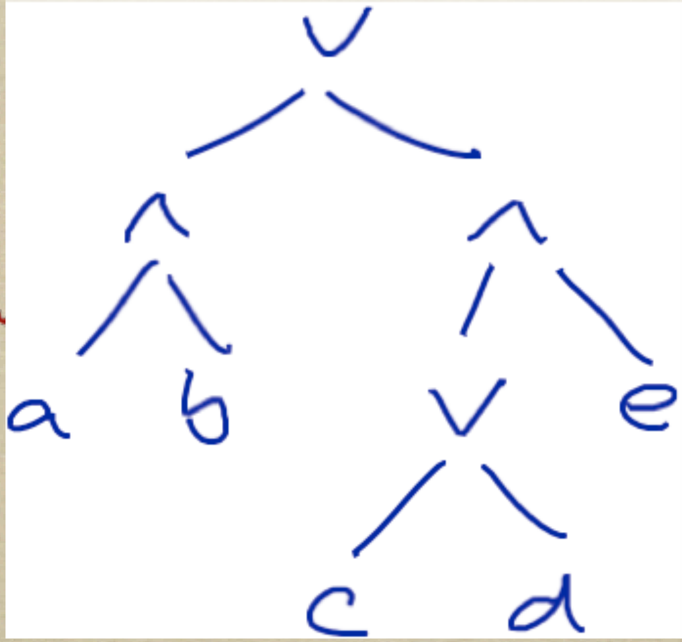


Tseitin transformation

- *Finally*, $z = (y \wedge e)$
- $z \Rightarrow (y \wedge e) \equiv (\neg z \vee y) \wedge (\neg z \vee e)$
- $(y \wedge e) \Rightarrow z \equiv (\neg y \vee \neg e \vee z)$



Tseitin end result



$$(a \wedge b) \vee ((c \vee d) \wedge e) \equiv$$

$$(\neg x \vee a) \wedge (\neg x \vee b) \wedge (\neg a \vee \neg b \vee x) \wedge$$
$$(\neg y \vee c \vee d) \wedge (\neg c \vee y) \wedge (\neg d \vee y) \wedge$$
$$(\neg z \vee y) \wedge (\neg z \vee e) \wedge (\neg y \vee \neg e \vee z) \wedge$$
$$(x \vee z)$$

Compositional Semantics

Semantics

- *Recall: meaning of a formula is a function*

  models $\mapsto$ *{T, F}*

- *Why this choice? So that meanings are* ***compositional***

- *Write [α] for meaning of formula α*

- $[\alpha \wedge \beta](M) = [\alpha](M) \wedge [\beta](M)$

- *Similarly for* ∨*,* ¬*, etc.*

Proofs

Entailment

- *Sentence A **entails** sentence B, $A \models B$, if B is true in every model where A is*
    - *same as saying that $(A \Rightarrow B)$ is valid*

Proof tree

- *A tree with a formula at each node*
- *At each internal node, children* $\models$ *parent*
- *Leaves:* ***assumptions*** *or* ***premises***
- *Root:* ***consequence***
- *If we believe assumptions, we should also believe consequence*

Proof tree example

rains $\Rightarrow$ pours

pours $\wedge$ outside $\Rightarrow$ rusty

rains

outside

Proof by contradiction

- *Assume opposite of what we want to prove, show it leads to a contradiction*
- *Suppose we want to show* $KB \vDash S$
- *Write KB' for* $(KB \wedge \neg S)$
- *Build a proof tree with*
  - *assumptions drawn from clauses of KB'*
  - *conclusion = F*
  - *so,* $(KB \wedge \neg S) \vDash F$ *(contradiction)*

Proof by contradiction

KB

rains ⇒ pours

pours ∧ outside ⇒ rusty

rains

outside

¬ rusty

← negation of desired conclusion

Proof by contradiction