15-780: Grad AI
Lec. 9: Linear programs, Duality

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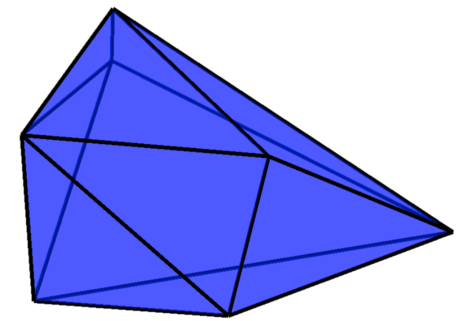
Admin

- Have you tested your handin directories?
  - /afs/cs/user/aothman/dropbox/USERID/
  - where USERID is your Andrew ID
- Poster session:
  - ??? May 9 1:30–4:30 7th floor atrium GHC

Review

- LPs, ILPs, MILPs
  - $\mathbb{R}$ or $\mathbb{Z}$ variables
  - linear $\leq$ $\geq$ $=$
  - linear objective
  - LP relaxations, integrality gap
  - relation to SAT, MAXSAT, PBI
  - complexity (LP: P; ILP: NP & no approx)
  - (in)feasible, (sub)optimal, (in)active

Review



- Standard form: all vars ≥ 0, all = constraints
- Nonsingular: n vars ≥ m constraints, rank m
- Basis
  - spans Rng(A) (m × m invertible submatrix)
  - corresponds to "corner"
  - using row ops to make basic variables into "slacks" → ***tableau*** notation
- Degeneracy: distinct bases yield same corner
- Naïve algorithm: check all bases

Finding corners

x	y	u	v	w	RHS
1	1	1	0	0	4
2	5	0	1	0	12
1	2	0	0	1	5

set x, y = 0

u = 4 v = 12 w = 5

1	1	1	0	0	4
2	5	0	1	0	12
1	2	0	0	1	5

set v, w = 0

x = 1 y = 2 u = 1

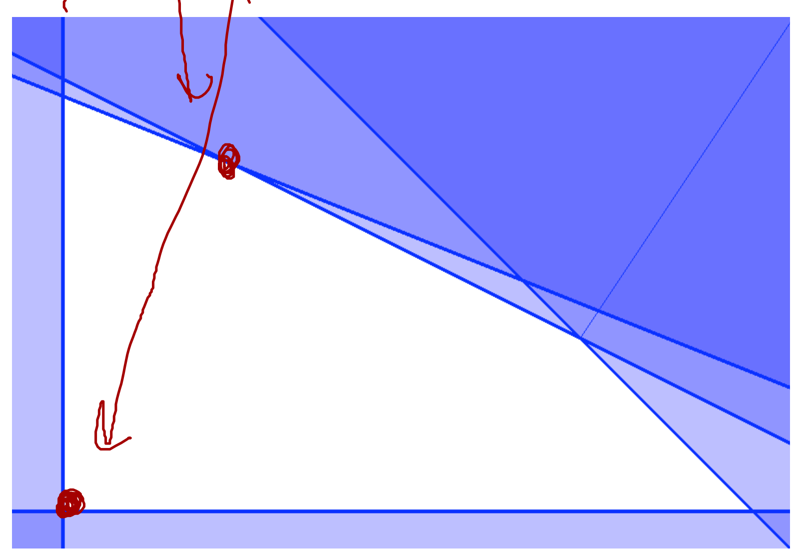
1	1	1	0	0	4
2	5	0	1	0	12
1	2	0	0	1	5

set x, u = 0

y = 4

v = -8

z = -3



Simplex in one slide

(ignoring degeneracy, which is actually important)

- Given a nonsingular standard-form LP
  - make it nonsingular if needed
- Start from a feasible basis and its tableau
  - big-M if needed
- Pick non-basic variable w/ objective > 0 (max)
- Pivot it into basis, getting neighboring basis
  - select exiting variable to keep feasibility
- Repeat until all non-basic variables have objective < 0 (max)

Example

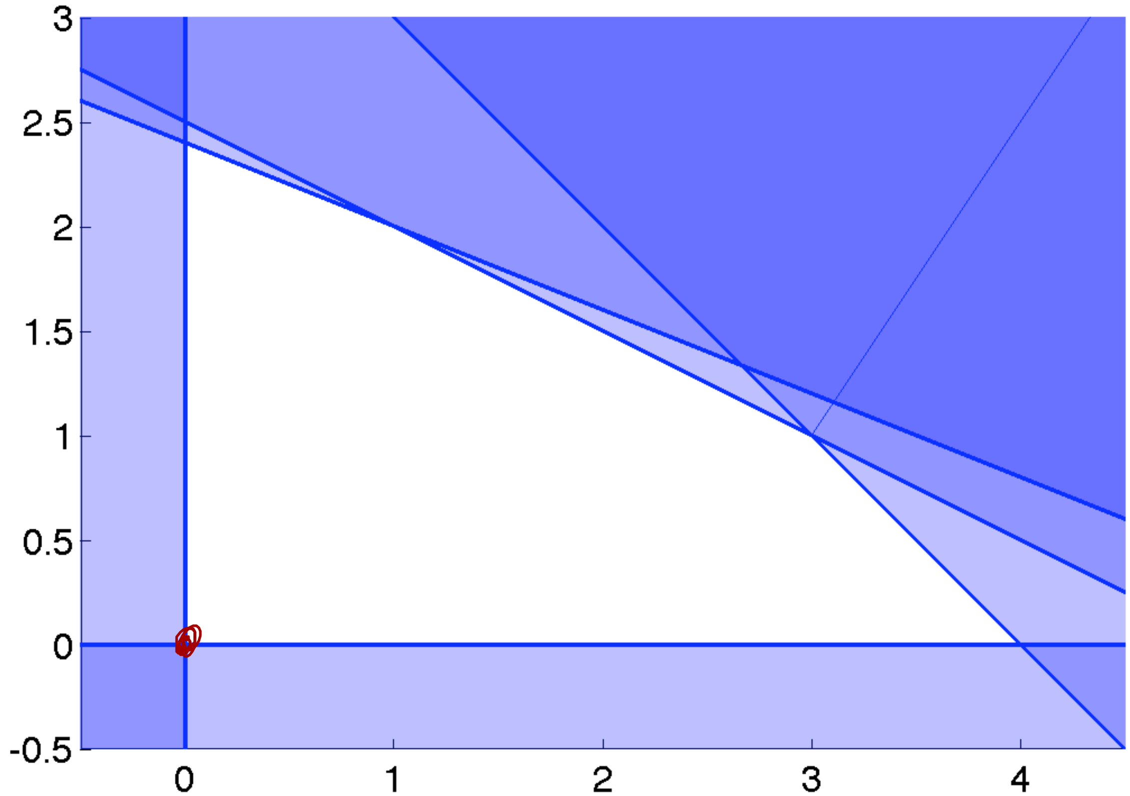
max $2x + 3y$ s.t.

