Learning about State

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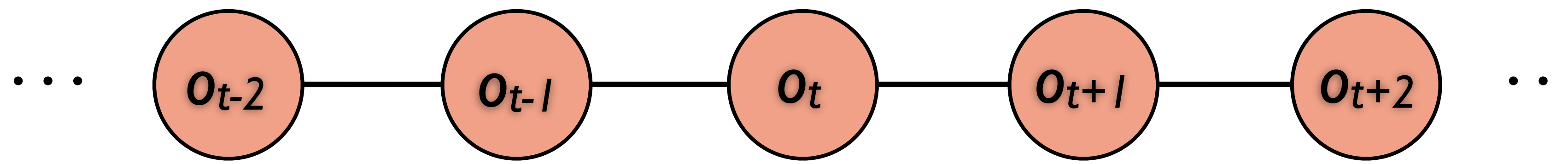
What's out there?



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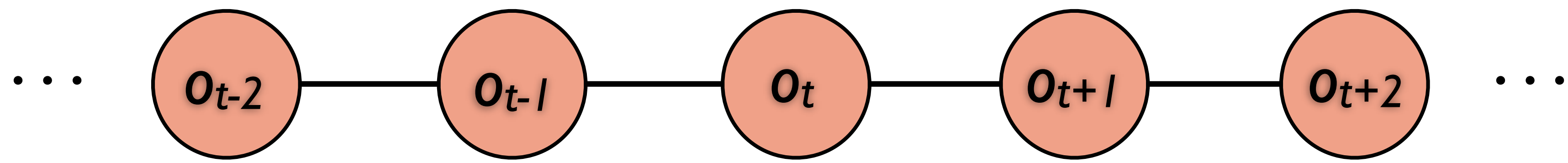

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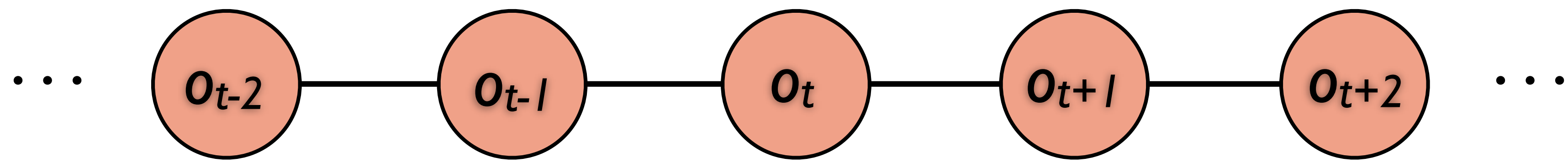
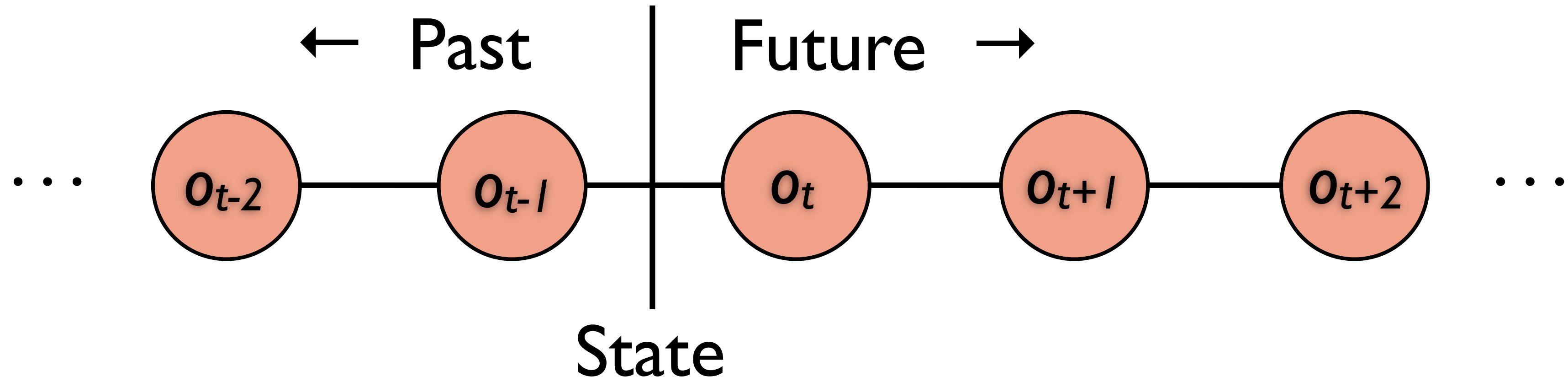


