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EFFICIENT AUCTIONS*

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We exhibit an *efficient auction* (an auction that maximizes surplus conditional on all available information). For private values, the Vickrey auction (for one good) or its Groves-Clarke extension (for multiple goods) is efficient. We show that the Vickrey and Groves-Clarke auctions can be generalized to attain efficiency when there are common values, if each buyer's information can be represented as a one-dimensional signal. When a buyer's information is multidimensional, *no* auction is generally efficient. Nevertheless, in a broad class of cases, our auction is *constrained-efficient* in the sense of being efficient subject to incentive constraints.

I. INTRODUCTION

We study *efficient auctions*—auctions that put goods into the hands of the buyers who value them the most.[1] Our interest in them is not purely theoretical but also practical: a leading rationale for the widespread privatization of state-owned assets in

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1. In this respect, we depart from most of the theoretical literature on auctions, which primarily concentrates on *revenue-maximization* (see for example, Myerson [1981], Riley and Samuelson [1981], and Milgrom and Weber [1982]). Ausubel and Cramton [1998b] draw a formal connection between the two objectives.

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recent years is to enhance efficiency.[2] Of course, if, when assets are privatized, there are a sufficiently large number of potential buyers, the question of which auction is most efficient may not matter very much, since competition will render virtually any kind of auction approximately efficient (see Swinkels [1997]). But, in practice, the number of serious bidders is often severely limited,[3] and so the choice of auction form may well be important for efficiency.

When the value that a given buyer attaches to the good (or goods) being sold is independent of information that other buyers may have (the case of private values), there is a well-known and simple answer to the question of how to achieve efficiency: the Vickrey (second-price) auction does so (this is the auction in which the high bidder wins but pays only the second highest bid, see Vickrey [1961]). Furthermore, one of the attractive features of the Vickrey auction is that it extends—via the Groves [1973]-Clarke [1971] mechanism—to the sale of any number of goods.[4]

Unfortunately, the Vickrey auction is no longer efficient once we leave the private-values setting (see Example 3). Yet, for practical applications, it is the case of *common* (or interdependent) *values*—where one buyer's valuation can depend on the private information of another buyer—that is usually pertinent. Suppose, for example, that several wildcatters are bidding for the right to drill for oil on a given tract of land. If the amount of oil under the ground is unknown, then as long as at least one bidder has some private information about this quantity (say, from performing a geological test), we are already in the realm of common values.[5]

2. For example, the U. S. Congress explicitly mandated the Federal Communications Commission to promote efficiency in its auctions of frequency bands for telecommunications.

3. For many properties sold in the FCC spectrum auctions, the number of bidders submitting realistic bids was as low as two or three.

4. Engelbrecht-Wiggans [1988] and Krishna and Perry [1997] show that the Vickrey-Clarke-Groves auction is essentially the *unique* efficient auction, where uniqueness means that any other efficient auction would induce each buyer to make the same expected payments (see also Williams [1994]).

5. We shall use the term "common values" to refer to any situation of such interdependency; that is, we shall invoke the *broad* denotation of the term. The *narrow* denotation—or the expression "pure common values"—applies to the case where all buyers share the *same* valuation. Much of the literature on common values, including seminal papers by Wilson [1977] and Milgrom [1979] and the recent major contribution by Pesendorfer and Swinkels [1997], is limited to *pure* common values. That is, these papers assume that all buyers have the *same* valuation, in which case the issue of efficiency is trivial: allocating the good to any buyer is equally efficient. Finally, those (relatively few) papers, such as Milgrom and Weber [1982] and Ausubel [1997], that accommodate heterogeneous valua-

The principal contribution of this paper is to show that, under standard conditions, the Vickrey auction can be generalized so as to attain efficiency even when there are common values (Propositions 1 and 2). It is also shown that our auction remains efficient regardless of the number of goods being sold[6] and of the nature of those goods, e.g., whether they are substitutes or complements (Proposition 5).

However, our generalized Vickrey auction will be *fully* (i.e., first best) efficient only if each buyer's private information can be summarized by a one-dimensional signal. We show (Proposition 3) that, if buyers' signals are multidimensional, full efficiency is, in general, unattainable by *any* auction (Maskin [1992] establishes a version of this proposition; an even more general impossibility result is developed in Jehiel and Moldovanu [1998]). This impossibility result suggests, however, that the relevant optimality criterion in the multidimensional case is *constrained efficiency* (i.e., efficiency subject to the buyers' incentive constraints—what Holmstrom and Myerson [1983] call "incentive efficiency"). We show that, in a broad class of cases, our generalized Vickrey auction is efficient in this sense (Propositions 4–6).