$x + y \leq 4$

$2x + 5y \leq 12$

$x + 2y \leq 5$

$x, y \geq 0$



pivot in (y) → pivot out (t)

x	y	s	t	u	RHS
1	1	1	0	0	4
2	5	0	1	0	12
1	2	0	0	1	5
2	3	0	0	0	↑

x	y	s	t	u	RHS
3/5	0	1	-1/5	0	8/5

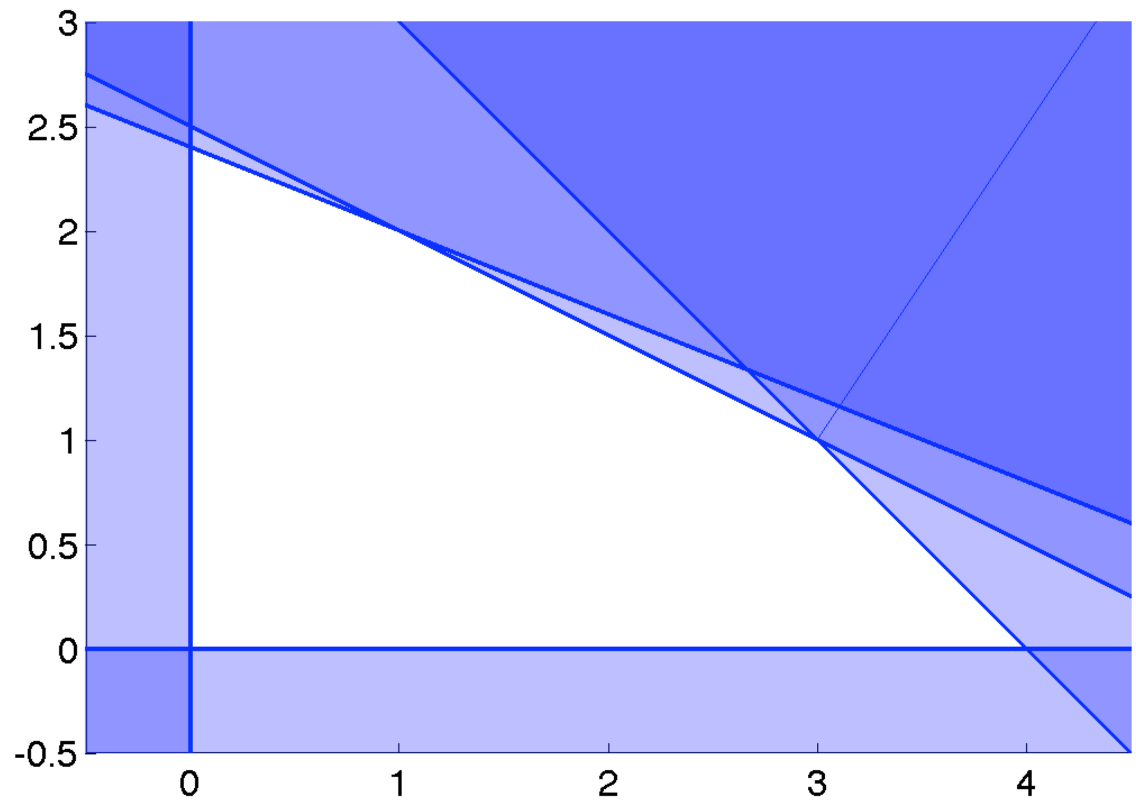
$y \leq 4$

$y \leq 12/5$ ←

$y \leq 5/2$

Example

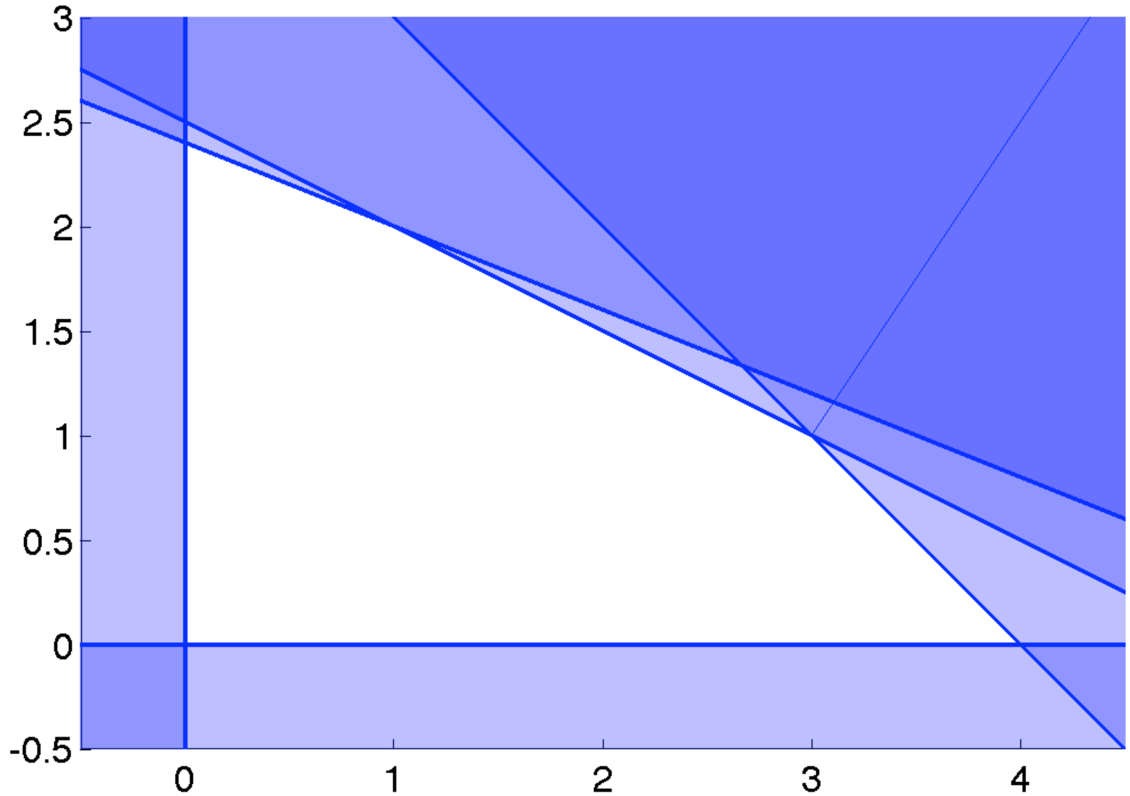
max 2x + 3y s.t.
x + y ≤ 4
2x + 5y ≤ 12
x + 2y ≤ 5



x	y	s	t	u	RHS
0.4	1	0	0.2	0	2.4
0.6	0	1	-0.2	0	1.6
0.2	0	0	-0.4	1	0.2
0.8	0	0	-0.6	0	↑

