15-780: Grad AI
Lec. 9: Linear programs, Duality

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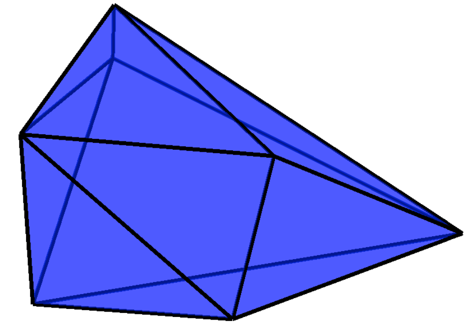
Admin

- Have you tested your handin directories?
  - /afs/cs/user/aothman/dropbox/USERID/
  - where USERID is your Andrew ID
- Poster session:
  - ???

Review

- LPs, ILPs, MILPs
  - $\mathbb{R}$ or $\mathbb{Z}$ variables
  - linear $\leq$ $\geq$ $=$
  - linear objective
  - LP relaxations, integrality gap
  - relation to SAT, MAXSAT, PBI
  - complexity (LP: P; ILP: NP & no approx)
  - (in)feasible, (sub)optimal, (in)active

Review



- Standard form: all vars ≥ 0, all = constraints
- Nonsingular: n vars ≥ m constraints, rank m
- Basis
  - spans Rng(A) (m × m invertible submatrix)
  - corresponds to "corner"
  - using row ops to make basic variables into "slacks" → ***tableau*** notation
- Degeneracy: distinct bases yield same corner
- Naïve algorithm: check all bases

Finding corners

x	y	u	v	w	RHS
1	1	1	0	0	4
2	5	0	1	0	12
1	2	0	0	1	5

set x, y = 0

u = 4 v = 12 w = 5

1	1	1	0	0	4
2	5	0	1	0	12
1	2	0	0	1	5

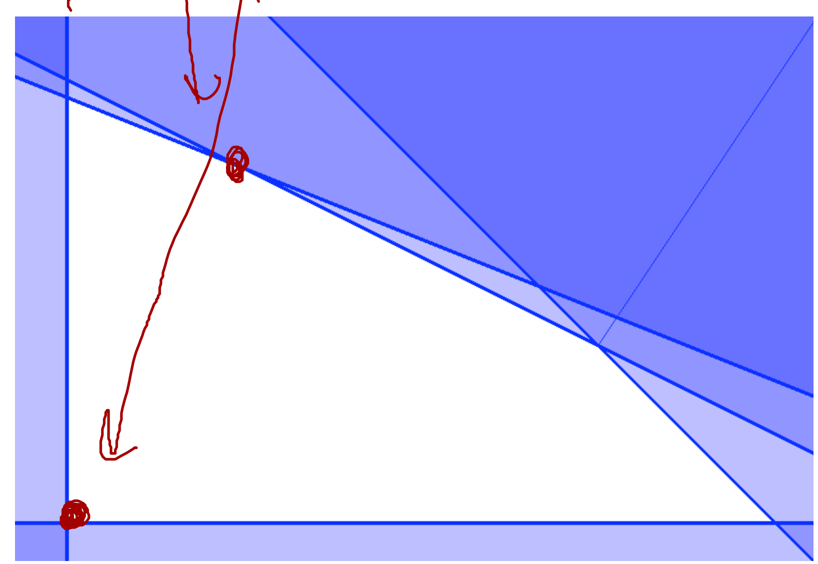
set v, w = 0

x = 1 y = 2 u = 1

1	1	1	0	0	4
2	5	0	1	0	12
1	2	0	0	1	5

set x, u = 0

y = 4
v = -8
z = -3



Simplex in one slide

(ignoring degeneracy, which is actually important)

- Given a nonsingular standard-form LP
  - make it nonsingular if needed
- Start from a feasible basis and its tableau
  - big-M if needed
- Pick non-basic variable w/ objective > 0 (max)
- Pivot it into basis, getting neighboring basis
  - select exiting variable to keep feasibility
- Repeat until all non-basic variables have objective < 0 (max)

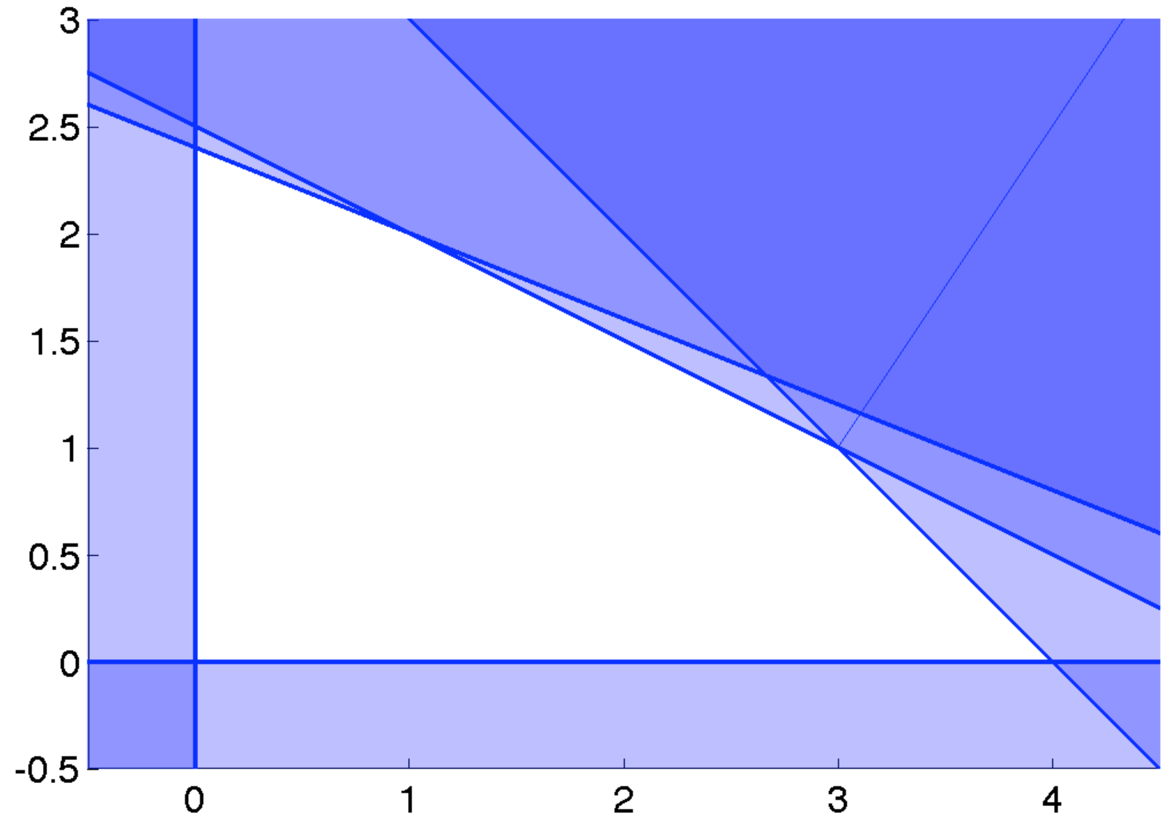
Example

max $2x + 3y$ s.t.