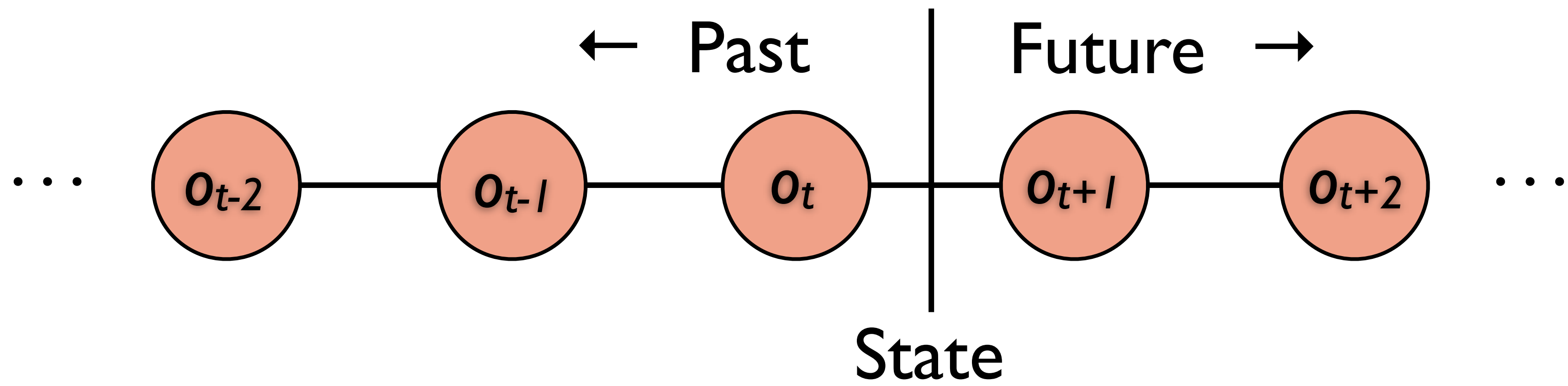
⇒ steam rising from a grate

What's out there? A dynamical system



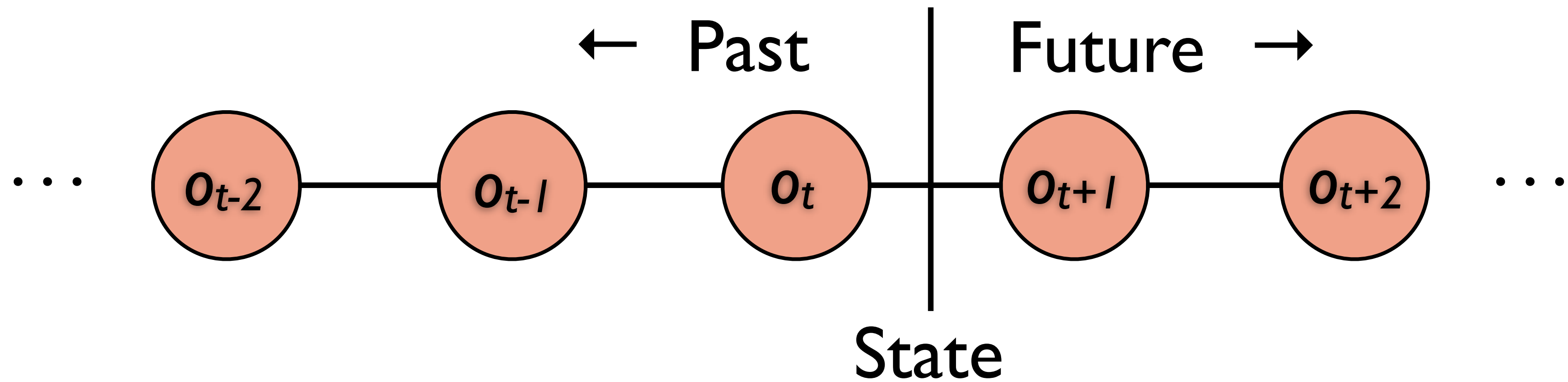
Dynamical system = recursive rule for updating state based on observations

What's out there? A dynamical system



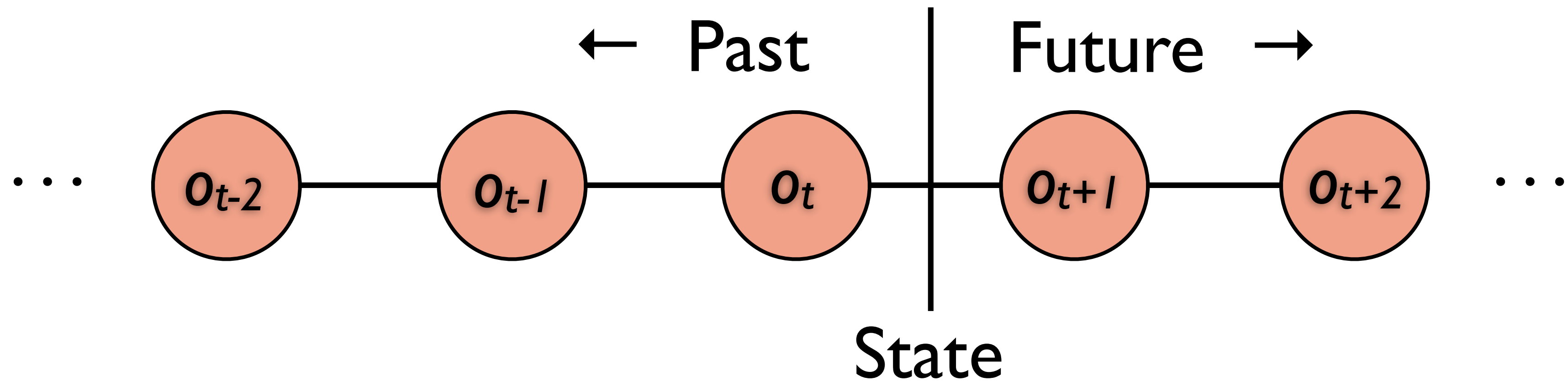
Dynamical system = recursive rule for updating state based on observations