Example

max 2x + 3y s.t.
x + y ≤ 4
2x + 5y ≤ 12
x + 2y ≤ 5



x	y	s	t	u	RHS
1	0	0	−2	5	1
0	1	0	1	−2	2
0	0	1	1	−3	1
0	0	0	1	−4	↑

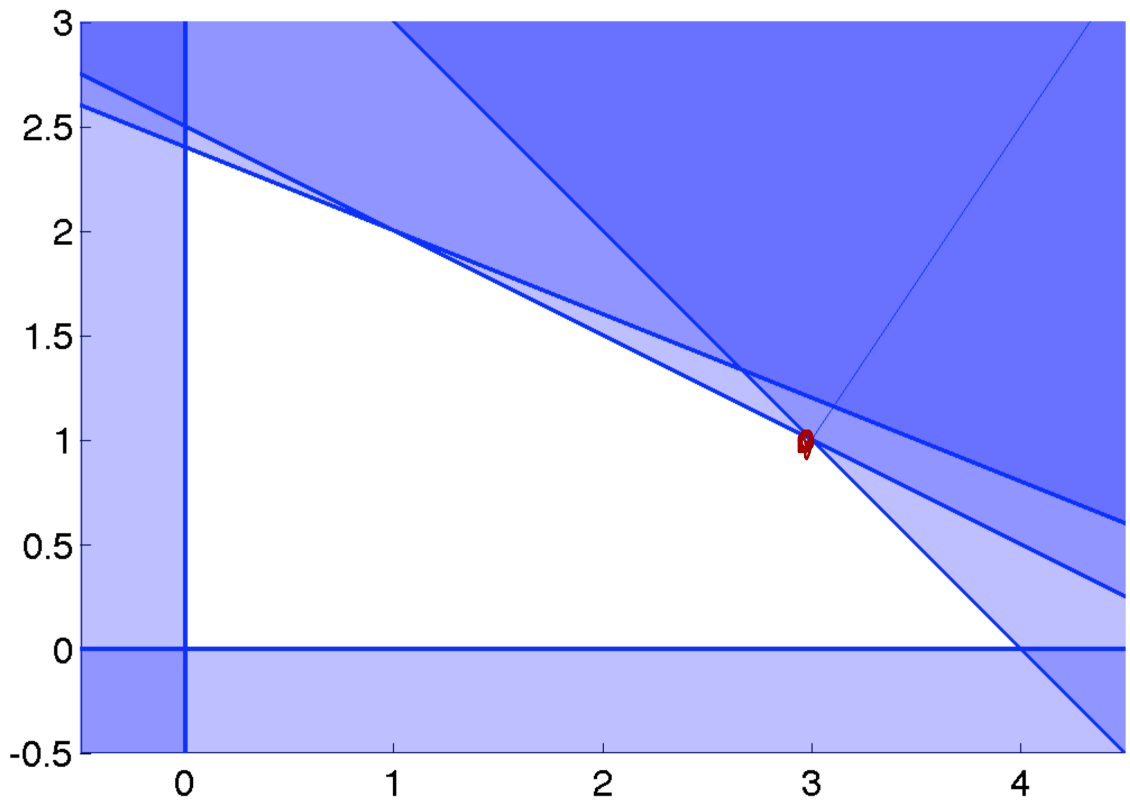
Example

max $2x + 3y$ s.t.

$x + y \leq 4$

$2x + 5y \leq 12$

$x + 2y \leq 5$

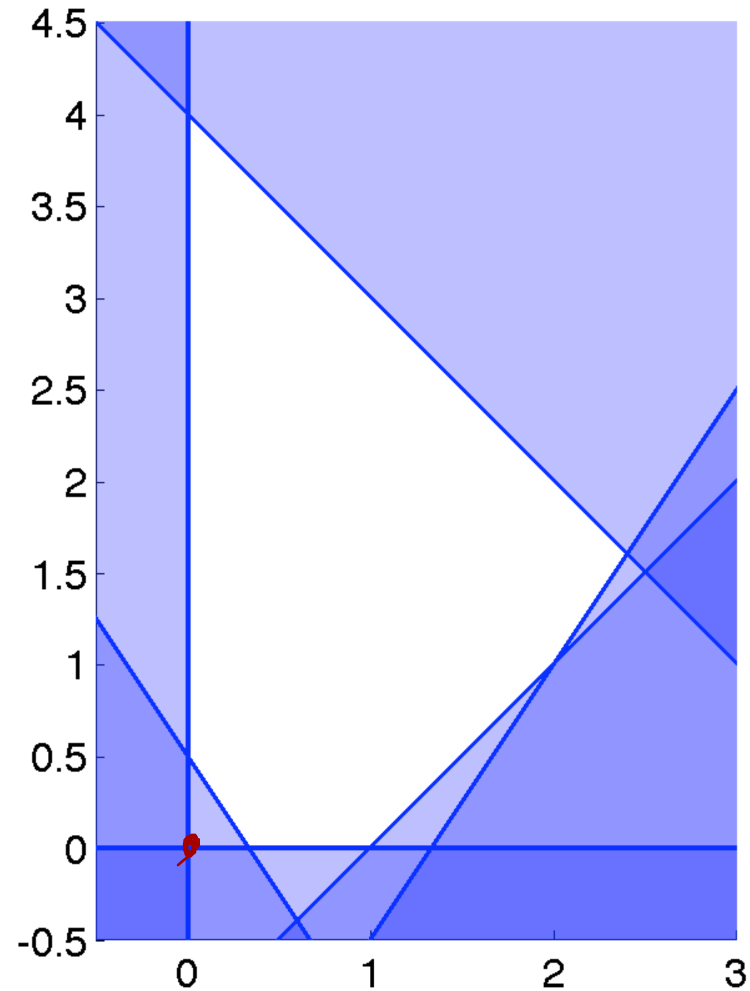


x	y	s	t	u	RHS
1	0	2	0	−1	3
0	1	−1	0	1	1
0	0	1	1	−3	1
0	0	−1	0	−1	↑

multiple constraints violated ==> multiple extra variables, each with –M in objective

Big M

x	y	τ	t	u	v	w	RHS
1	1	0	1	0	0	0	4
3	−2	0	0	1	0	0	4
1	−1	0	0	0	1	0	1
−3	−2	1	0	0	0	1	−1
1	−2	−M	0	0	0	0	↑



- So far, assumed we started w/ initial feasible basis
- How do we get one?
  - for each violated constraint, add var w/ coeff –1
  - penalize in objective, include in initial basis

Ex: combinatorial auctions

- Goods: Newspaper, Magazine, L shoe, R shoe
- Bids (note use of bidding language: 7 rt 16 numbers for B_1 and 1 rt 16 for B_2):