$x + y \leq 4$

$2x + 5y \leq 12$

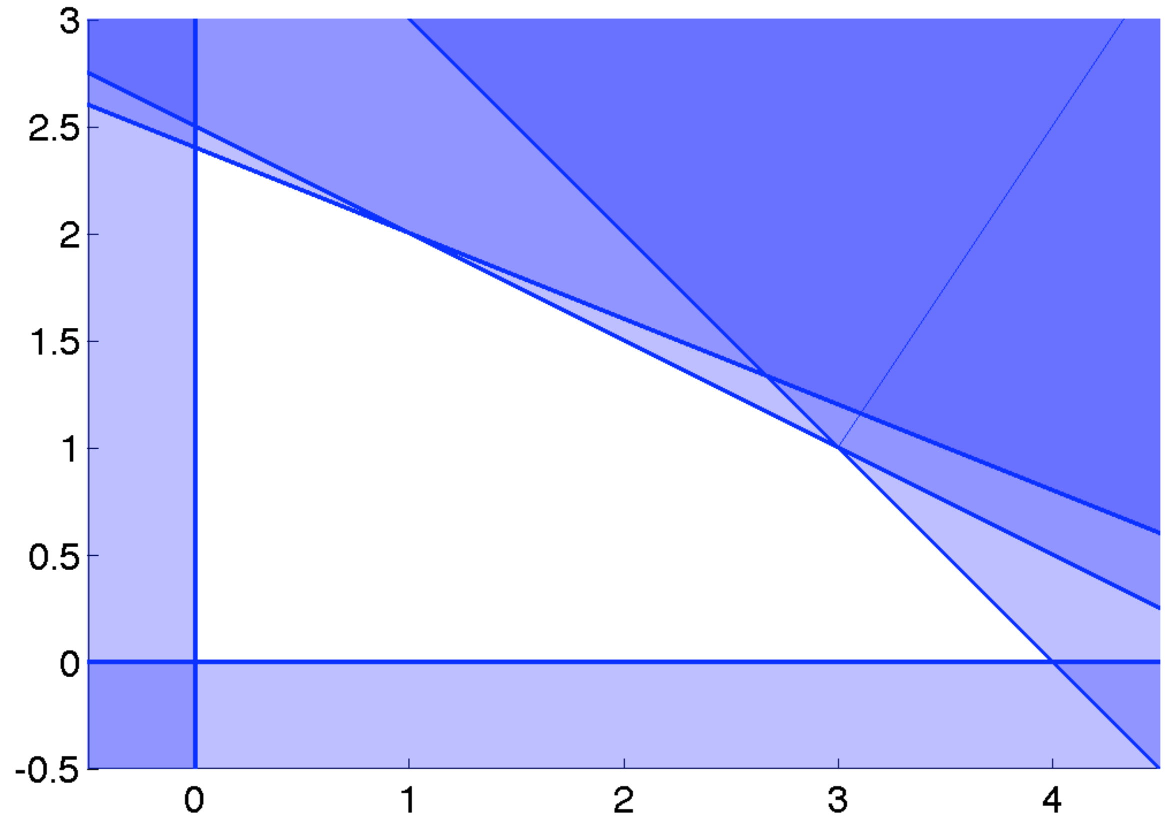
$x + 2y \leq 5$



x	y	s	t	u	RHS
1	1	1	0	0	4
2	5	0	1	0	12
1	2	0	0	1	5
2	3	0	0	0	↑

Example

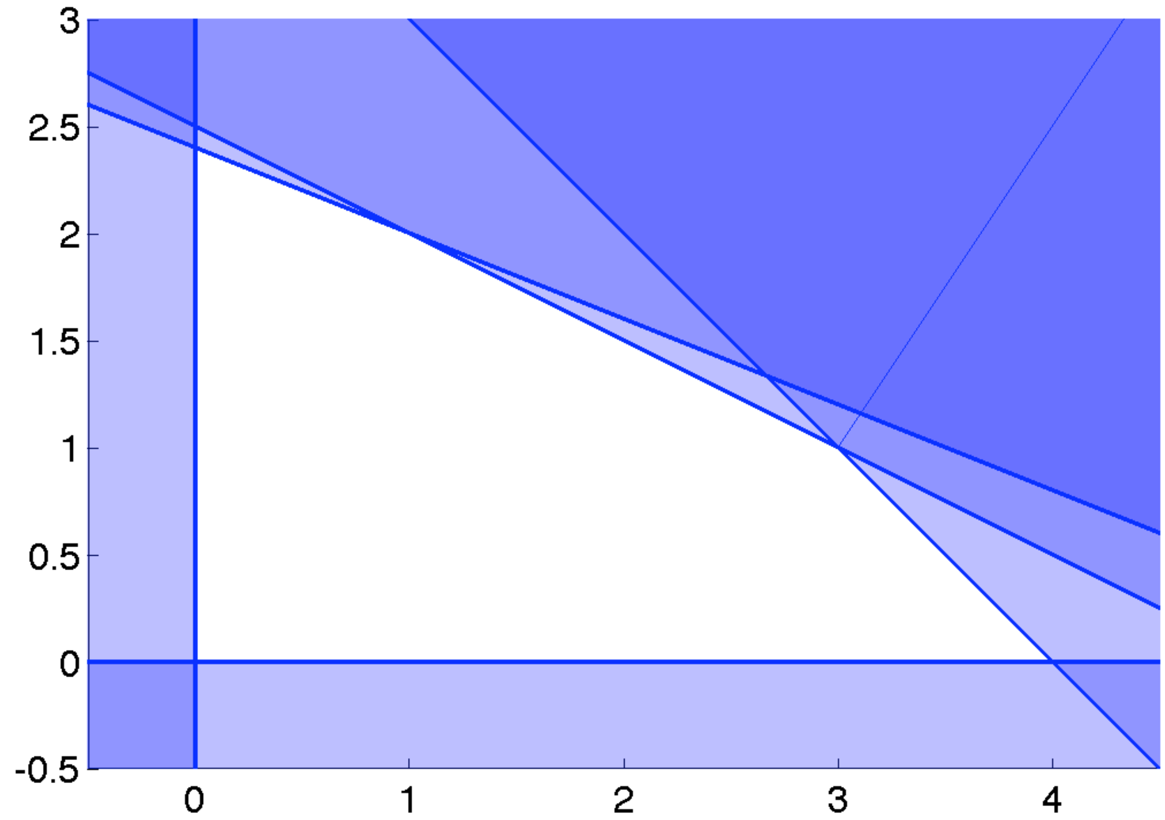
max 2x + 3y s.t.
x + y ≤ 4
2x + 5y ≤ 12
x + 2y ≤ 5



x	y	s	t	u	RHS
0.4	1	0	0.2	0	2.4
0.6	0	1	-0.2	0	1.6
0.2	0	0	-0.4	1	0.2
0.8	0	0	-0.6	0	↑