Learning a dynamical system



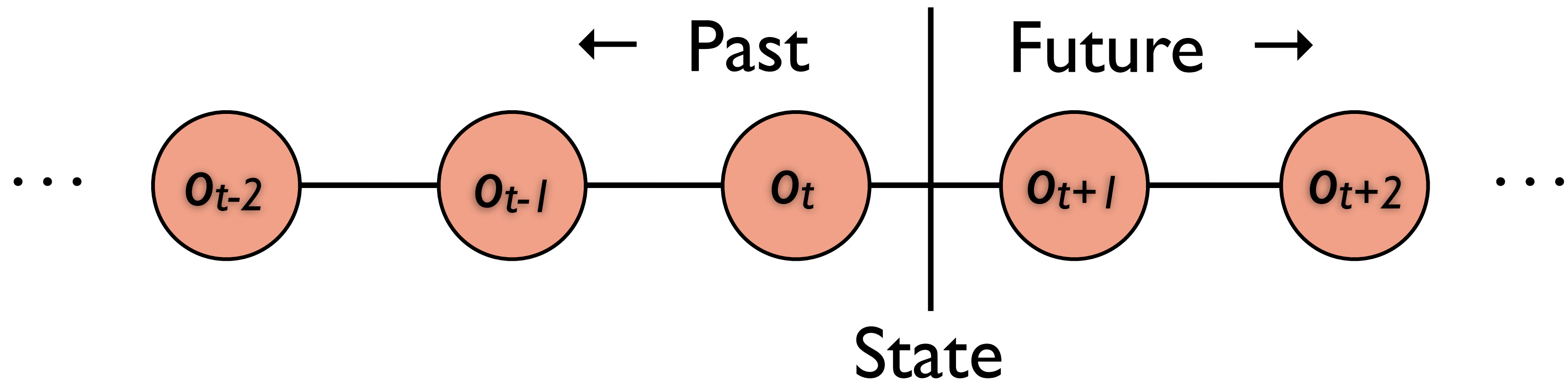
Given past observations from a partially observable system

Learning a dynamical system



Given past observations from a partially observable system

Learning a dynamical system



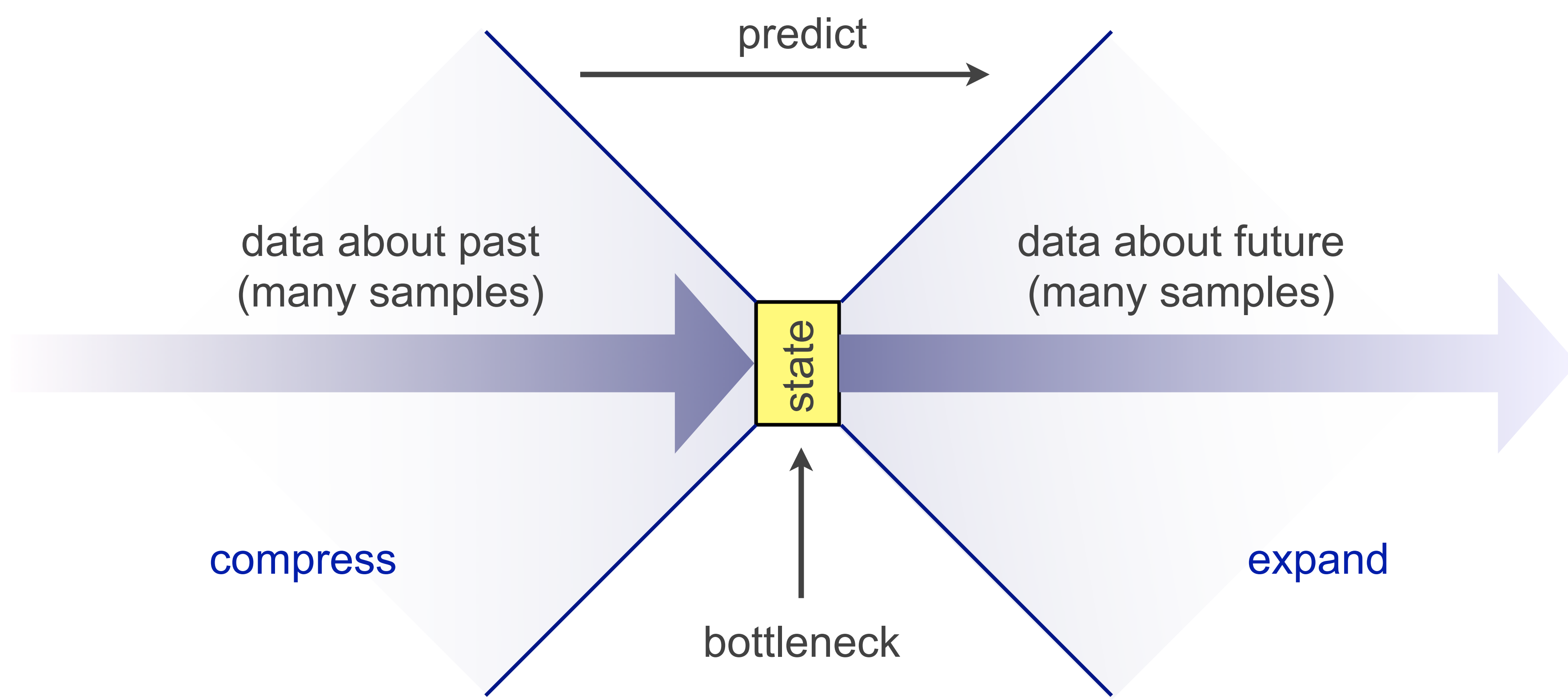
Given past observations from a partially observable system