  - N: +5; M: +4
  - N, M: –3
  - L, R: +10
  - N, L, R: –5;
    M, L, R: –4;
    N, M, L, R: +3

  Bidder 1

  - M: +10

  Bidder 2

Winner determination

- Goods: Newspaper, Magazine
- Bids:

$$(n_1 \wedge m_1) = \overline{(\bar{n}_1 \vee \bar{m}_1)}$$

  - N: +5; M: +4
  - N, M: –3

Bidder 1

  - N, M: +4

Bidder 2

$$0 \le n_1, n_2, m_1, m_2 \le 1$$

$$n_1 + n_2 \le 1 \qquad y \le n_2$$

$$m_1 + m_2 \le 1 \qquad y \le m_2$$

$$(1-x) \le (1-n_1) + (1-m_1)$$

$$0 \le x, y \le 1$$

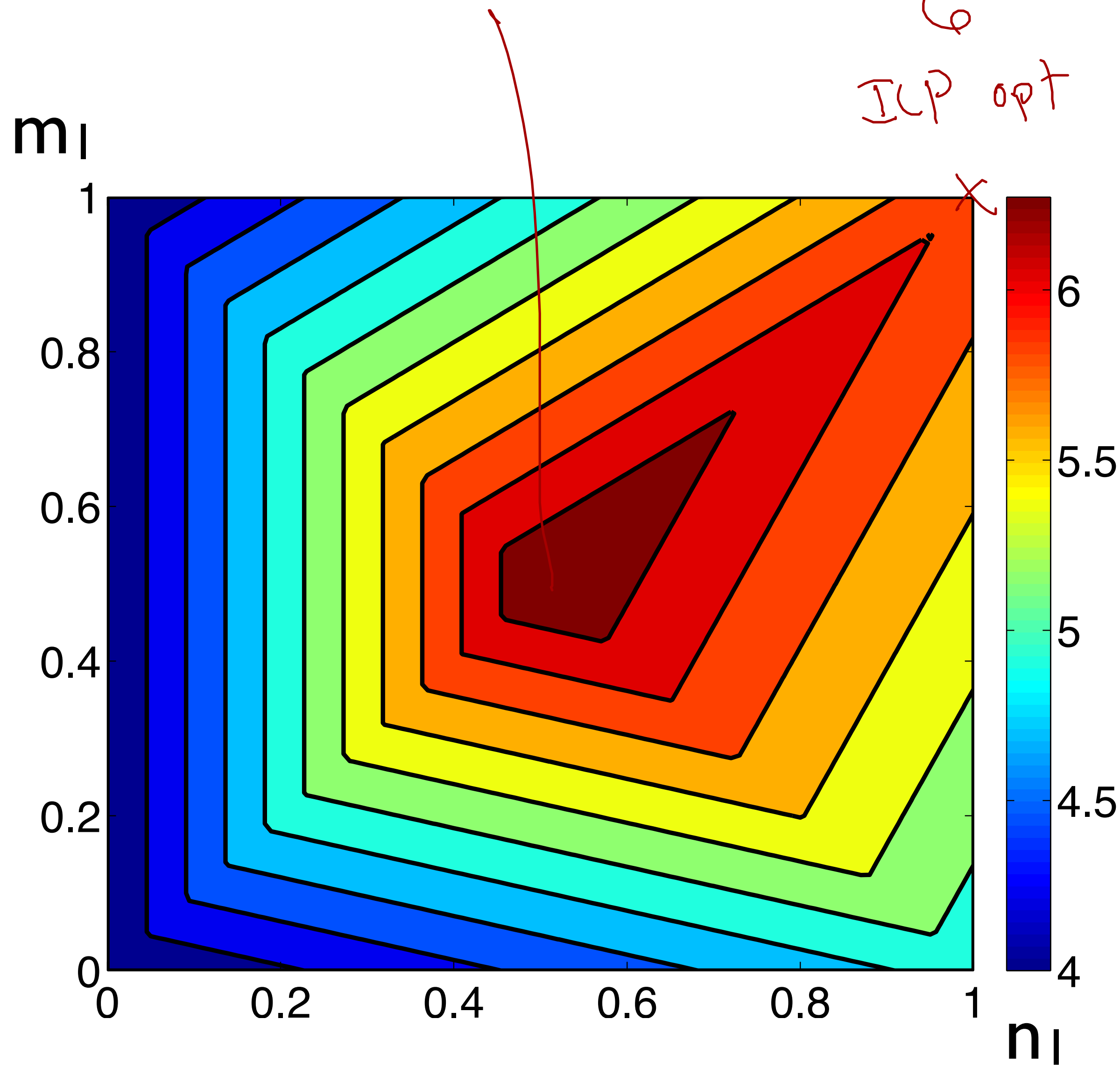
$$\max \; 5n_1 + 4m_1 + 4y - 3x$$

(all $\in \mathbb{Z}$) → delete for LP relax

Bounds

- Any feasible point yields lower bd: (N to B_1, keep M) → 5
- Upper bound: solve LP relaxation
  - a bit expensive
  - can we be lazier?

LP OPT = 6.5



Being lazy

- A “hard” LP:

  max $x + y$ s.t.

  $x + y \leq 3$

  $x \leq 1$

  $y \leq 1$

V. Vazirani. Approximation Algorithms. Ch 12.

OK, we got lucky

- What if it were:

  max $x + 3y$ s.t.

  $x + y \leq 3$

  $x \leq 1$

  $y \leq 1$

$2 \cdot$

$x + 3y \leq 5$

How general is this?

primal

- What if it were:

  $\max \; px + qy \;$ s.t.

  $a \; (x + y \leq 3)$

  $b \; (x \leq 1)$

  $c \; (y \leq 1)$

$a(x+y-3) + b(x-1) + c(y-1) \leq 0$

$(a+b)x + (a+c)y \leq 3a+b+c$

$px + qy \leq 3a+b+c$

$a, b, c \geq 0$

$a+b = p$

$a+c = q$

$\min \; 3a+b+c$

Dual

Let's do it again

- Note ≥, ≤, = constraints, min obj

  min $x - 2y$ s.t.

a ($x + y \geq 2$)

b ($y \leq 3$)

c ($2x - y = 0$)

d ($x \geq 0$)

e ($y \geq 0$)