We proceed as follows. In Section II we consider the sale of a single good. Maskin [1992] shows that the English auction is often efficient in this setting (provided that signals are one-dimensional). However, there is no known general extension of the English auction to more than one good.[7] The Vickrey auction, by contrast, *does* extend to any number of goods. Thus, in this paper we concentrate on Vickrey. In subsection II.A we show that there exists a generalization of the Vickrey auction that attains full efficiency in the allocation of a single good, provided that buyers' signals are one-dimensional. In subsection II.B we give conditions under which this same auction achieves constrained efficiency (efficiency subject to incentive compatibility constraints) in the multidimensional case. Then in Section III we extend our efficient

tions in a common values setting almost invariably suppose that the distribution of signals is *symmetric* across buyers. This implies, in standard mechanisms like the high-bid or second-price auction, that the winner in equilibrium will be the buyer with the highest valuation. That is, given symmetry, the standard auctions turn out to be efficient, even though they fail to remain so once symmetry is relaxed.

6. In recent work, Perry and Reny [1998] have obtained a beautiful extension of the Vickrey auction for the case in which all goods are identical.

7. Ausubel's [1997] elegant extension requires that all goods be identical and that marginal valuations be decreasing. Moreover, it does not generally attain efficiency except in the case of private values.

auction to multiple goods. We conclude, in Section IV, with our view of the most important remaining issue.

II. Single-Good Auctions

II.A. One-Dimensional Signals

A.1. *Formulation*

Suppose that there is a single (indivisible) unit of a good available for auction. There are n risk-neutral buyers. Buyer i observes a *private* real-valued signal $s_i \in S_i \equiv [\underline{s}_i, \overline{s}_i]$. From an ex ante standpoint, the signals $s_1, \ldots, s_n$ can be thought of as random variables. Let $F(s_1, \ldots, s_n)$ be their joint distribution.

Let $v_i(s_1, \ldots, s_n)$ be buyer i's expected *valuation* for the good, conditional on all the signals $(s_1, \ldots, s_n)$.[8] Because we permit $v_i(\cdot)$ to depend on s_j, with $j \neq i$, we are allowing for the possibility of *common values* (i.e., interdependence of valuations). If buyer i is awarded the good and pays price p, his net payoff is

$$v_i(s_1, \ldots, s_n) - p$$

conditional on the vector of signals $(s_1, \ldots, s_n)$. Assume that, for all i, $v_i(\cdot)$ is continuously differentiable in its arguments and that a higher signal value s_i corresponds to a higher valuation:

$$\frac{\partial v_i}{\partial s_i} > 0. \tag{1}$$

We shall assume that a buyer who is not awarded the good (and pays nothing) has zero utility. For now, we will assume that, although signals are private information, the functional forms $v_1(\cdot), \ldots, v_n(\cdot)$ are common knowledge (this assumption can be relaxed; see Remark 1 following Proposition 1).

We give two examples of this formulation.

Example 1. If

$$v_i(s_1, \ldots, s_n) = s_i, \tag{2}$$

then we are in the realm of *private values.*

Example 2. Suppose that buyer i's true valuation is given by the random variable y_i, which he does not observe. Assume,

8. In fact, if each buyer i has no residual uncertainty about his valuation conditional on all the signal values, our analysis extends immediately to the case of risk-averse buyers.

moreover, that

$$y_i = z + z_i + \alpha_i, \tag{3}$$

where z is a normal random variable common to all buyers, z_i is a normal random variable idiosyncratic to buyer i, and α_i is a constant (the constant is introduced to create some possible ex ante asymmetry across buyers). Finally, suppose that buyer i observes signal s_i, where

$$s_i = y_i + \epsilon_i, \tag{4}$$

where ϵ_i is a normal random variable; z, all the z_i's, and all the ϵ_i's are jointly independent; all the z_i's are identically distributed, and all the ϵ_i's are identically distributed. In this case,

$$v_i(s_1, \ldots, s_n) = E[\, y_i | s_1, \ldots, s_n]. \tag{5}$$

We now turn to *auctions*, which are selling procedures in which the good is awarded to (at most) one buyer, and transfers are made between the buyers and the seller, all on the basis of the buyers' bidding behavior. Familiar examples include the *high-bid auction*, in which buyers submit sealed bids, the winner is the high bidder (ties may be broken by some stochastic device such as flipping a coin), he pays his bid, and all losers pay nothing. The *second-price* (or Vickrey) auction has the same rules as the high-bid, except that the winner, instead of paying his own bid pays only the second-highest bid. Finally, in the open or *English auction*, buyers call out bids publicly, with the stipulation that each successive bid should be higher than its immediate predecessor. The winner is the last buyer to bid, and he pays his bid.