Example

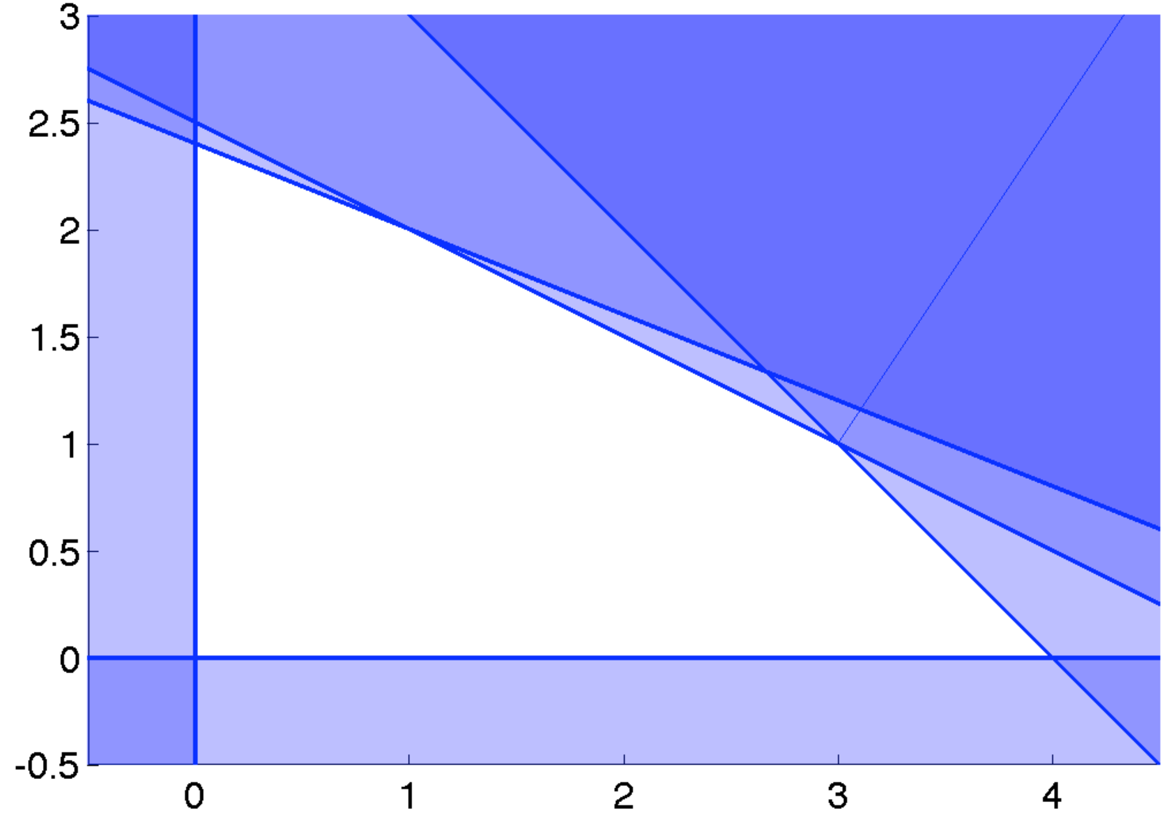
max 2x + 3y s.t.
x + y ≤ 4
2x + 5y ≤ 12
x + 2y ≤ 5



x	y	s	t	u	RHS
1	0	0	−2	5	1
0	1	0	1	−2	2
0	0	1	1	−3	1
0	0	0	1	−4	↑

Example

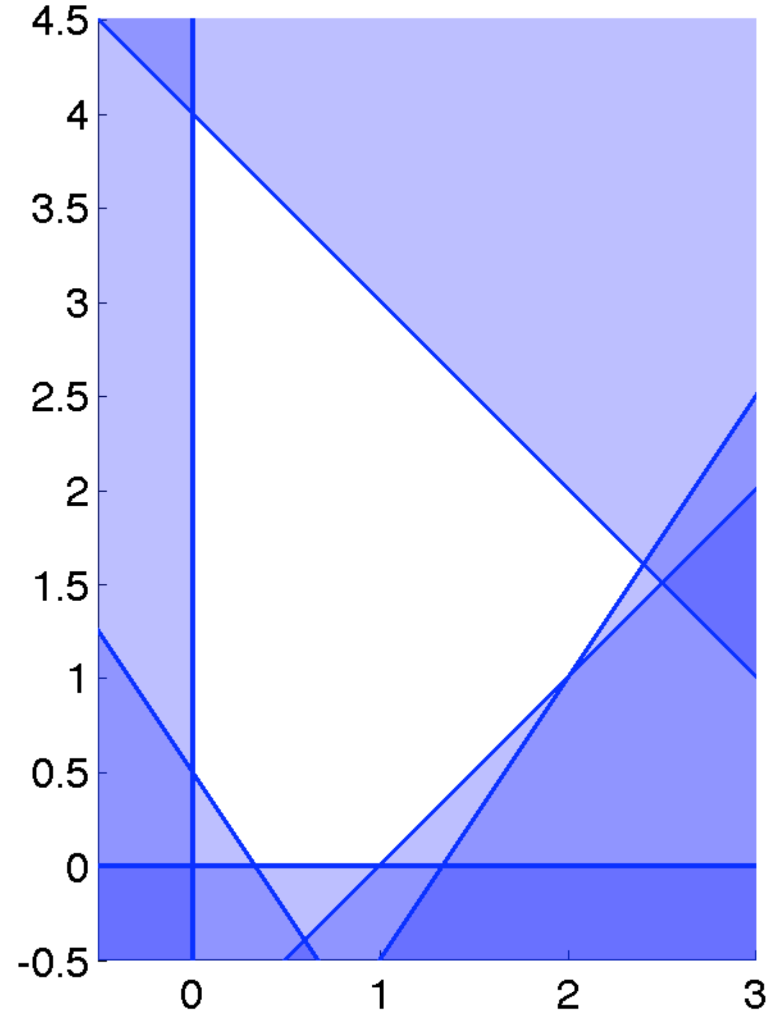
max 2x + 3y s.t.
x + y ≤ 4
2x + 5y ≤ 12
x + 2y ≤ 5



x	y	s	t	u	RHS
1	0	2	0	−1	3
0	1	−1	0	1	1
0	0	1	1	−3	1
0	0	−1	0	−1	↑

Big M

x	y	t	u	v	w	RHS
1	1	1	0	0	0	4
3	–2	0	1	0	0	4
1	–1	0	0	1	0	1
–3	–2	0	0	0	1	–1
1	–2	0	0	0	0	↑



- So far, assumed we started w/ initial feasible basis
- How do we get one?
  - for each violated constraint, add var w/ coeff –1
  - penalize in objective, include in initial basis