$a(x+y-2) + b(y-3) + c(2x-y)$

$+ dx + ey \geq 0$

$(a+2c+d)x + (a+b-c+e)y \geq 2a+3b$

$= x - 2y$

$a \geq 0$

$b \leq 0$

$c \in \mathbb{R}$

$d \geq 0$

$e \geq 0$

$a + 2c + d = 1$

$a + b - c + e = -2$

max $2a + 3b$ Dual

Summary of LP duality

- Use multipliers to write combined constraints — ~~[illegible]~~ problem min

  $\geq$ → +ve
  $\leq$ → −ve
  $=$ → $\mathbb{R}$
  } multiplier

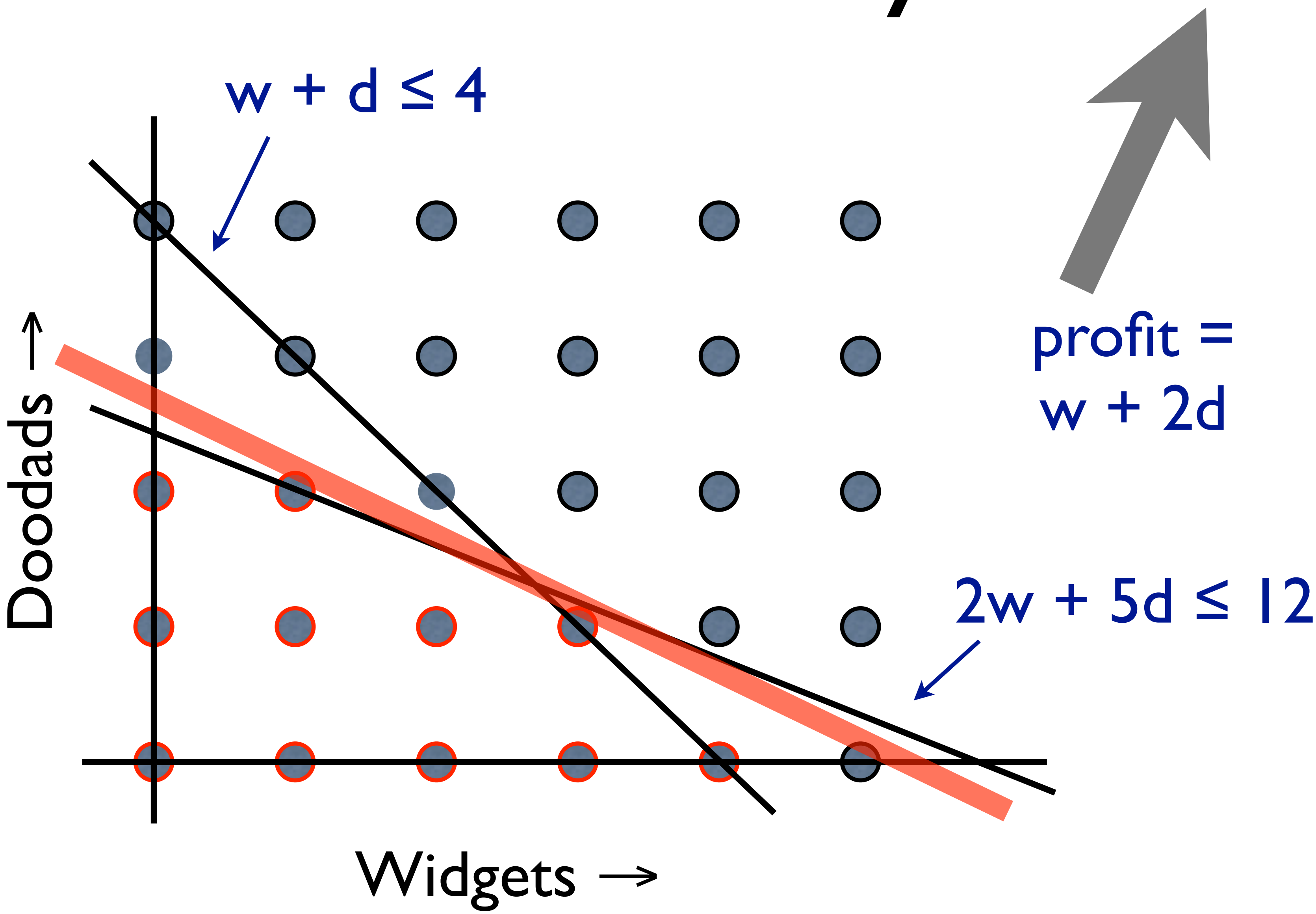
- Constrain multipliers to give us a bound on objective (by matching coefficients)
- Optimize to get tightest bound
- Q: what happens if we take dual of dual?

Ordering

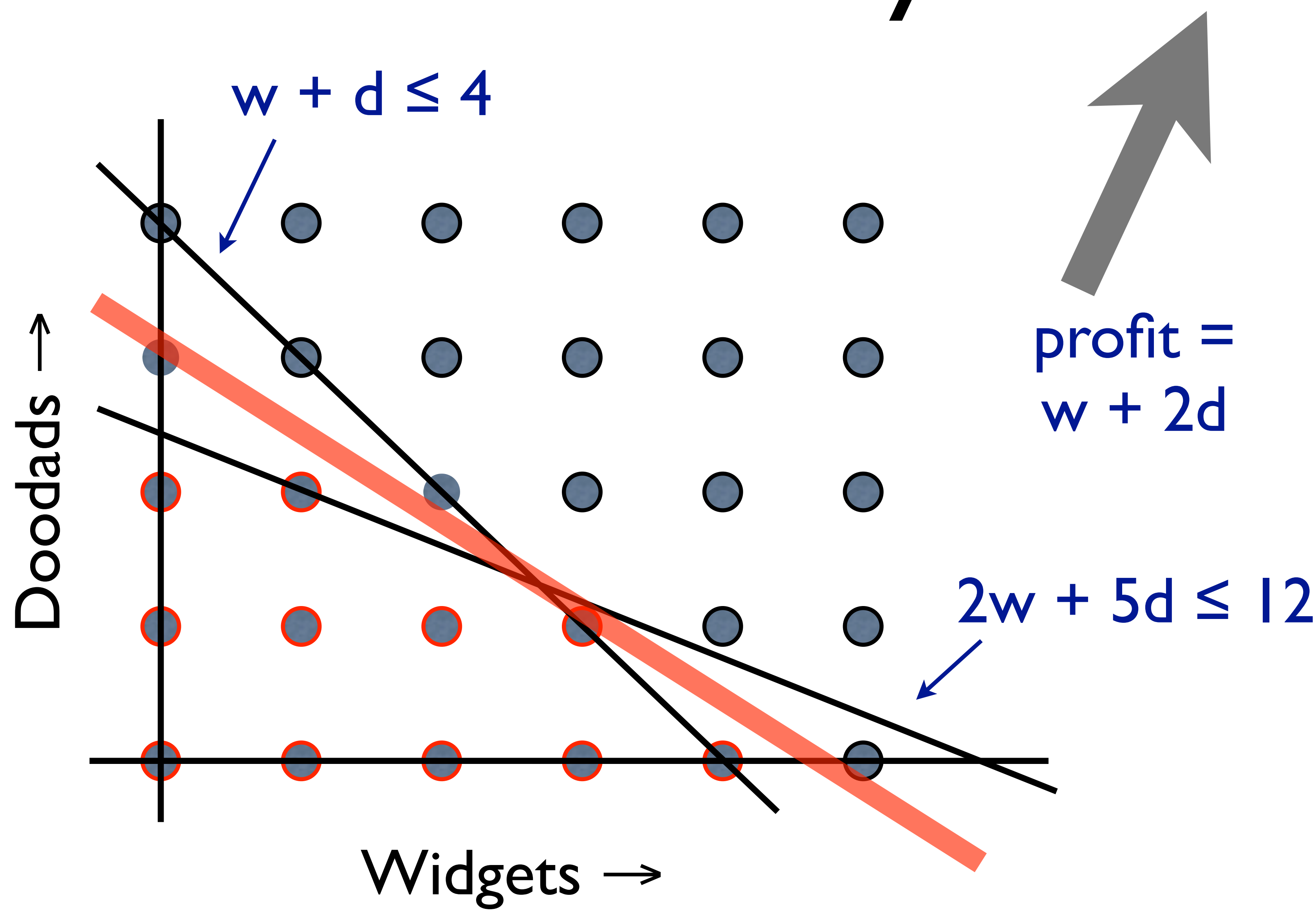
- For primal max problem (dual min):
  - primal feas $\leq$ primal opt $\leq$ dual opt $\leq$ dual feas
- For primal min problem (dual max):
  - primal feas $\geq$ primal opt $\geq$ dual opt $\geq$ dual feas