We seek auctions for which, in Bayesian equilibrium, the good is allocated to the buyer who, conditional on all available information, values it the most. We call an auction *efficient* if, for all signal values $(s_1, \ldots, s_n)$, the winner in equilibrium is buyer i such that

$$v_i(s_1, \ldots, s_n) \geq v_j(s_1, \ldots, s_n) \qquad \text{for all } j.$$

It is readily seen that, even with private values, the high-bid auction is not, in general, efficient. Actually, if the distribution F is affiliated (see Milgrom and Weber [1982]) and *symmetric*, then, with private values, there exists a symmetric equilibrium in which all buyers use the same bidding function $b(s_i)$, which is increasing in s_i. Hence, the winner is the buyer with the highest signal, which, again under standard assumptions, implies that he is the buyer with the highest valuation. However, this conclusion rests heavily on symmetry.[9] To see this, consider a simple two-buyer example with private values in which s_1 is drawn from a continuous distribution on [0,1] whereas s_2 is drawn (independently) from a continuous distribution on [0,10]. It is easy to see that the equilibrium bid functions $(b_1(\cdot),b_2(\cdot))$ in the high-bid auction satisfy $b_1(1) = b_2(10)$, where $b_2(\cdot)$ is strictly increasing at $s_2 = 10$.[10] But these properties imply that buyer 2 with valuation slightly less than 10 will bid strictly less than $b_1(1)$ and so will lose to buyer 1 with valuation 1. The equilibrium is thus inefficient.

The second-price auction *is* efficient in the case of private values. This is because, in that case, it is a dominant strategy for a buyer to bid his valuation, and so the winner will be the buyer with the highest valuation. However, once we drop the private values assumption, the efficiency of the second-price auction breaks down. To see this, consider the following example.

Example 3. Suppose that there are three buyers, whose valuations are

$$v_1(s_1,s_2,s_3) = s_1 + \tfrac{1}{2}s_2 + \tfrac{1}{4}s_3$$

$$v_2(s_1,s_2,s_3) = s_2 + \tfrac{1}{4}s_1 + \tfrac{1}{2}s_3$$

$$v_3(s_1,s_2,s_3) = s_3.$$

Assume that $s_1 = s_2 = 1$. Note that if s_3 is slightly less than 1, then v_1 is the biggest valuation, but if s_3 is slightly greater than 1, v_2 is biggest. Thus, in a neighborhood of $(s_1,s_2,s_3) = (1,1,1)$, efficient allocation of the good between buyers 1 and 2 (it is not efficient for buyer 3 to be allocated the good) depends on the value of s_3. But in the second-price auction, a buyer's bid can depend only on his *own* signal (the others' signals are private information). Hence, when $s_1 = s_2 = 1$, which of buyers 1 and 2 wins cannot depend on

9. Back and Zender [1993] and Ausubel and Cramton [1998a], show that, even *with* symmetry, the natural extension of the high-bid auction to the case of multiple goods fails to be efficient.

10. To see that $b_1(1) = b_2(10)$, note that if instead $b_2(10) > b_1(1)$, buyer 2 with valuation 10 could reduce his bid slightly from $b_2(10)$ and still win with probability 1, contradicting the assumption that $b_2(10)$ is an equilibrium bid. If $b_2(\cdot)$ is not strictly increasing at $s_2 = 10$, then there is an interval of signal values for buyer 2 for all of which he bids $b_2(10)$, that is, he bids $b_2(10)$ with positive ex ante probability. But then buyer 1 is strictly better off bidding slightly more than $b_1(1)$ than $b_1(1)$ itself, since with positive probability he only ties as winner with $b_1(1)$, whereas he wins with probability 1 with a bid more than $b_1(1) < 1$.

whether s_3 is greater or less than 1, and so the second-price auction is inefficient.

One straightforward way to construct an efficient auction for a setting like that of Example 3 would be to invoke the methods of the mechanism-design literature and consider "direct revelation mechanisms" in which each buyer i reports a signal value $\hat{s}_i$, the good is awarded to the buyer i for whom $v_i(\hat{s}_1, \ldots, \hat{s}_n) \geq \max_{j \neq i} v_j(\hat{s}_1, \ldots, \hat{s}_n)$, and in equilibrium, $\hat{s}_i$ equals the true value s_i. A serious objection to such an approach, however, is that it would, in effect, require the mechanism designer (or auctioneer) to know the physical signal spaces S_1, S_2, and S_3 and the functional forms of the valuation functions $v_1(\cdot)$, $v_2(\cdot)$, and $v_3(\cdot)$, a strong assumption (indeed, in our view it remains a strong assumption even to suppose that the buyers themselves know all this information; see Remark 2 following Proposition 1). Instead, we will seek auction rules that are independent of the details—such as functional forms or distributions of signals—of any particular application and that work well (i.e., attain efficiency or constrained efficiency) in a broad range of circumstances.[11]

We will show, in fact, that the Vickrey auction can be extended (in a detail-free way) to ensure efficiency when, as in Example 3, there are common values. To do this, we require, in addition to (1), the following condition on valuations:[12]

$$\text{for all } i \text{ and } j \neq i, \frac{\partial v_i}{\partial s_i}(s_1, \ldots, s_n) > \frac{\partial v_j}{\partial s_i}(s_1, \ldots, s_n)$$

$$\text{(6)} \quad \text{at any point where } v_i(s_1, \ldots, s_n) = v_j(s_1, \ldots, s_n) = \max_k v_k(s_1, \ldots, s_n).$$

Condition (6) says that (if buyers i and j have equal and maximal valuations) buyer i's signal must have a greater marginal effect on his own valuation than on that of buyer j. Notice that this condition is satisfied trivially in the case of private values (in which buyers' valuation functions satisfy (2)), since the right-hand