Predict future observations

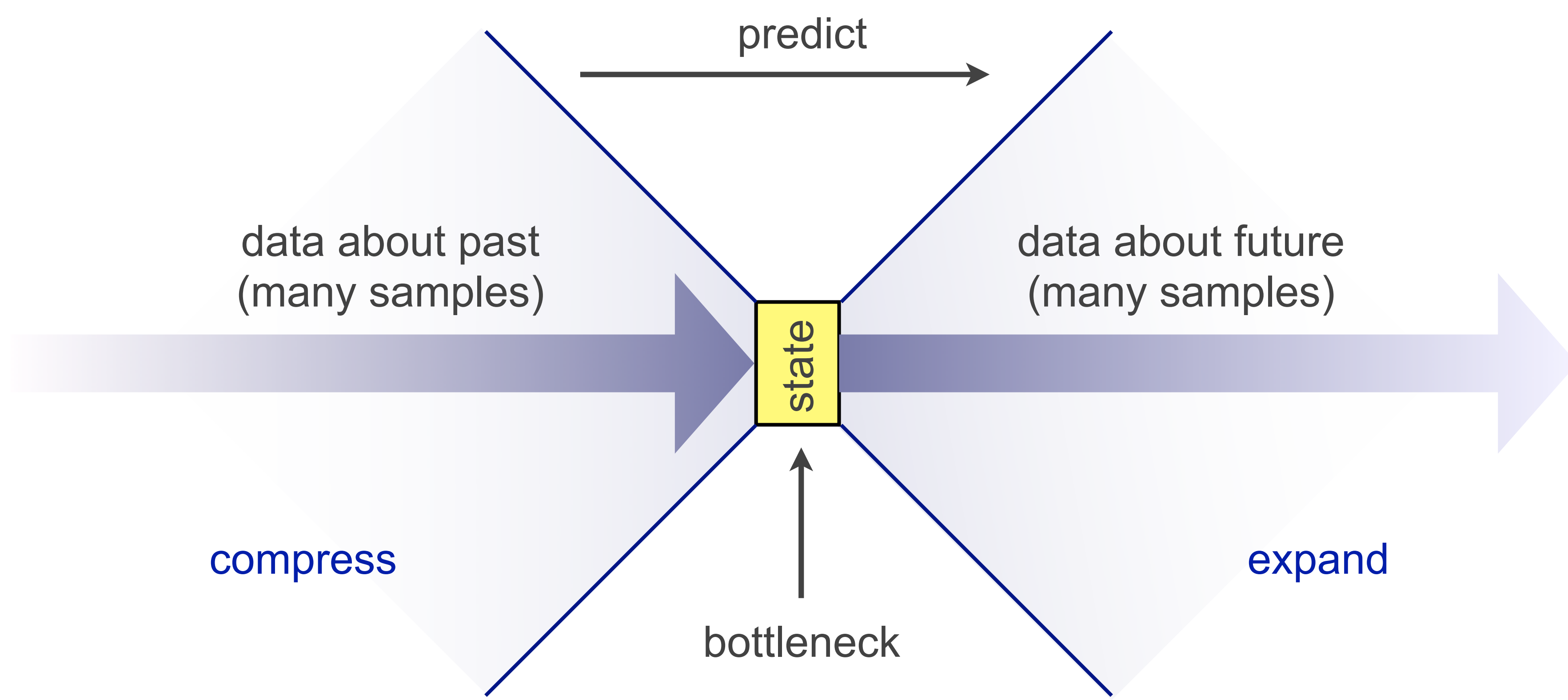
Examples

- Baum-Welsh EM algorithm for HMMs
- Tomasi-Kanade structure from motion
- Black-Scholes model of stock price
- SLAM (from lidars, cameras, beacons, …)
- System identification for Kalman filters
- …

A general principle



A general principle



If bottleneck = rank constraint, get a ***spectral*** method

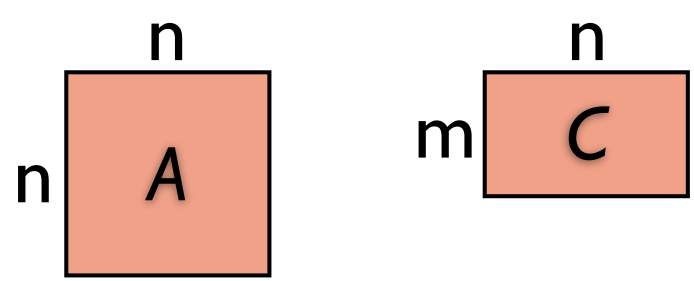
Why spectral methods?

- Many ways to learn models of dynamical systems
  - max likelihood via EM, gradient descent, …
  - Bayesian inference via Gibbs, MH, …
- In contrast to these, spectral methods give
  - ***no local optima!***

    ⇒ huge gain in computational efficiency
  - slight loss in statistical efficiency

Example: SSID for Kalman filter

$$x = A\,x_{-} + \textit{noise}$$
$$o = C\,x + \textit{noise}$$

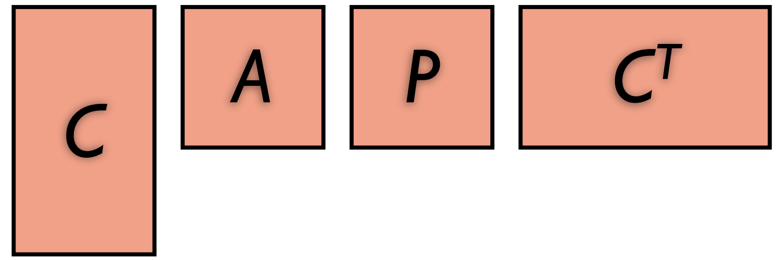


- Past data = last *k* observations
- Future data = next *k'* observations
  - must have *k* and *k'* big enough
- Prediction = linear regression
  - look at empirical covariance of past & future
- Spectral: bottleneck = SVD of covariance

[van Overschee & de Moor, 1996]

Kalman SSID

x = A x₋ + *noise*
o = C x + *noise*



- Assume for simplicity m ≥ n, both A and C full rank
- For k ≥ 1,

$$
\begin{aligned}
\mathbb{E}[o_{t+k} o_t^\top] &= \mathbb{E}[\mathbb{E}[o_{t+k} o_t^\top \mid x_t]] \\
&= \mathbb{E}[\mathbb{E}[o_{t+k} \mid x_t]\mathbb{E}[o_t^\top \mid x_t]] \\
&= \mathbb{E}[C A^k x_t (C x_t)^\top] \\
&= C A^k \mathbb{E}[x_t x_t^\top] C^\top \\
&= C A^k P C^\top
\end{aligned}
$$