Ex: combinatorial auctions

- Goods: Newspaper, Magazine, L shoe, R shoe
- Bids (note use of bidding language: 7 rt 16 numbers for B_1 and 1 rt 16 for B_2):

Bidder 1

- N: +5; M: +4
- N, M: –3
- L, R: +10
- N, L, R: –5;
  M, L, R: –4;
  N, M, L, R: +3

Bidder 2

- M: +10

Winner determination

- Goods: Newspaper, Magazine
- Bids:

Bidder 1:
  - N: +5; M: +4
  - N, M: –3

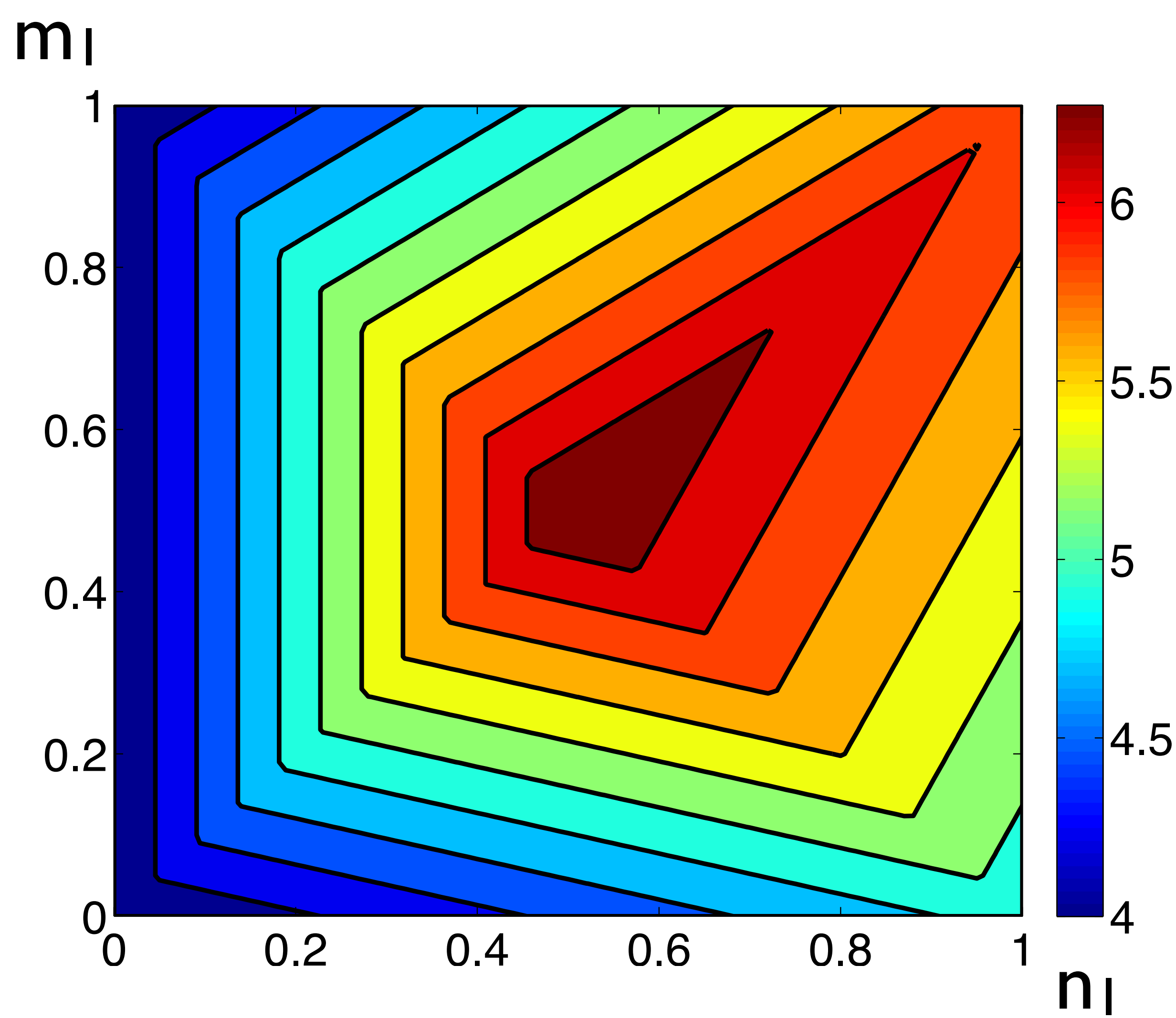
Bidder 1

Bidder 2:
  - N, M: +4

Bidder 2

Bounds

- Any feasible point yields lower bd: (N to B_1, keep M) → 5
- Upper bound: solve LP relaxation
  - a bit expensive
  - can we be lazier?



Being lazy

- A “hard” LP:

  max $x + y$ s.t.

  $x + y \leq 3$

  $x \leq 1$

  $y \leq 1$

V. Vazirani. Approximation Algorithms. Ch 12.

OK, we got lucky

- What if it were:

  max $x + 3y$ s.t.

  $x + y \leq 3$

  $x \leq 1$

  $y \leq 1$

How general is this?

- What if it were:

  $\max \; px + qy$ s.t.

  $x + y \leq 3$

  $x \leq 1$

  $y \leq 1$

Let's do it again

- Note ≥, ≤, = constraints, min obj

  min $x - 2y$ s.t.