11. The insistence that an auction institution be "detail-free" has been called the "Wilson Doctrine" after R. Wilson.

12. Formula (6) is a "single-crossing" condition in the sense of Mirrlees [1971] or Spence [1973]. We are indebted to P. Milgrom, who urged us to adopt essentially this condition in place of an earlier, more stringent requirement. Gresik [1993] introduced a stronger version of this condition in his study of trading mechanisms with common values.

side of the inequality in (6) is then zero. Moreover, it holds for Example 2. In that model, s_i is a more informative signal about y_i than about y_j, $j \neq i$, since s_i conveys information about both the idiosyncratic and common components (z_i and z) of y_i but only about the common component z of y_j. Hence, a small change in s_i will affect the expected value of y_i more than that of y_j.

Condition (6) is in fact necessary if *any* auction, let alone our generalized Vickrey auction, is to be efficient.[13] To see this, consider the following example.

Example 4. Consider two wildcatters who are competing for the right to drill for oil on a given tract of land. The wildcatters' costs of drilling differ. Wildcatter 1 has a fixed cost of 1 and a marginal cost of 2 (per unit of oil extracted). Wildcatter 2's fixed cost is 2 and marginal cost is 1. Oil can be sold at a price of 4. Wildcatter 1 performs a (private) test and discovers that the expected size of the oil reserve is s_1 units. Wildcatter 2's private information s_2 does not affect either driller's payoff. We have

$$v_1(s_1,s_2) = (4 - 2)s_1 - 1 = 2s_1 - 1$$

and

$$v_2(s_1,s_2) = (4 - 1)s_1 - 2 = 3s_1 - 2.$$

Notice that

$$\frac{\partial v_1}{\partial s_1} < \frac{\partial v_2}{\partial s_1},$$

and so (6) is violated. Moreover, we claim that there is no way to induce wildcatter 1 to reveal his information while maintaining efficiency (assuming that, even ex post, s_1 cannot be measured directly and that nobody but the winning wildcatter can monitor how much oil there turns out to be). Efficiency dictates that wildcatter 1 get the drilling rights if $\frac{1}{2} < s_1 < 1$ and that wildcatter 2 get the drilling rights if $s_1 > 1$ (if $s_1 < \frac{1}{2}$, it is inefficient to drill at all; and because of the fixed costs, it would always be inefficient to give both wildcatters drilling rights).

13. More precisely, if (6) fails to hold, then we can find a joint distribution $F(s_1, \ldots, s_n)$ with respect to which there is no auction that is efficient. Signals in this joint distribution will be *independent*; otherwise we could use the methods of Crémer and McLean [1988] to construct a fully efficient auction regardless of whether (6) holds.

Suppose that wildcatter 1 is given a reward $R(\hat{s}_1)$ if he claims that there are $\hat{s}_1$ units of oil. Then if $s_1 > 1 > s'_1 > ½$, incentive compatability and efficiency demand that

(7) $$R\ (s_1) \geq 2s_1 - 1 + R(s'_1)$$

and

(8) $$2s'_1 - 1 + R\ (s'_1) \geq R(s_1).$$

Subtracting (7) from (8), we obtain

$$2(s'_1 - s_1) \geq 0,$$

a contradiction. Hence efficiency is impossible.

It is easy to see what is going wrong here. As s_1 rises, the drilling rights become more valuable to wildcatter 1. But, from the standpoint of efficiency, they are increasingly likely to be awarded to 2, thanks to the violation of (6). It is this conflict between 1's personal objective and overall efficiency that creates the incentive problem.

Notice that the three buyers' valuations in Example 3 satisfy condition (6). We saw, however, that the Vickrey auction fails to be efficient in that example because buyers 1 and 2 cannot embody information about buyer 3's valuation in their bids. This suggests that generalizing the Vickrey auction to allow buyers to make *contingent* bids—bids that depend on other buyers' valuations—may overcome this problem.

For simplicity of exposition, we first consider the case of two buyers.

II.A.2. Auctions with Two Buyers

Instead of a single bid, we will have each buyer i report a bid *function,*

$$\hat{b}_i : \hat{V}_j \rightarrow \mathbb{R}_+,$$

where $j \neq i$ and $\hat{V}_j$ is an interval $[0,\bar{v}_j]$ in $\mathbb{R}_+$. For each $v_j \in \hat{V}_j$ we can interpret $\hat{b}_i(v_j)$ as buyer i's bid if the other buyer's valuation turns out to be v_j. Given the bid functions $(\hat{b}_1(\cdot),\hat{b}_2(\cdot))$, let us look for a *fixed point*, i.e., a pair (v_1°,v_2°) such that

(9) $$(v_1^\circ,v_2^\circ) = (\hat{b}_1(v_2^\circ),\hat{b}_2(v_1^\circ)).$$

(We shall deal with the issue of nonexistent or multiple fixed points below.) Then we will suppose that

$$\text{(10)}\quad \text{buyer } i \text{ is the winner} \Leftrightarrow \hat{b}_i(v_j^\circ) > \hat{b}_j(v_i^\circ) \quad \text{(break ties by flipping a coin).}$$