"usually" =

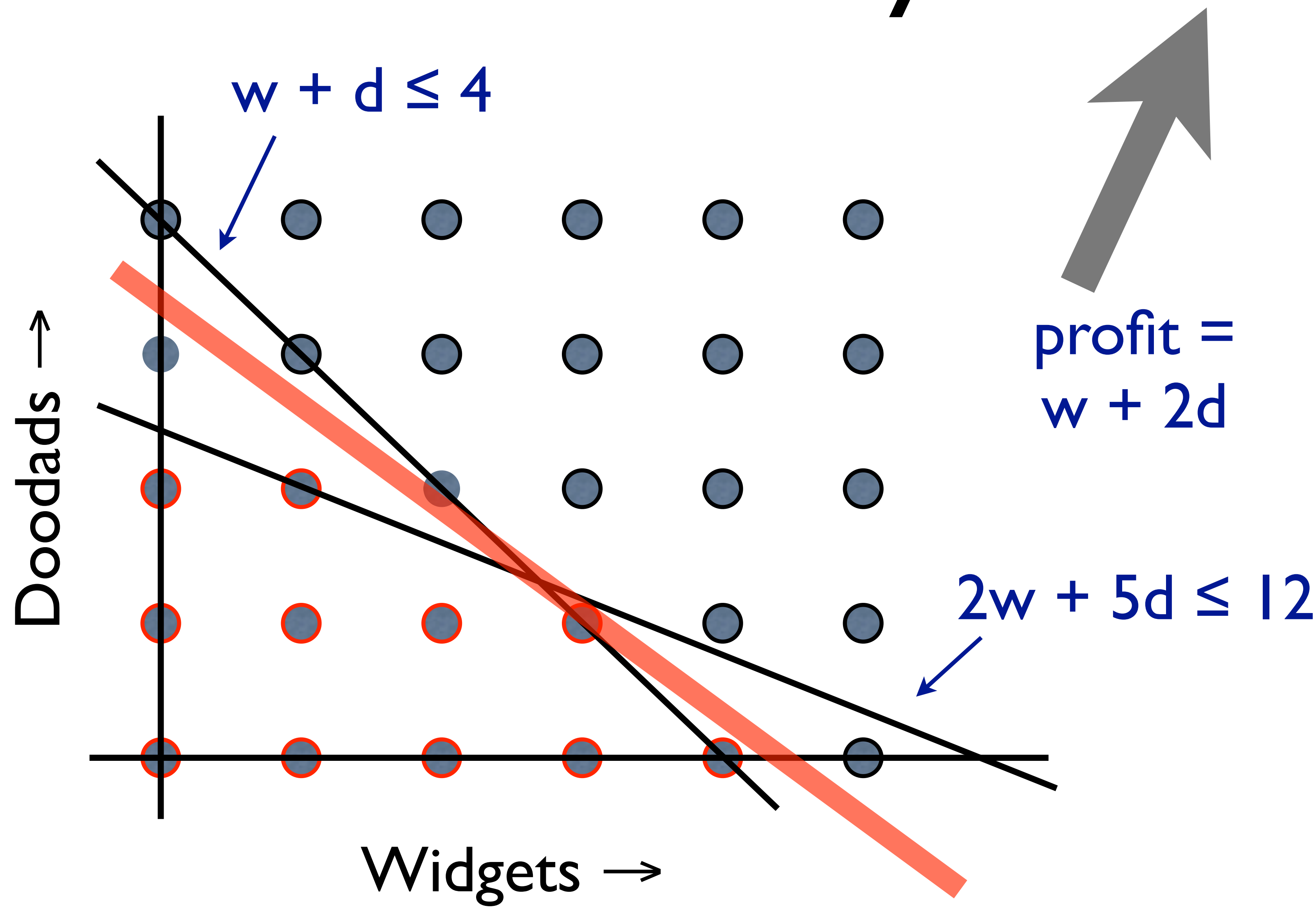
Geometrically



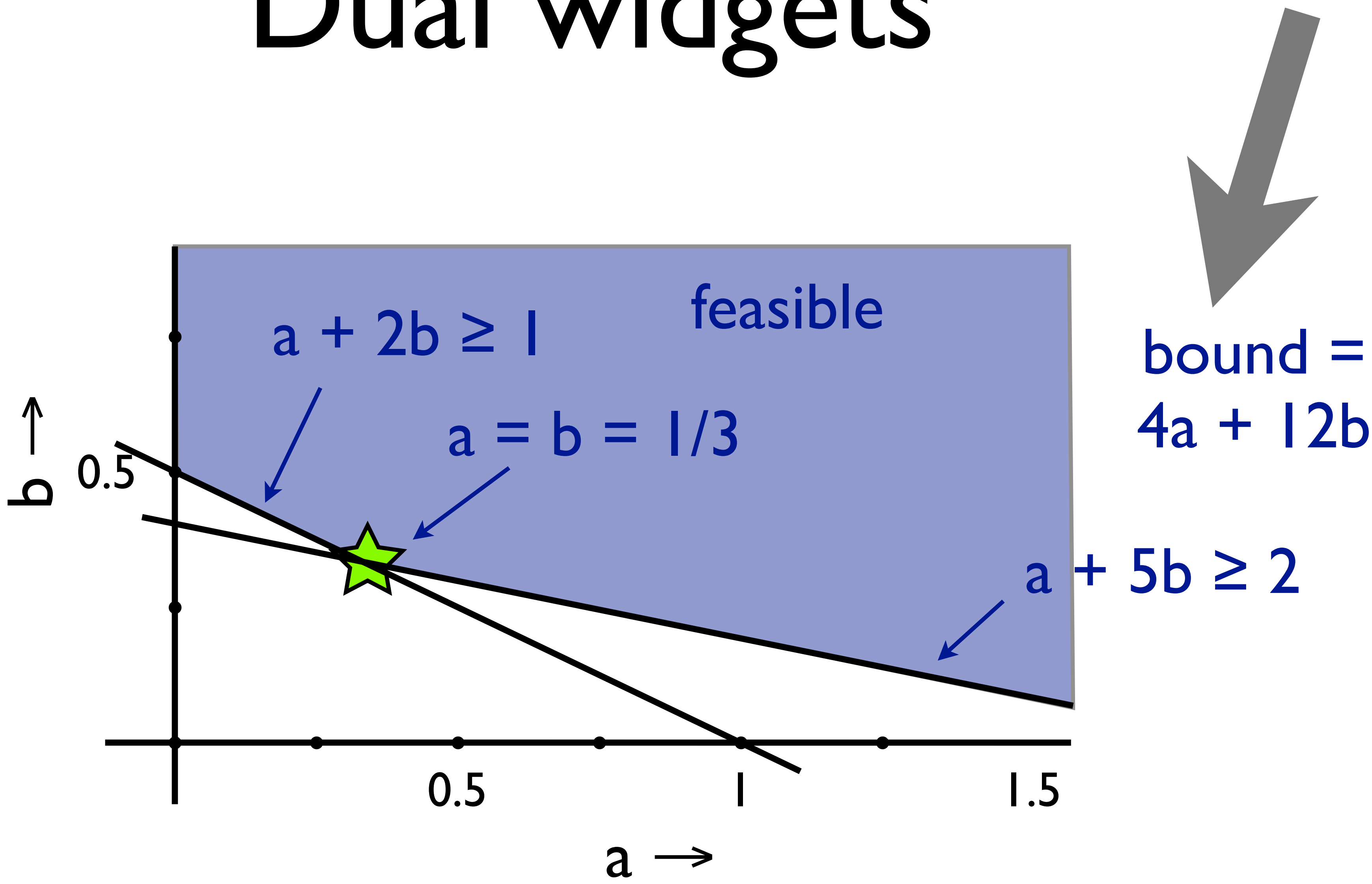
Geometrically



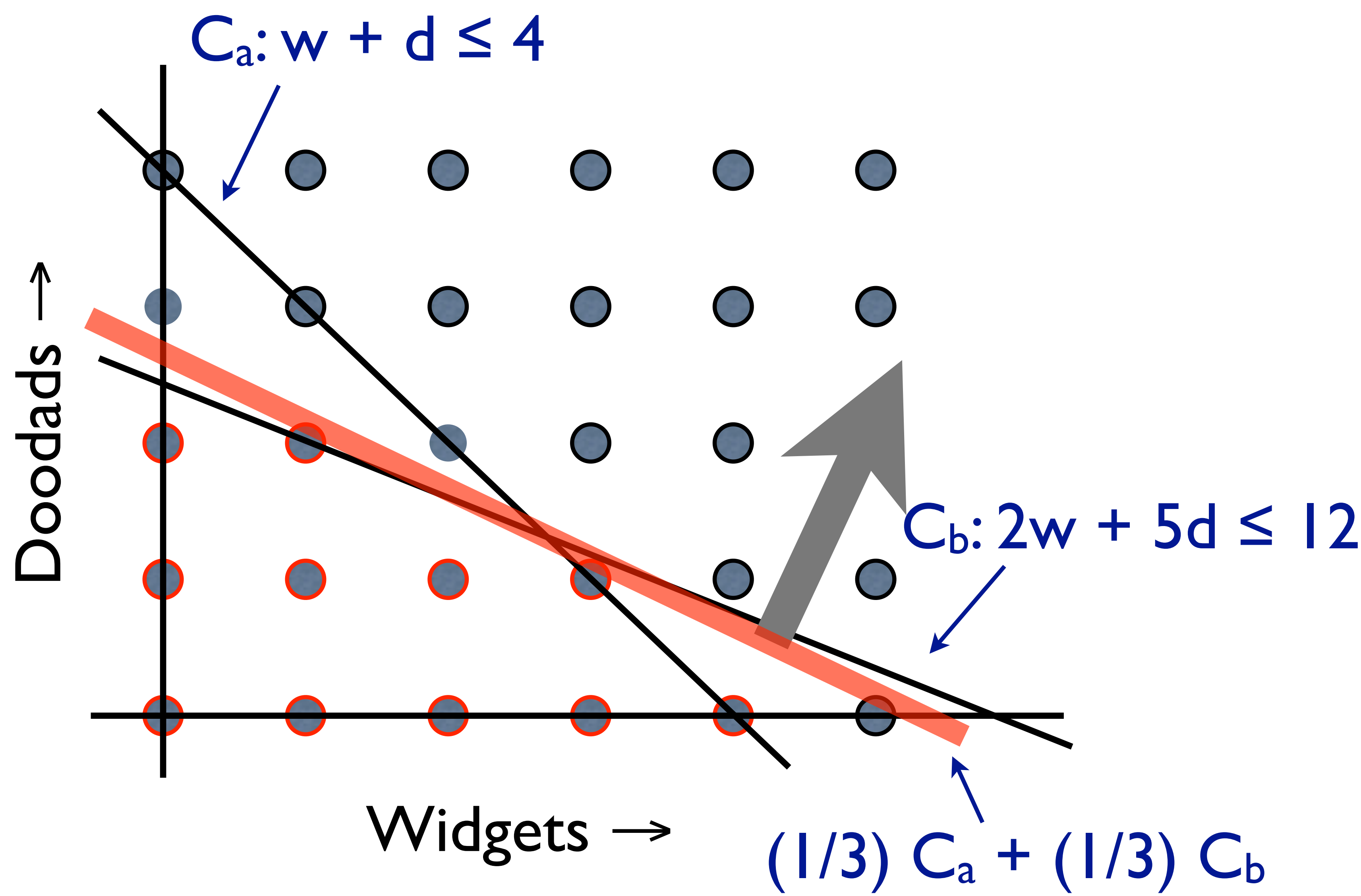
Geometrically



Dual widgets



Dual variables as multipliers



So why bother?

- Reason 1: any feasible solution to dual yields upper bound (compared with only optimal solution to primal)
- Reason 2: dual might be easier to work with
- Reason 3: solvers can often work w/ primal and dual at the same time for no extra cost

Interpreting the dual variables

- Primal variables in the factory LP were how many widgets and doodads to produce
- Interpreted dual variables as multipliers for primal constraints—not much intuition
- Often possible to interpret dual variables as ***prices*** for primal constraints

Dual variables as prices

- Suppose someone offered us a quantity ε of wood, loosening constraint to

  $w + d \leq 4 + \varepsilon$

- How much should we be willing to pay for this wood?