  $x + y \geq 2$

  $y \leq 3$

  $2x - y = 0$

Summary of LP duality

- Use multipliers to write combined constraints

  $\geq$

  $\leq$

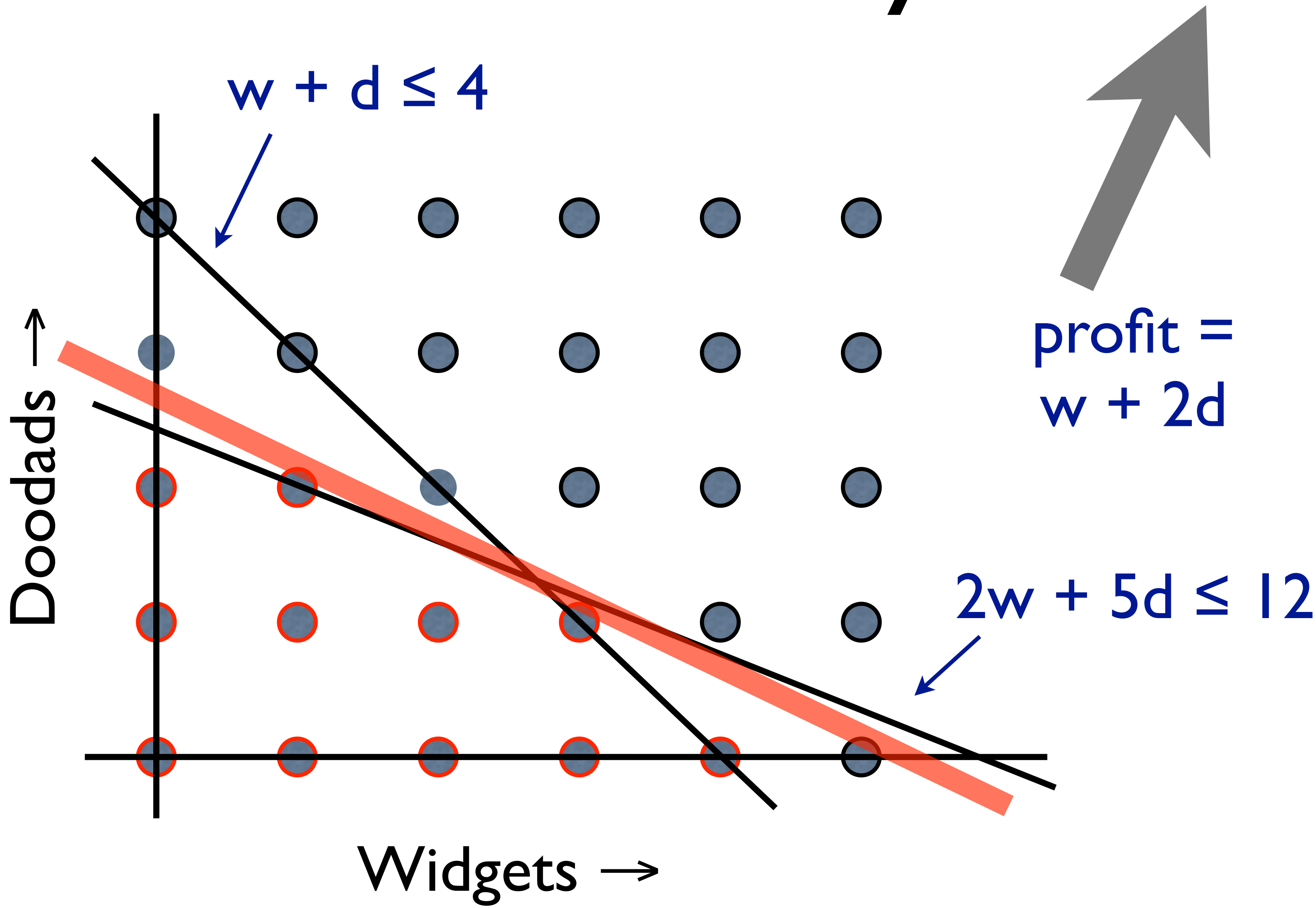
  $=$

- Constrain multipliers to give us a bound on objective (by matching coefficients)
- Optimize to get tightest bound
- Q: what happens if we take dual of dual?

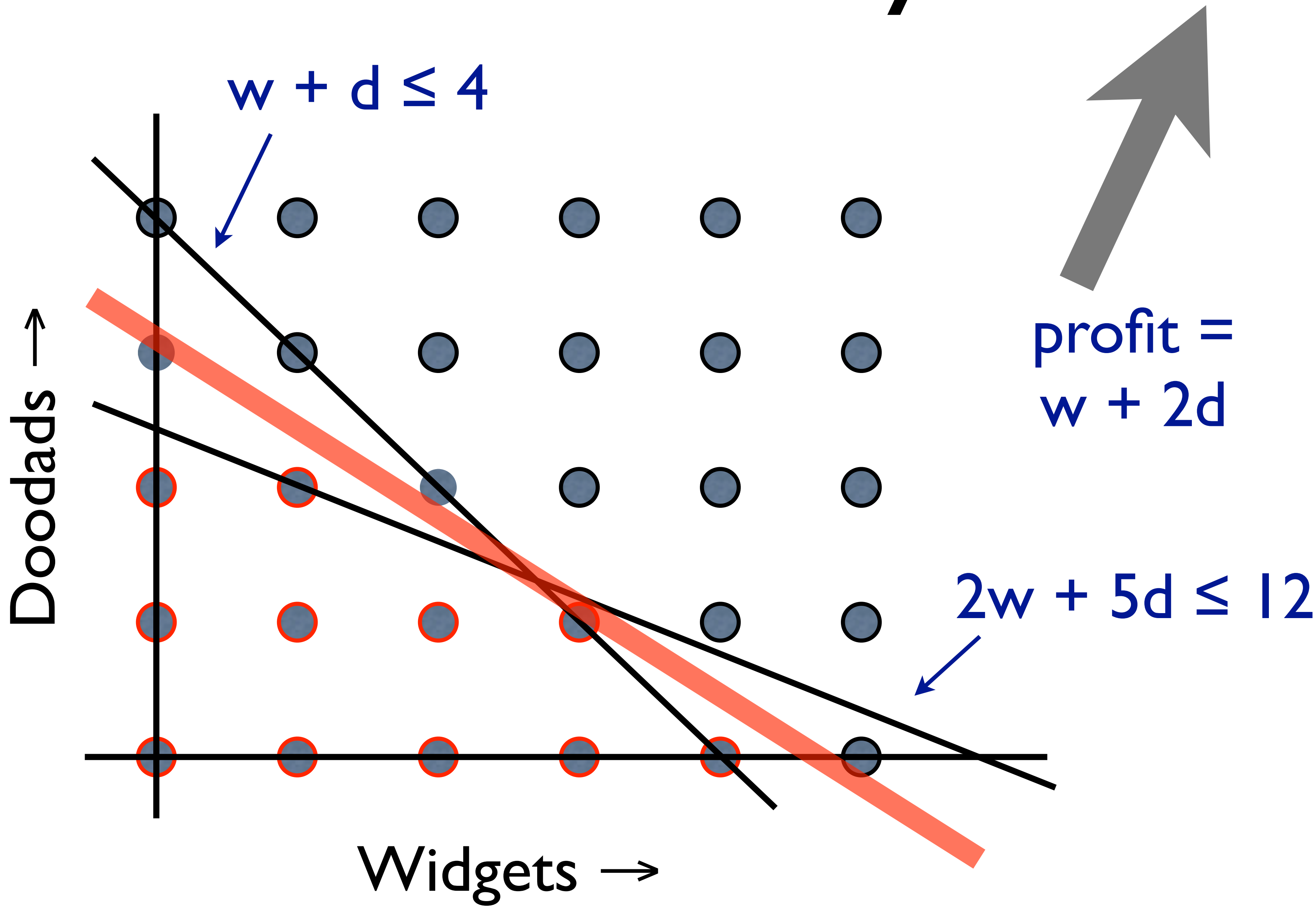
Ordering

- For primal max problem (dual min):
  - primal feas   primal opt   dual opt   dual feas
- For primal min problem (dual max):
  - primal feas   primal opt   dual opt   dual feas