To see that this allocation rule is the "right" one, consider what happens when buyers bid "truthfully." In the standard Vickrey auction with private values, bidding truthfully means bidding one's true valuation. In our setting, a buyer does not actually know his valuation because he does not know the other buyer's signal value. So, bidding truthfully means making a bid contingent on the other buyer's valuation so that, whatever that other signal value (and hence other valuation) turns out to be, his corresponding bid will equal his own valuation. That is, if buyer 1's signal value is s_1, the truthful bid function is $b_1(\cdot)$ such that

$$\text{(11a)}\qquad b_1(v_2(s_1,s'_2)) = v_1(s_1,s'_2) \qquad \text{for all } s'_2.$$

Similarly,

$$\text{(11b)}\qquad b_2(v_1(s'_1,s_2)) = v_2(s'_1,s_2) \qquad \text{for all } s'_1$$

(note that $b_1(\cdot)$ is well-defined, i.e., if $v_2(s_1,s'_2) = v_2$, there cannot exist $s''_2 \neq s'_2$ such that $v_2(s_1,s''_2) = v_2$ because $v_2(\cdot)$ satisfies (1); similarly, $b_2(\cdot)$ is well-defined). Observe that

$$\text{(12)}\qquad (v_1^\circ,v_2^\circ) = (v_1(s_1,s_2),v_2(s_1,s_2))$$

is a fixed point of the mapping

$$\text{(13)}\qquad (v_1,v_2) \mapsto (b_1(v_2),b_2(v_1)), \qquad \text{for } (v_1,v_2) \in V_1 \times V_2,$$

where

$$\text{(14)}\qquad \begin{aligned} V_1 &= \{v_1 \mid \text{there exists } s'_1 \text{ such that } v_1(s'_1,s_2) \geq v_1 \text{ with strict inequality only if } s'_1 = \underline{s}_1\} \\ &\text{and} \\ V_2 &= \{v_2 \mid \text{there exists } s'_2 \text{ such that } v_2(s_1,s'_2) \geq v_2 \text{ with strict inequality only if } s'_2 = \underline{s}_2\} \end{aligned}$$

(note that V_1 consists of all possible valuations v_1 that buyer 1 could have that are consistent with *some* signal value s'_1; for completeness, it also includes valuations that are smaller than those consistent with any signal value. V_2 is defined symmetrically). This means that, if buyers bid truthfully, our allocation rule (9)–(10) ensures that buyer 1 wins if and only if $v_1(s_1,s_2) > v_2(s_1,s_2)$, which is precisely what is entailed by efficiency. We conclude that, if buyers make truthful reports, the allocation rule (9)–(10) is *efficient*. There is, however, one technical caveat to this conclusion: conditions (1) and (6) are not strong enough to rule out other fixed points of (13) besides (12).[14] We shall confront the issue of multiple fixed points in detail below (see Proposition 2). For now, let us assume that the inequality in (6) holds at *all* points (s_1,s_2) and that its right-hand side is nonnegative. Then (12) is indeed the unique fixed point of (13). To see this, note from (11a) and (11b) that

$$\frac{db_2}{dv_1}(v_1(s'_1,s'_2))\,\frac{\partial v_1}{\partial s_1}(s'_1,s'_2) = \frac{\partial v_2}{\partial s_1}(s'_1,s'_2), \qquad \text{for all } (s'_1,s'_2),$$

and so, from (1) and the stronger version of (6), we obtain

$$\left|\frac{db_2}{dv_1}(v_1)\right| < 1 \qquad \text{for all } v_1$$

(15) and

$$\left|\frac{db_1}{dv_2}(v_2)\right| < 1 \qquad \text{for all } v_2.$$

The "contraction mapping" property (15) ensures that any fixed point is unique.

We have shown that the outcome is efficient if buyers bid truthfully, but it remains to establish that there exists a payment scheme that induces truthful bidding. The way that the Vickrey auction induces truthfulness in the private-values case is to make a winning buyer's payment equal to the lowest bid that he could have made for which he would still have won the auction. So, for example, if buyer 1 bids 5 and buyer 2 bids 3, buyer 1 should win but pay 3, since that is the lowest bid (ignoring ties) that would have won him the good. Let us try to adhere to this principle as

14. Suppose, for example, that $v_1(s_1,s_2) = s_1^2 + s_1s_2 - s_2^2 + s_1 - 2s_2 + 24$ and $v_2(s_1,s_2) = s_2^2 + s_1s_2 - s_1^2 - 9s_1 + 13$. Then if $(s_1,s_2) = (2,3)$, one fixed point is the pair of true valuations $(v_1(2,3),v_2(2,3)) = (21,6)$. However, for these signal values, $(v_1(2,4),v_2(1,3)) = (14,15)$ also constitutes a fixed point, because $v_i(2,4) = v_i(1,3)$, $i = 1,2$, and so $v_2(1,3) = b_2(v_1(1,3))$, and $v_1(2,4) = b_1(v_2(2,4))$.