Kalman SSID

$$\Sigma_k \;=\; \mathbb{E}[o_{t+k} o_t^\top] \;=\; C A^k P C^\top$$

- Let U = left n leading singular vectors of Σ_1

$$\begin{aligned}
\hat{A} &\stackrel{\text{def}}{=} U^\top \Sigma_2 (U^\top \Sigma_1)^\dagger \\
&= U^\top C A^2 P C^\top (U^\top C A P C^\top)^\dagger \\
&= (U^\top C A) A P C^\top (P C^\top)^\dagger (U^\top C A)^{-1} \\
&= S A S^{-1} \\
\hat{C} &\stackrel{\text{def}}{=} U (S A S^{-1})^{-1} \\
&= U S A^{-1} S^{-1} \\
&= U (U^\top C A) A^{-1} S^{-1} \;=\; C S^{-1}
\end{aligned}$$

Kalman SSID

- Algorithm: estimate Σ_1 and Σ_2 from data, get $\hat{U}$ by SVD, plug in for $\hat{A}$ and $\hat{C}$
- ***Consistent***: continuity of formulas for $\hat{A}$ and $\hat{C}$, law of large numbers for Σ_1 and Σ_2
  - wrinkle: SVD for $\hat{U}$ isn't continuous, but range($\hat{U}$) is
- Can also recover steady-state x

Variations

- Use arbitrary features of length-k window of past and future observations
  - work from covariance of past, future features
  - good features make a big difference in practice
- Impose constraints on learned model (e.g., stability)

Kalman SSID: example

- Works well for ***video textures***
  - steam grate example above
  - fountain:

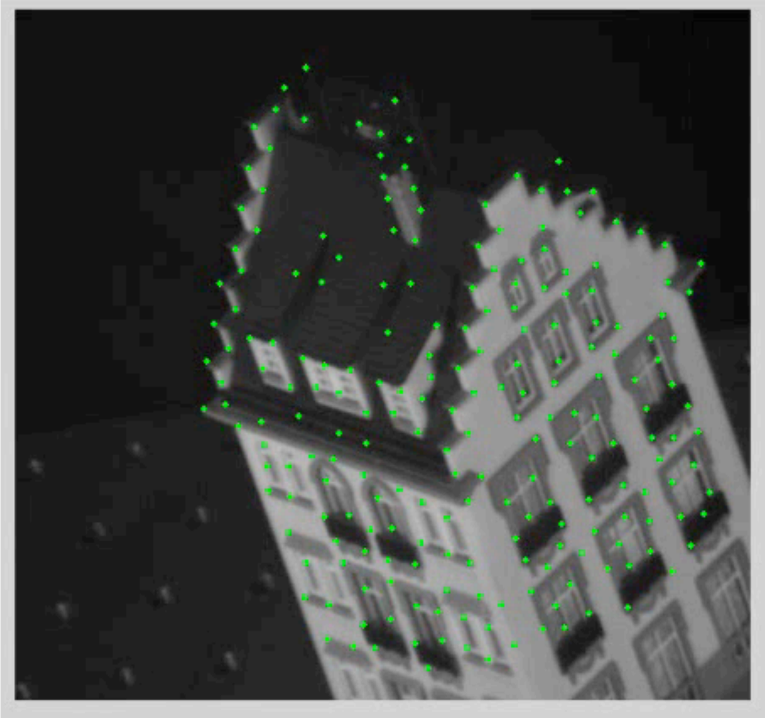
observation = raw pixels (vector of reals over time)

Structure from motion

feature 1, step 2

$$\begin{bmatrix} x_{11} & y_{11} & x_{12} & y_{12} & \cdots & x_{1T} & y_{1T} \\ x_{21} & y_{21} & x_{22} & y_{22} & \cdots & x_{2T} & y_{2T} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_{N1} & y_{N1} & x_{N2} & y_{N2} & \cdots & x_{NT} & y_{NT} \end{bmatrix}$$

- Track N features over T steps

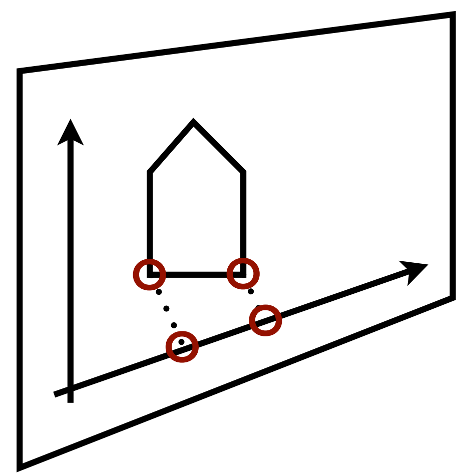


[Tomasi & Kanade, 1992]

Structure from motion

$$\begin{bmatrix} x_{11} & y_{11} & x_{12} & y_{12} & \cdots & x_{1T} & y_{1T} \\ x_{21} & y_{21} & x_{22} & y_{22} & \cdots & x_{2T} & y_{2T} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_{N1} & y_{N1} & x_{N2} & y_{N2} & \cdots & x_{NT} & y_{NT} \end{bmatrix}$$

- x_{it} is projection of feature i onto camera's horizontal axis at time t (and y_{it}, vertical)
  - $[u_i, v_i, w_i]$ = feature i coordinates
  - $[h_{1t}, h_{2t}, h_{3t}]$ = camera horizontal axis @t
  - $[v_{1t}, v_{2t}, v_{3t}]$ = camera vertical axis @t