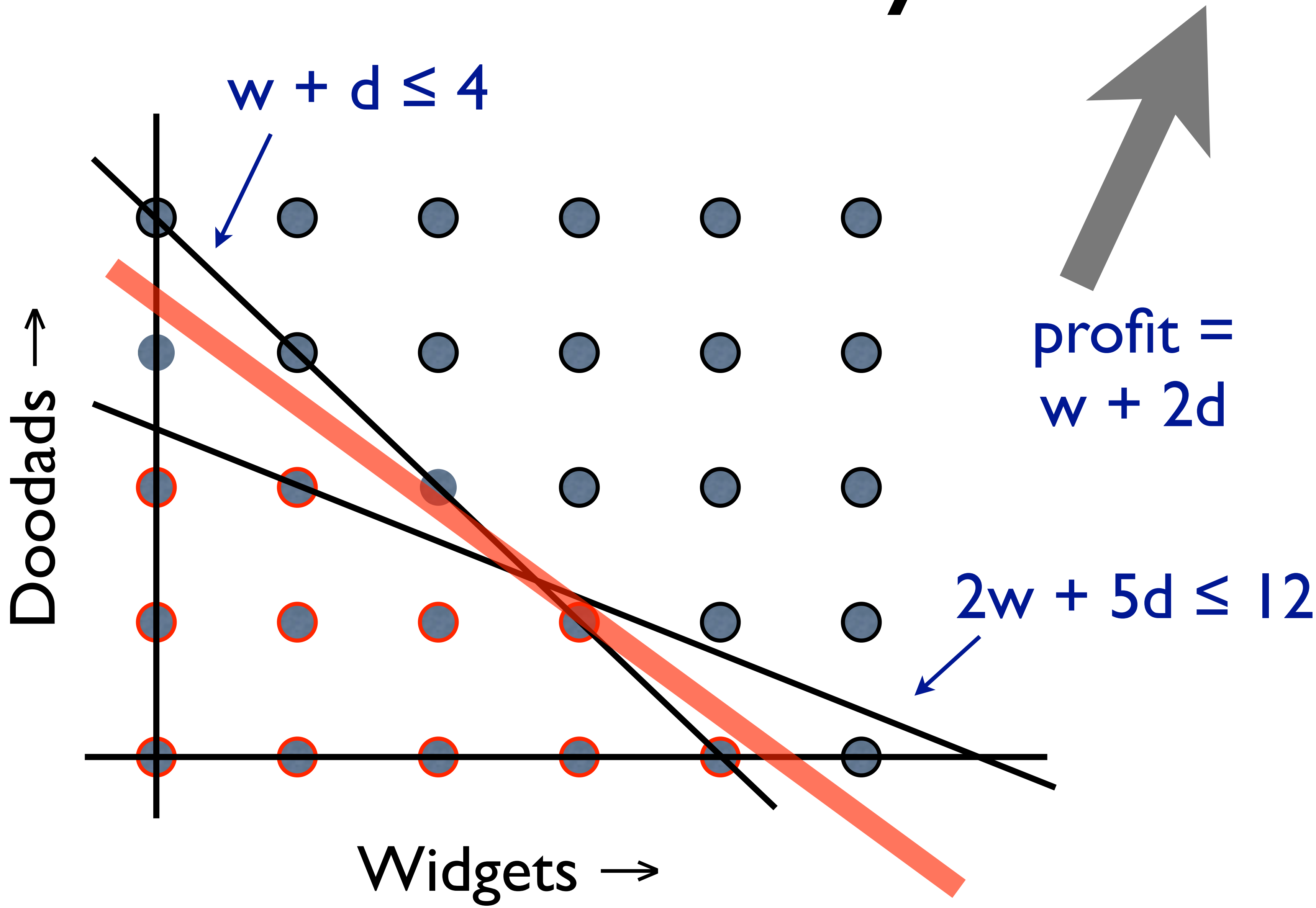
Geometrically



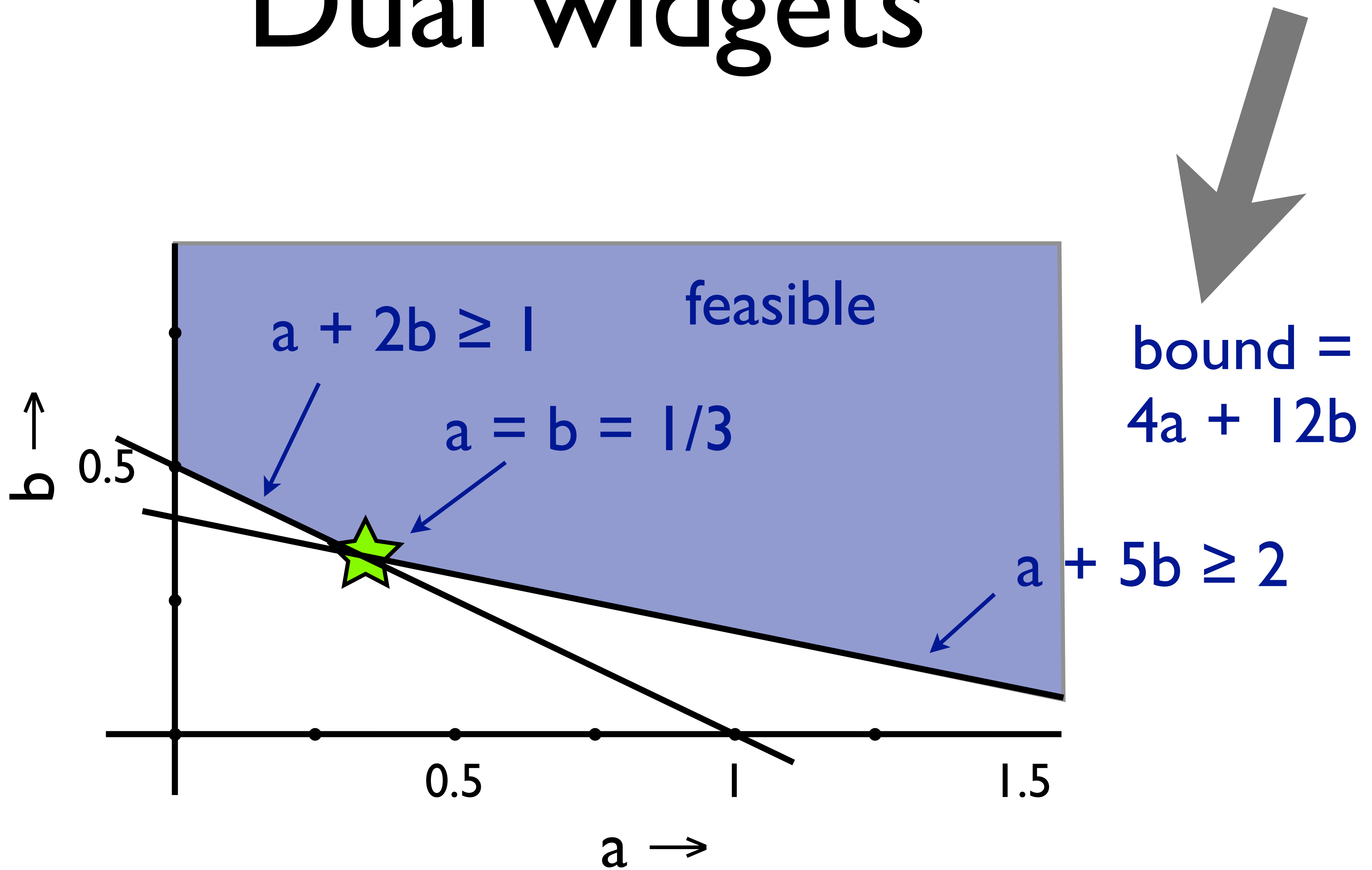
Geometrically



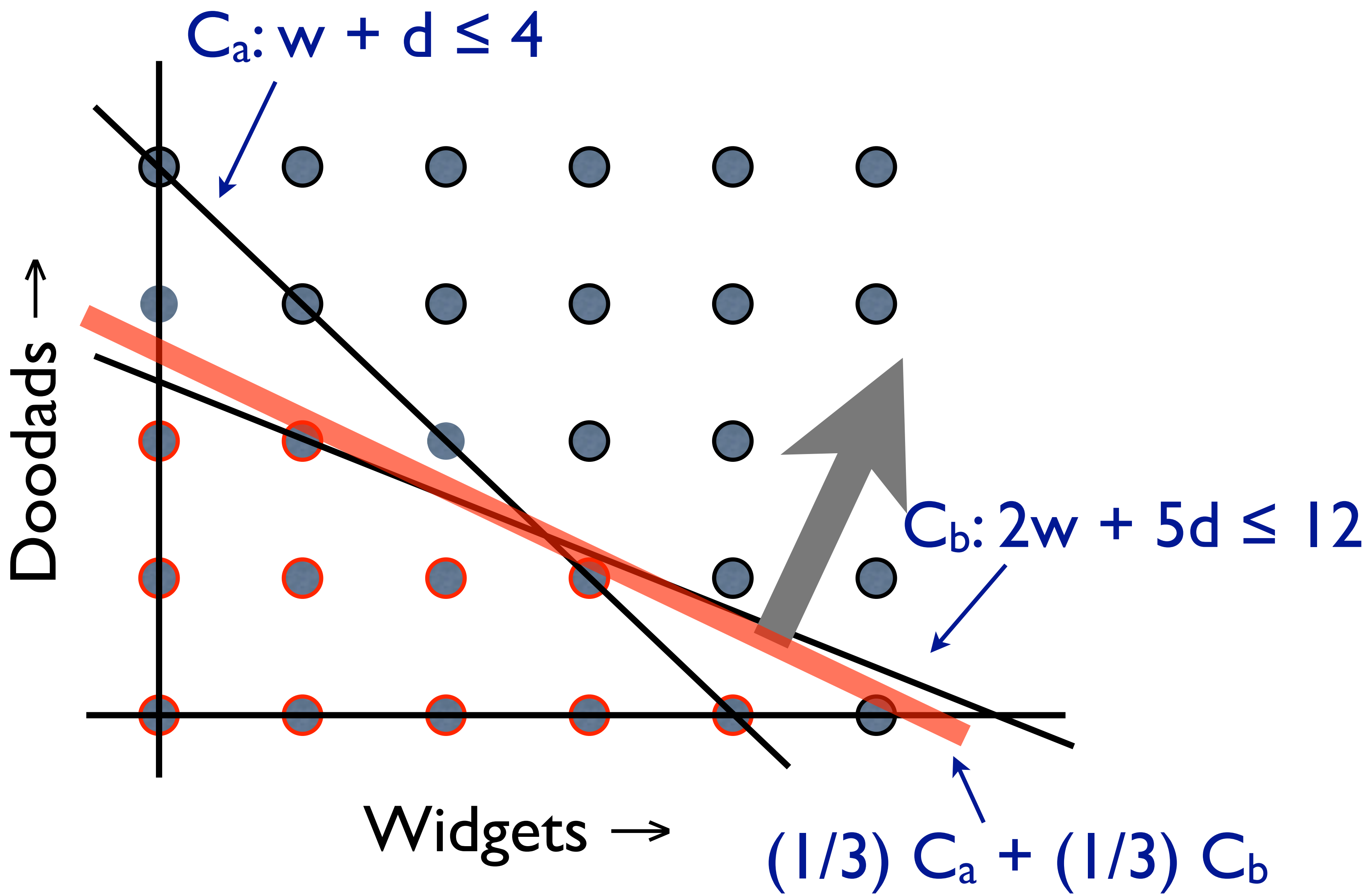
Geometrically



Dual widgets



Dual variables as multipliers



So why bother?

- Reason 1: any feasible solution to dual yields upper bound (compared with only optimal solution to primal)
- Reason 2: dual might be easier to work with
- Reason 3: solvers can often work w/ primal and dual at the same time for no extra cost

Interpreting the dual variables

- Primal variables in the factory LP were how many widgets and doodads to produce
- Interpreted dual variables as multipliers for primal constraints—not much intuition
- Often possible to interpret dual variables as ***prices*** for primal constraints

Dual variables as prices

- Suppose someone offered us a quantity ε of wood, loosening constraint to

  $w + d \leq 4 + \varepsilon$

- How much should we be willing to pay for this wood?