closely as possible. In the two-buyer case this means that, if buyer 1 is the winner (i.e., $\hat{b}_1(v_2^\circ) > \hat{b}_2(v_1^\circ)$, where $(v_1^\circ, v_2^\circ) = (\hat{b}_1(v_2^\circ), \hat{b}_2(v_1^\circ))$), then he should pay

$$\hat{b}_2(v_1^*) \tag{16}$$

where[15]

$$v_1^* = \hat{b}_2(v_1^*). \tag{17}$$

This is because if buyer 1 were restricted to constant bids, v_1^* would be the lowest such bid for which, under our allocation rule, buyer 1 would still win the auction, given buyer 2's reported bid schedule $\hat{b}_2(\cdot)$. Note that if 1's valuation actually were v_1^*, his net payoff from winning would be zero.

To see that, provided that (1) and the stronger version of (6) hold, buyer 1 has an incentive to bid truthfully in equilibrium under payment rules (16) and (17), suppose that buyer 2 is truthful, i.e., he sets $\hat{b}_2(\cdot) = b_2(\cdot)$, where $b_2(\cdot)$ satisfies (11b). Then, if buyer 1 wins, his payoff is

$$v_1(s_1,s_2) - b_2(v_1^*), \tag{18}$$

where[16]

$$v_1^* = b_2(v_1^*). \tag{19}$$

It suffices to show that if buyer 1 sets $\hat{b}_1(\cdot) = b_1(\cdot)$ satisfying (11), then he wins if and only if (18) is positive. (This is because buyer 1's payoff if he wins (i.e., (18)) is independent of how much he bids, and so the best he can do is to ensure that he wins precisely in those cases in which this payoff is positive.) Now from (15), (18) is positive if and only if, for any v'_1,

$$v_1(s_1,s_2) - v_1^* > \frac{db_2}{dv_1}(v'_1)(v_1(s_1,s_2) - v_1^*). \tag{20}$$

From the intermediate value theorem, there exists a value of v'_1 such that

$$b_2(v_1(s_1,s_2)) - b_2(v_1^*) = \frac{db_2}{dv_1}(v'_1)(v_1(s_1,s_2) - v_1^*).$$

15. Because $\hat{V}_1 = [0,\bar{v}_1]$ and $v_1^\circ > \hat{b}_2(v_1^\circ)$, there exists (at least) one point v_1^* satisfying (16) and (17). We will deal with the issue of multiple solutions in Proposition 2.

16. Because (15) holds and $v_1^\circ > b_2(v_1^\circ)$, there exists a unique point v_1^* satisfying (17).

Hence, (20) holds if and only if[17]

$$v_1(s_1,s_2) - v_1^* > b_2(v_1(s_1,s_2)) - b_2(v_1^*). \tag{21}$$

Now, from (11b), $b_2(v_1(s_1,s_2)) = v_2(s_1,s_2)$; and from (19), $v_1^* = b_2(v_1^*)$. Hence (21) holds if and only if

$$v_1(s_1,s_2) > v_2(s_1,s_2). \tag{22}$$

But, when he is truthful, buyer 1 wins if and only if (22) holds. Hence, if buyer 1 bids truthfully, (18) is indeed positive if and only if buyer 1 wins.

To summarize, we have shown

PROPOSITION 1. Consider the two-buyer auction in which, for $i = 1,2$,

(i) buyer i reports $\hat{V}_j = [0,\overline{v}_j]$ ($j \neq i$) and a contingent bid function $\hat{b}_i : \hat{V}_j \rightarrow \mathbb{R}_+$, with the stipulation that

$$\left|\frac{d\hat{b}_i}{dv_j}\right| < 1; \tag{23}$$

(ii) a fixed point (v_1°,v_2°) is taken according to (9) (the restriction (23) ensures that there can be at most one fixed point; if there is no fixed point at all, the good is not allocated);

(iii) the winner is determined according to (10);

(iv) if buyer 1 is the winner, he makes a payment according to (16)–(17) (buyer 2's payment when he is the winner is symmetric).

If (1) holds, the inequality of (6) is satisfied at all points (s_1,s_2), and the right-hand side of this inequality is nonnegative, then this auction is *efficient*. That is, it is an equilibrium for each buyer i to bid truthfully, i.e., to set $\hat{V}_j = V_j$ and $\hat{b}_i(\cdot) = b_i(\cdot)$, where V_j satisfies (14) and $b_i(\cdot)$ satisfies (11a) or (11b). Moreover, if both buyers do so, the auction results in an efficient outcome.[18]

17. If (21) holds, then, from the intermediate value theorem, (20) holds for some value of v_1'. Hence, from (15), we must have $v_1(s_1,s_2) - v_1^* > 0$. But then (20) holds for all values of v_1'.