Dual variables as prices

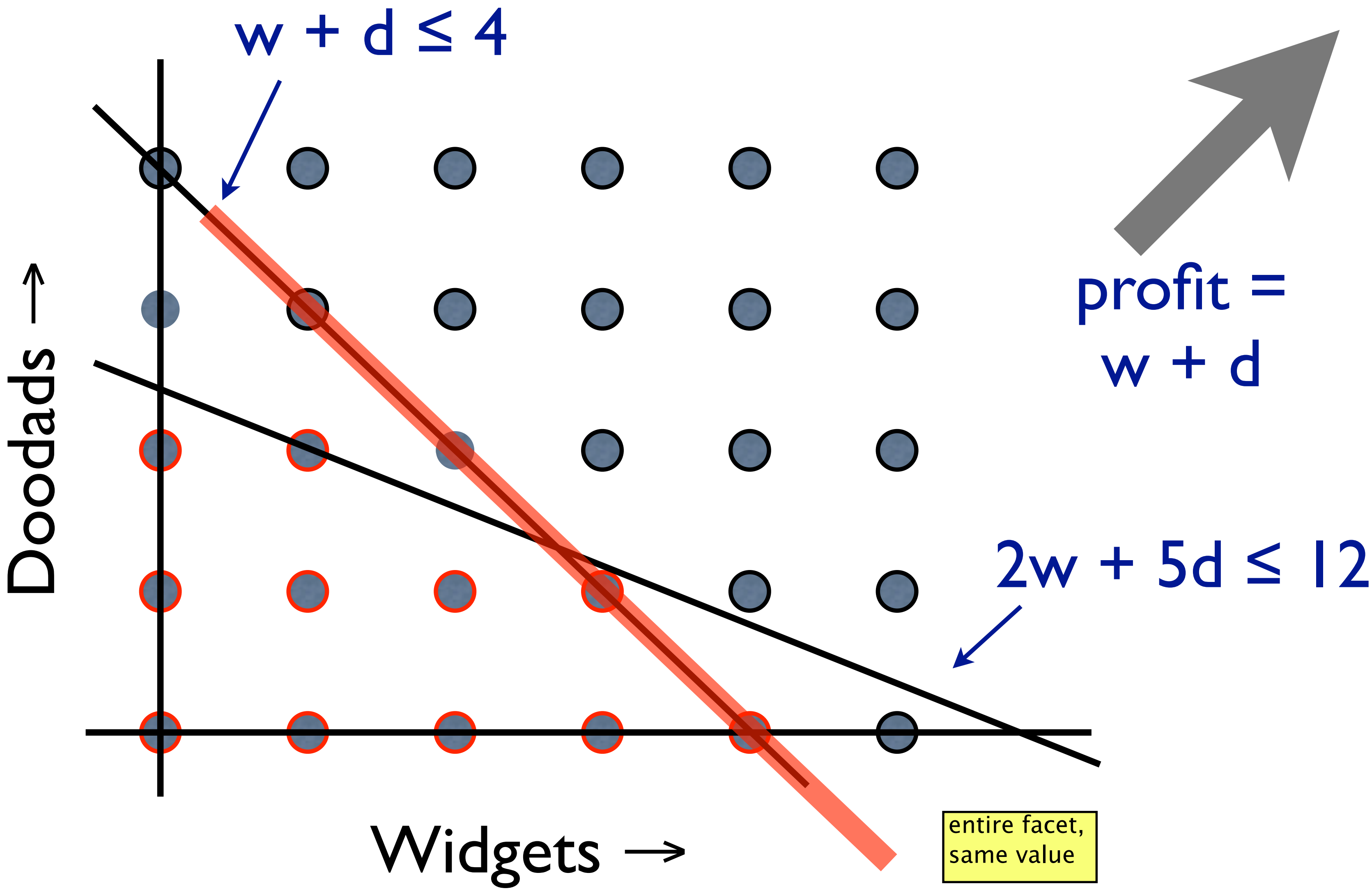
- Dual constrs stay same: $a + 2b \geq 1$, $a + 5b \geq 2$
- Dual objective becomes: min $(4+\varepsilon)a + 12b$
- Previous solution $a = b = 1/3$ still feasible
  - still optimal if ε small enough
- Bound changes to $(4+\varepsilon)a + 12b$, increase by $\varepsilon/3$
- So we should pay up to \$1/3 per unit of wood (in small quantities)

Same is true for steel: since $b=1/3$ at opt, we would pay \$1/3 per unit in small quantities

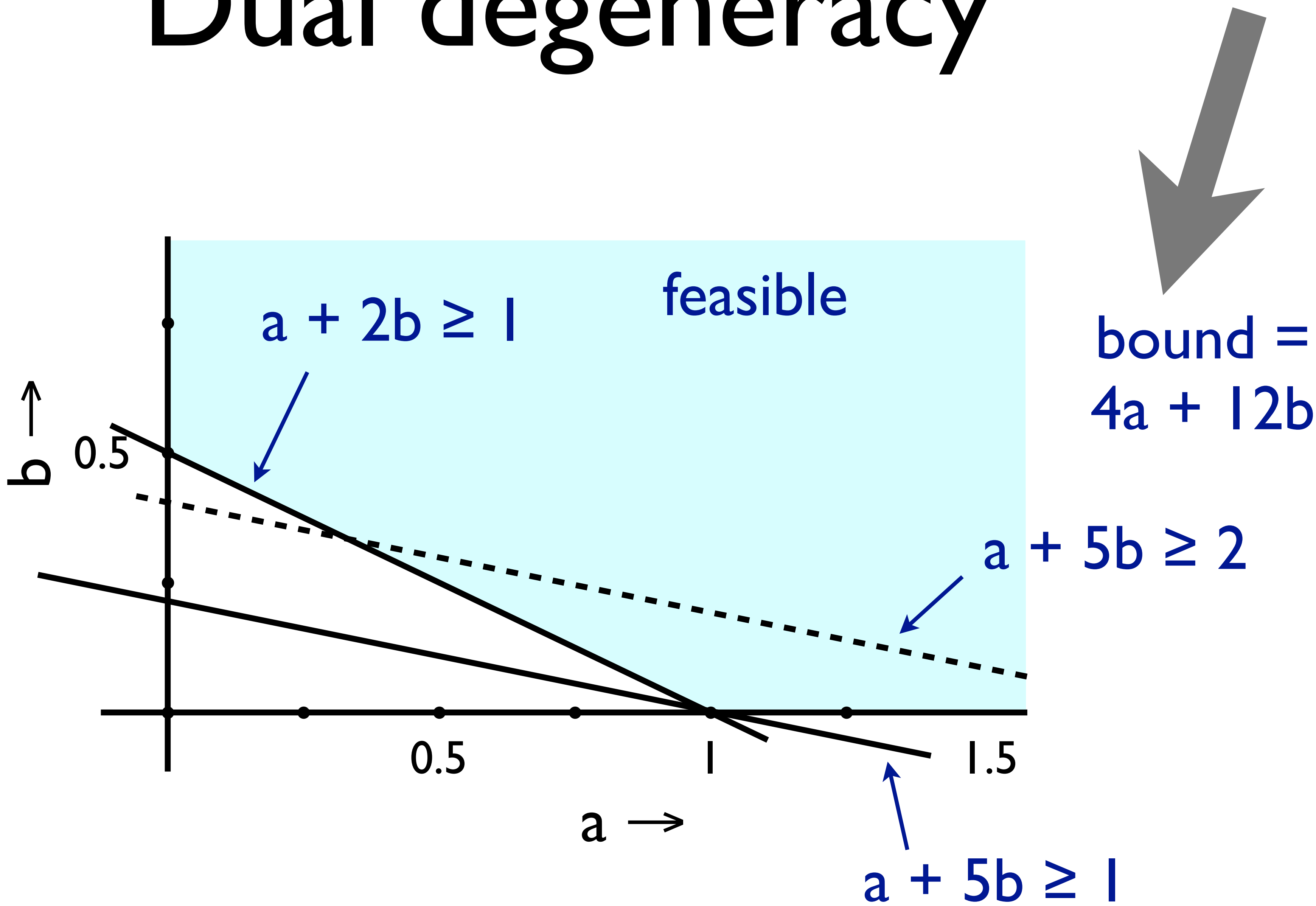
Dual degeneracy

- Primal degenerate = two bases, same corner
- Dual can be degenerate too
  - so, 4 possibilities for degeneracy
- E.g., what if objective were w+d (not w+2d)?

Dual degeneracy



Dual degeneracy



Complementary slackness

- Suppose a constraint is inactive. Would we pay anything to have it relaxed?
- Write $s_i \geq 0$ for slack in primal constraint j
- Write $d_j \geq 0$ for dual variable (multiplier, price) for constraint j
- CS: at optimal primal and dual solutions,

$$\sum_j s_j d_j = 0$$

- Uses: certificate of optimality, proving that optimal solution satisfies some property