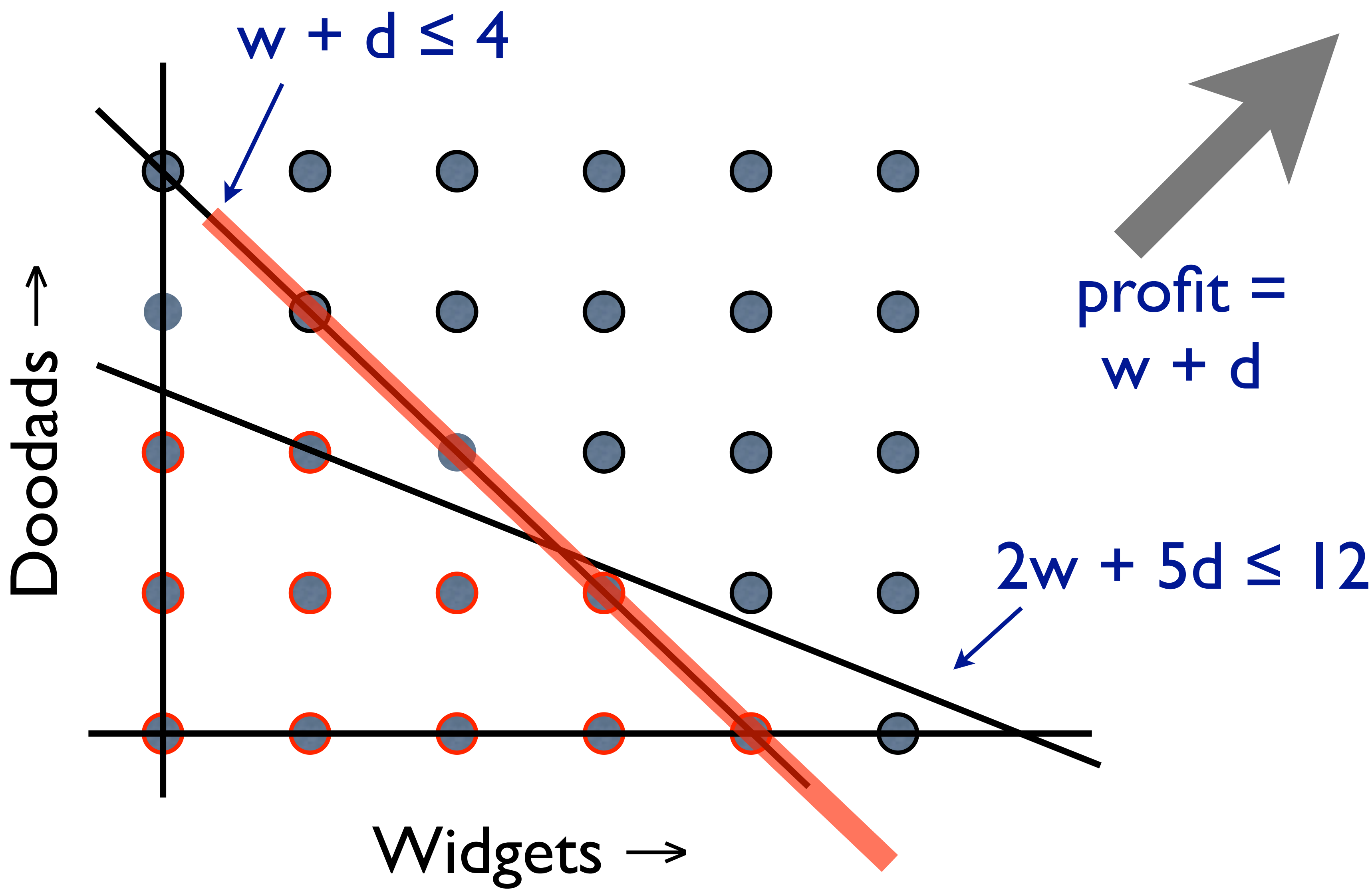
Dual variables as prices

- Dual constrs stay same: $a + 2b \geq 1$, $a + 5b \geq 2$
- Dual objective becomes: min $(4+\varepsilon)a + 12b$
- Previous solution $a = b = 1/3$ still feasible
  - still optimal if ε small enough
- Bound changes to $(4+\varepsilon)a + 12b$, increase by $\varepsilon/3$
- So we should pay up to \$1/3 per unit of wood (in small quantities)

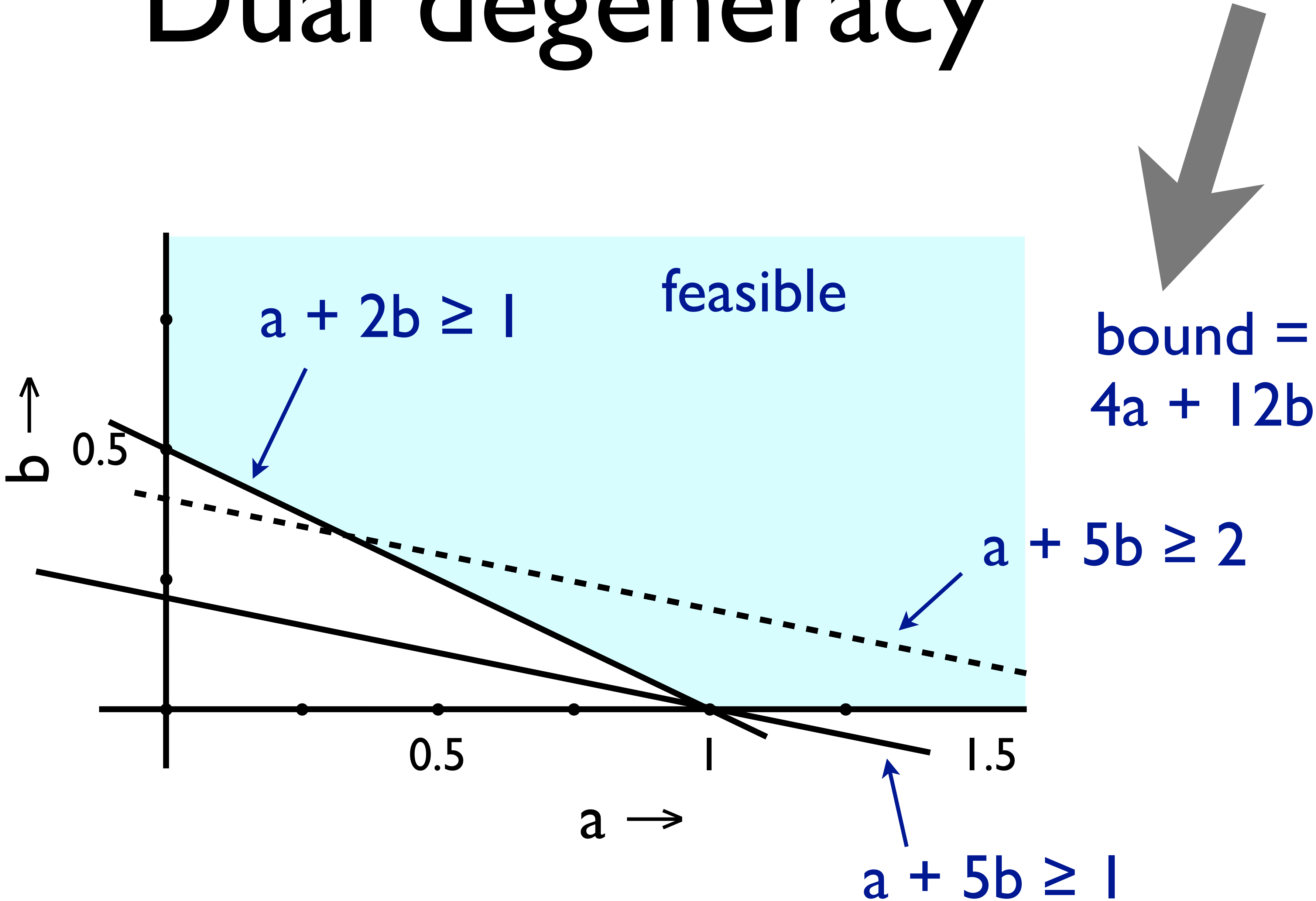
Dual degeneracy

- Primal degenerate = two bases, same corner
- Dual can be degenerate too
  - so, 4 possibilities for degeneracy
- E.g., what if objective were w+d (not w+2d)?

Dual degeneracy



Dual degeneracy



Complementary slackness

- Suppose a constraint is inactive. Would we pay anything to have it relaxed?
- Write $s_i \geq 0$ for slack in primal constraint j
- Write $d_j \geq 0$ for dual variable (multiplier, price) for constraint j
- CS: at optimal primal and dual solutions,

- Uses: certificate of optimality, proving that optimal solution satisfies some property