Structure from motion

$$\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ \vdots & \vdots & \vdots \\ u_N & v_N & w_N \end{bmatrix} \begin{bmatrix} h_{11} & v_{11} & h_{12} & v_{12} & \dots & h_{1T} & v_{1T} \\ h_{21} & v_{21} & h_{22} & v_{22} & \dots & h_{2T} & v_{2T} \\ h_{31} & v_{31} & h_{32} & v_{32} & \dots & h_{3T} & v_{3T} \end{bmatrix}$$

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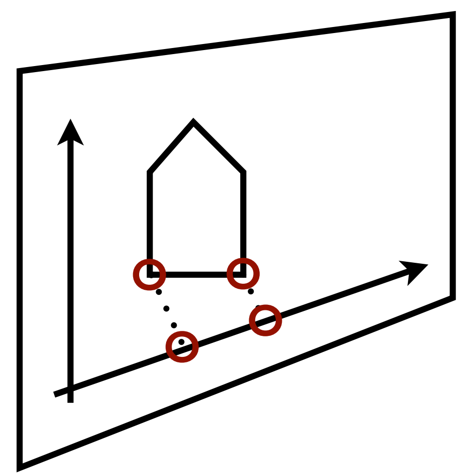
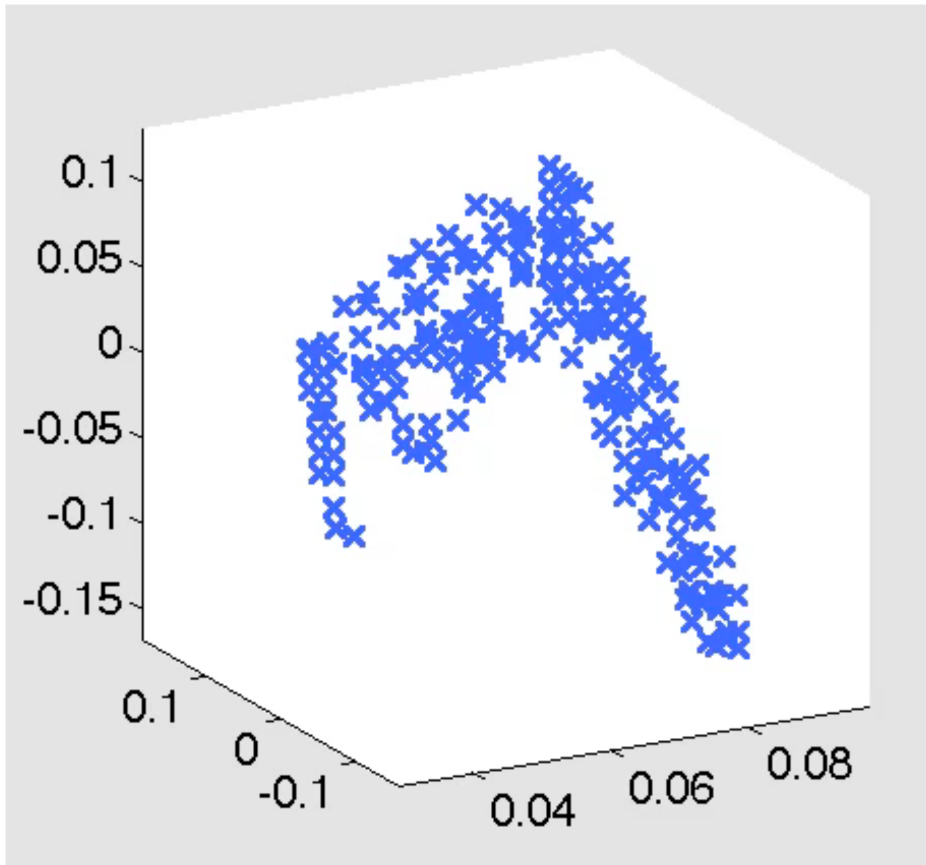


Structure from motion



- only determined up to an invertible transform

SfM as SSID

$$\text{cov} = \begin{bmatrix} x_{11} & y_{11} & x_{12} & y_{12} & \cdots & x_{1T} & y_{1T} \\ x_{21} & y_{21} & x_{22} & y_{22} & \cdots & x_{2T} & y_{2T} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_{N1} & y_{N1} & x_{N2} & y_{N2} & \cdots & x_{NT} & y_{NT} \end{bmatrix}$$

- Past data: indicator of time step & h/v axis
  - means we get to memorize each time step—no attempt to learn dynamics
- Future data: observed screen coordinates (column of matrix)

Kalman SSID: failure

HMM (Baum-Welsh) Kalman Filter (SSID) Preview…

all models: 10 latent dimensions

Kalman SSID: failure

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Kalman SSID: failure

HMM (Baum-Welsh)

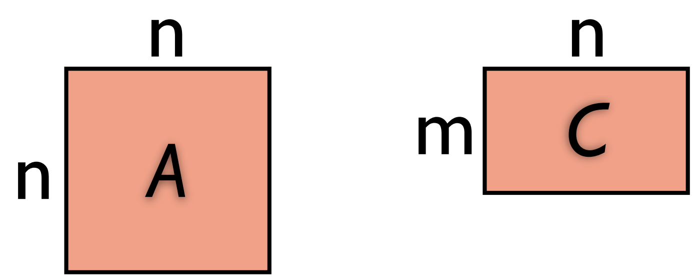
Kalman Filter (SSID)

Preview…

all models: 10 latent dimensions

Can we generalize?