18. Note that truthful bidding is not a dominant strategy in the auction of Proposition 1 (except in the case of private values). However, it constitutes a Bayesian equilibrium that is *robust* in the sense that, given $v_1(\cdot)$ and $v_2(\cdot)$, we can change the distribution of signals F arbitrarily without affecting equilibrium behavior. (Indeed, notice that none of our analysis has referred to F at all.) Thus, equilibrium strategies are invariant to changes in the distribution of signals.

We have not ruled out the possibility of "untruthful" equilibria in Proposition 1. However, we are confident that, by using techniques from the Bayesian

Remark 1. It may seem very demanding to insist that a buyer make his bid a function of the other buyer's valuation. In this two-buyer case, however, there is only a *single point* of each buyer i's schedule $\hat{b}_i(\cdot)$ that has to be correct (i.e., truthful) in order to ensure an efficient outcome,[19] namely, the point where $\hat{b}_i(\cdot)$ intersects the 45 degree line:

$$v_j^* = \hat{b}_i(v_j^*). \tag{24}$$

That is, when s_i is buyer i's signal value, all that we require of buyer i is that he report $\hat{b}_i(\cdot)$ so that, if (24) holds, then there exists s'_j such that

$$v_j^* = v_j(s_i, s'_j) = v_i(s_i, s'_j). \tag{25}$$

It can readily be checked that, given that $\hat{b}_2(\cdot)$ satisfies (23)–(25), buyer 1 will win if and only if his payoff from winning is positive, provided that he chooses $\hat{b}_1(\cdot)$ so that it too satisfies (23)–(25). Hence, this behavior constitutes an *equilibrium*.

Admittedly, even calculating the v_j^* that satisfies (25) requires buyer i to know something about buyer j's valuation function $v_j(\cdot)$, and, for an efficient equilibrium to occur, this knowledge must be *common* to the two buyers. However, such common knowledge is not necessary for the auction and buyers to perform reasonably well.

Indeed, to take the opposite extreme, suppose that buyer 1 knew *nothing* about the nature of $v_2(\cdot)$. He could, nevertheless, make an *uncontingent* bid $b_1(\cdot) \equiv b_1$, for some $b_1 \in \mathbb{R}_+$. With such a bid he would win if and only if $b_1 > \hat{b}_2(b_1)$, in which case he would pay $\hat{b}_2(v_1^*) = v_1^* < b_1$. In other words, by submitting a constant function, he can induce an outcome and payment very much like those in the ordinary second-price auction. And this places a lower bound on how badly he can do in our auction. But if he has even a vague idea of how his and buyer 2's valuations are interdependent, buyer 1 should be able to do strictly better than this by submitting a bid function with nonzero slope. In any case, because the fixed point is continuous in his report, he cannot go too badly wrong by doing so.

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implementation literature (see Palfrey [1993] for a survey), one could modify the auction to eliminate such equilibria (albeit at the cost of a more complex set of rules).

19. Indeed, if $n = 2$, the ordinary Vickrey auction is efficient (see Maskin [1992]). However, as Example 3 shows, its efficiency fails for $n \geq 3$; whereas our generalized Vickrey auction remains efficient for that case.

In this sense, having buyers report contingent bids should be viewed as giving them an *opportunity* to express their interdependencies—an opportunity that they can exploit to any degree that they wish or their knowledge permits—rather than as imposing an onerous requirement on them.

Remark 2. As noted in the previous remark, some degree of common knowledge about valuation functions is needed to ensure that players can calculate equilibrium (this is true not only of our auction, but of any other auction—high-bid, second-price, etc.—as well). One may inquire, therefore, why we do not go "all the way" and have each buyer i report a *pair* of valuation functions ($\hat{v}_1(\cdot)$, $\hat{v}_2(\cdot)$) and then (i) use a "direct revelation" mechanism (in which each buyer reports his signal value and these are then plugged into the reported valuation functions) if the two buyers' reports agree, and (ii) punish buyers in some way if their reports disagree, rather than resorting to the "indirect" device of having them make contingent bids. There is a difficulty, however, with having buyer 1 report $\hat{v}_2(\cdot)$, namely, he may not even know what buyer 2's physical signal space is (we have modeled S_2 as an interval of "numbers," but, to buyer 2, the signal s_2 presumably corresponds to something physical). To complicate matters further, s_2 may in fact be only a sufficient statistic for a variety of information parameters that buyer 2 receives. Notice that there is no contradiction in supposing that buyer 1 does not know $v_2(\cdot)$ but *does* know v_2^* satisfying (25); the latter requires less knowledge.[20] Indeed, that it is common knowledge that, for $i = 1,2$, buyer i can calculate v_j^* can be thought of as the *weakest* hypothesis that ensures efficiency in equilibrium. In that sense, the efficient auction we are proposing is the "simplest" possible one.[21]

II.A.3. Auctions with More than Two Buyers

Our two-buyer auction can readily be extended to the case of three or more buyers. There are two minor complications that arise in this case. First, it may no longer suffice for buyers to submit single-valued bid functions. In the two-buyer case, there is

20. Imagine, for example, that from previous experience with similar goods buyer 1 has learned that, when buyer 2's valuation is less than v_2^* his own valuation v_1 is greater than v_2, whereas when $v_2 > v_2^*$, $v_1 \leq v_2$. Notice that this information entails knowledge neither of the functional form $v_2(\cdot)$ nor of buyer 2's signal space.