$$x = A\,x_{-} + \textit{noise}$$
$$o = C\,x + \textit{noise}$$



- Get rid of Gaussian noise assumption
- HMM: same form as Kalman filter, but
  - $A \geq 0,\ A\mathbb{1} = 1,\ C \geq 0,\ C\mathbb{1} = 1$
  - noise ~ multinomial
  - x, o are indicators: e.g., "4" = $[0\ 0\ 0\ 1\ 0]^T$

Derivations for Kalman v. HMM

Kalman filter

HMM

$$\begin{aligned}
&\mathbb{E}[o_{t+k}o_t^\top] \\
&\quad = \mathbb{E}[\mathbb{E}[o_{t+k}o_t^\top \mid x_t]] \\
&\quad = \mathbb{E}[\mathbb{E}[o_{t+k} \mid x_t]\mathbb{E}[o_t^\top \mid x_t]] \\
&\quad = \mathbb{E}[CA^k x_t (Cx_t)^\top] \\
&\quad = CA^k\mathbb{E}[x_t x_t^\top]C^\top \\
&\quad = CA^kPC^\top
\end{aligned}$$

- Assume for simplicity m ≥ n, both A and C full rank

Derivations for Kalman v. HMM

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&\quad= CA^k \mathbb{E}[x_t x_t^\top] C^\top \\
&\quad= CA^k P C^\top
\end{aligned}$$

HMM

$$\begin{aligned}
&\mathbb{E}[o_{t+k}o_t^\top] \\
&\quad= \mathbb{E}[\mathbb{E}[o_{t+k}o_t^\top \mid x_t]] \\
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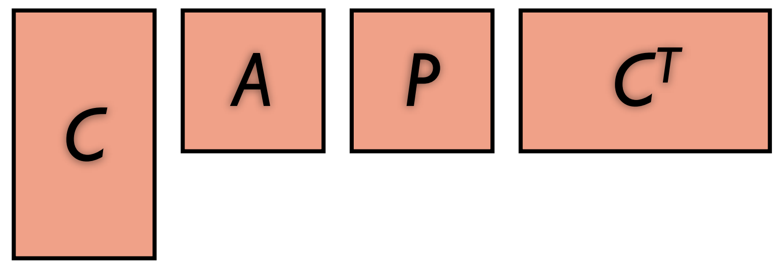
&\quad= \mathbb{E}[CA^k x_t (Cx_t)^\top] \\
&\quad= CA^k \mathbb{E}[x_t x_t^\top] C^\top \\
&\quad= CA^k P C^\top
\end{aligned}$$

- Assume for simplicity $m \geq n$, both A and C full rank

HMM SSID: first try

$$\begin{aligned} U^\top \Sigma_2 (U^\top \Sigma_1)^\dagger &= U^\top C A^2 P C^\top (U^\top C A P C^\top)^\dagger \\ &= (U^\top C A) A P C^\top (P C^\top)^\dagger (U^\top C A)^{-1} \\ &= S A S^{-1} \end{aligned}$$

- As before, recover Â and Ĉ from $E[o_{t+1} o_t^T]$ & $E[o_{t+2} o_t^T]$
- ***Doesn't*** satisfy A ≥ 0, A**1** = **1**, C ≥ 0, C**1** = **1**
  - is this a problem?



Merging A & C

- HMM tracking: write $b_t = P[x_t \mid o_{1:t}]$
  - $P[x_t \mid o_{1:t-1}] = b_{t-0.5} = A\, b_{t-1}$
  - $P[o_t \mid o_{1:t-1}] = C\, b_{t-0.5}$
  - if $o_t = o$:
    - $P(x_t{=}x \mid o_{1:t}) = P(o \mid x_t{=}x)\, P(x_t{=}x \mid o_{1:t-1}) / Z$
    - i.e., $b_t = \text{diag}(C_{o,:})\, b_{t-0.5} / Z$

where $Z = P(o_t{=}o \mid b_{t-1})$

- Write $A_o = \text{diag}(C_{o,:})\, A$
  - $b_t = A_o\, b_{t-1} / Z$

Merging A & C

- HMM tracking: write $b_t = P[x_t \mid o_{1:t}]$
  - $P[x_t \mid o_{1:t-1}] = b_{t-0.5} = A\, b_{t-1}$
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where $Z = P(o_t{=}o \mid b_{t-1})$

- Write $A_o = \text{diag}(C_{o,:})\, A$
  - $b_t = A_o\, b_{t-1} / Z$

It's enough to estimate A_o

Merging A & C

- HMM tracking: write $b_t = P[x_t \mid o_{1:t}]$
  - $P[x_t \mid o_{1:t-1}] = b_{t-0.5} = A\, b_{t-1}$
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    - i.e., $b_t = \mathrm{diag}(C_{o,:})\, b_{t-0.5} \,/\, Z$

where $Z = P(o_t=o \mid b_{t-1})$

- Write $A_o = \mathrm{diag}(C_{o,:})\, A$
  - $b_t = A_o\, b_{t-1} \,/\, Z$

It's enough to estimate A_o

$P(o_t=o \mid b_{t-1}) = \mathbf{1}^T A_o\, b_{t-1}$

HMM SSID: try #2

$$
\begin{aligned}
\Sigma_2^o &\overset{\text{def}}{=} \mathbb{E}[o_{t+2}(\delta_o^\top o_{t+1})o_t^\top] \\
&= \mathbb{E}[\mathbb{E}[o_{t+2}(\delta_o^\top o_{t+1})o_t^\top \mid x_t]] \\
&= \mathbb{E}[\mathbb{E}[o_{t+2}(\delta_o^\top o_{t+1}) \mid x_t]\mathbb{E}[o_t^\top \mid x_t]] \\
&= \mathbb{E}[\mathbb{E}[o_{t+2} \mid x_t, o_{t+1} = o]\mathbb{P}[o_{t+1} = o \mid x_t](Cx_t)^\top] \\
&= \mathbb{E}[\mathbb{E}[o_{t+2} \mid x_t, o_{t+1} = o](\mathbf{1}^\top A_o x_t)(Cx_t)^\top] \\
&= \mathbb{E}\left[CA\left[\frac{A_o x_t}{\mathbf{1}^\top A_o x_t}\right](\mathbf{1}^\top A_o x_t)(Cx_t)^\top\right] \\
&= \mathbb{E}[CAA_o x_t(Cx_t)^\top] \\
&= CAA_o\mathbb{E}[x_t x_t^\top]C^\top \\
&= CAA_o PC^\top
\end{aligned}
$$

HMM SSID: *try #2*

$x = A_o x_- / P(o)$
$o \sim Cx_-$
$C_{o,:} = e^T A_o$

$$\begin{aligned}\hat{A}_o &\stackrel{\text{def}}{=} U^\top \Sigma_2^o (U^\top \Sigma_1)^\dagger \\ &= U^\top C A A_o P C^\top (U^\top C A P C^\top)^\dagger \\ &= (U^\top C A) A_o P C^\top (P C^\top)^\dagger (U^\top C A)^{-1} \\ &= S A_o S^{-1}\end{aligned}$$

- Estimate Σ_1 and Σ_2^o from data; get $\hat{U} = \text{SVD}(\Sigma_1)$
- Plug in to get $\hat{A}_o$ (for each o)
- Also need $e = S^{-1}\mathbf{1}$ = leading left eigenvector of $A_1 + A_2 + \ldots$

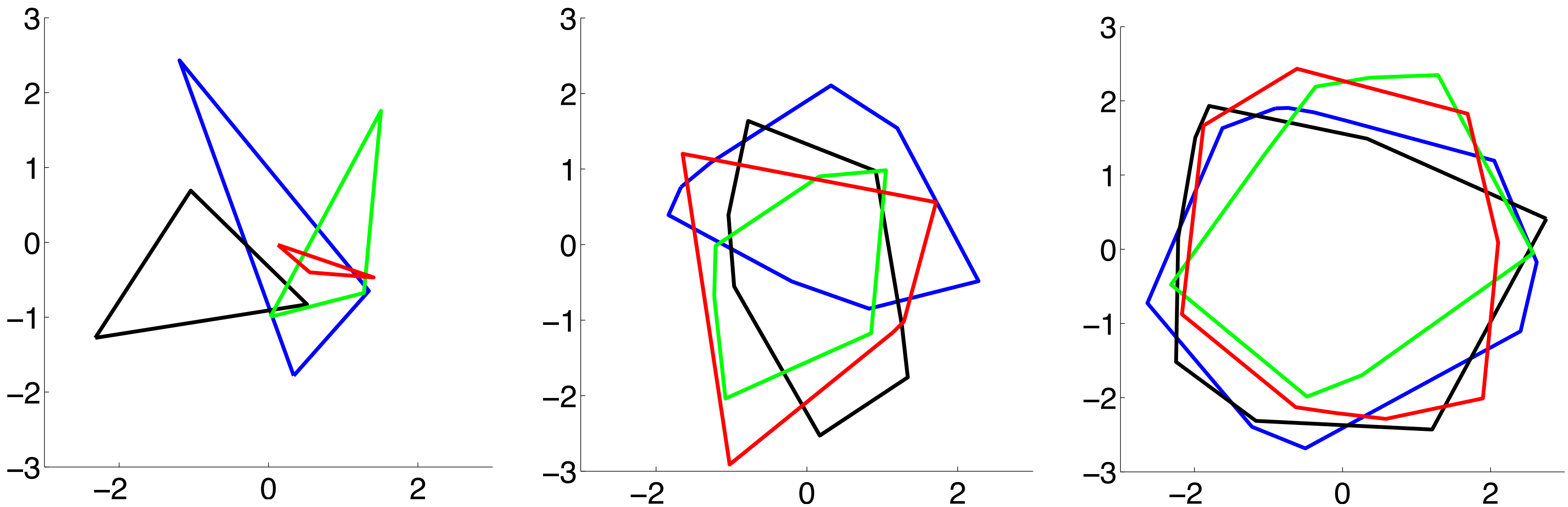
Example: clock

- Discrete observations: sampled frames from training video
  - when tracking: nearest neighbor or Parzen windows (mixture of Gaussians HMM)
- 10 latent dimensions

Can we generalize?

- HMMs had $x \in \Delta$
  - intuition: number of discrete states = number of dimensions
- We now have $x \in S\Delta$
  - essentially equally restrictive
- Can we allow $x \in X$ for general X?
  - # states > # dims

states > # dims: the picture



N=3 N=15 N=100

Random projections of N-dimensional simplex

SSID for OOMs

≈ PSRs without actions, multiplicity automata, …

OOM:
$$x = A_o x_- / P(o)$$
$$o \sim C x_-$$
$$C_{o,:} = e^T A_o$$

HMM:
$$x = A_o x_- / P(o)$$
$$o \sim C x_-$$
$$C_{o,:} = e^T A_o$$

- OOM: defined by transition matrices A_o, normalization vector e
  - like HMM, but lift restriction of $X = S\Delta$
  - instead of $A_o x \geq 0$, have $A_o x \in \lambda X$, $\lambda \geq 0$
  - includes HMM as special case

OOM SSID

- No change!

OOM example

- No change!
  - our HMM SSID was actually learning OOMs all along…