21. It also has the desirable property that equilibrium strategies remain in equilibrium even ex post when buyers' signal values have become common knowledge.

a *unique* truthful bid $b_1(v_2)$ by buyer 1 for each possible buyer 2 valuation v_2 (because, given s_1, there is a unique s'_2 such that $v_2(s_1,s'_2) = v_2$). In the case of three buyers, however, there could exist, for given s_1, two different pairs (s'_2,s'_3) and (s''_2,s''_3) such that $v_2(s_1,s'_2,s'_3) = v_2(s_1,s''_2,s''_3) \equiv v_2$ and $v_3(s_1,s'_2,s'_3) = v_3(s_1,s''_2,s''_3) \equiv v_3$. Then, a unique truthful bid $b_1(v_2,v_3)$ would not be well-defined if $v_1(s_1,s'_2,s'_3) \neq v_1(s_1,s''_2,s''_3)$. Accordingly, we shall have buyers report bid *correspondences*.

Second, for $n \geq 3$, conditions to rule out the possibility of multiple fixed points become too restrictive to be palatable. With two buyers, the strong version of condition (6), which is still fairly mild, suffices. But this condition is no longer sufficient with three or more buyers, even if all buyers bid truthfully (with truthful bidding one fixed point will be the true valuations, but there could be others). To deal with this problem, we will introduce a potential second stage of the auction in which buyers, in effect, choose among the different fixed points that may have arisen in the first stage.

To simplify matters, let us assume that, for all i, $\bar{s}_i = \infty$ and that[22]

(26) for all $s_{-i} \in S_{-i}$ there exists $s'_i \in S_i$ such that $v_i(s'_i,s_{-i})$

$$> \max_{j \neq i} v_j(s'_i,s_{-i}).$$

That is, regardless of the other buyers' signal values, buyer i has a signal value that gives him the highest valuation. (This assumption is not required, but helps keep complications to a minimum.)

Consider the following auction defined in six steps (steps (a)–(d) are the heart of the auction; steps (e) and (f) serve only to deal with multiple fixed points and can be ignored by readers uninterested in these technicalities):

(a) each buyer i ($i = 1, \ldots, n$) submits a bid correspondence $\hat{b}_i : \hat{V}_{-i} \longrightarrow \mathbb{R}_+$, where $\hat{V}_{-i} \subseteq \mathbb{R}_+^{n-1}$;

(b) a fixed point $(v_1^\circ, \ldots, v_n^\circ)$ is calculated so that

$$(27) \qquad v_i^\circ \in \hat{b}_i(v_{-i}^\circ) \qquad \text{for all } i$$

(if there is no fixed point, the good is not allocated, and no buyer makes a payment; if there are multiple fixed points, go to (e) below);

(c) if $v_i^\circ \geq \max_{j \neq i} v_j^\circ$, the good is awarded to buyer i;

22. The notation "s_{-i}" denotes a vector of signals for all buyers other than i.

(d) if buyer i is the winner, he makes a payment

$$\max_{j \neq i} v_j^*, \tag{28}$$

where $(v_1^*, \ldots, v_n^*)$ is a vector such that

$$v_i^* = \max_{j \neq i} v_j^* \tag{29}$$

and

$$v_k^* \in \hat{b}_k(v_{-k}^*) \qquad \text{for all } k \neq i \tag{30}$$

(if there is no vector $(v_1^*, \ldots, v_n^*)$ satisfying (29) and (30), the good is not awarded, and no buyer makes a payment; if there are multiple such vectors, go to (f));

(e) if there are multiple vectors

$$(v_1^{\circ 1}, \ldots, v_n^{\circ 1}), \ldots, (v_1^{\circ J}, \ldots, v_n^{\circ J})$$

satisfying (27), then the submitted bid functions $(\hat{b}_1(\cdot), \ldots, \hat{b}_n(\cdot))$ are made public, and each buyer i chooses $v_i^{\circ} \in \{v_i^{\circ 1}, \ldots, v_i^{\circ J}\}$; if there exists $j \in \{1, \ldots, J\}$ such that

$$v_i^{\circ} = v_i^{\circ j} \qquad \text{for all } i,$$

then the auction returns to step (c) using $(v_1^{\circ j}, \ldots, v_n^{\circ j})$ to determine the winner; otherwise the good is not awarded and no buyer makes a payment;

(f) if there are multiple vectors $(v_1^{*1}, \ldots, v_n^{*1}), \ldots, (v_1^{*K}, \ldots, v_n^{*K})$ satisfying (29) and (30), then the schedules $(\hat{b}_1(\cdot), \ldots, \hat{b}_n(\cdot))$ are made public, and each buyer $j \neq i$ chooses $v_j^* \in \{v_j^{*1}, \ldots, v_j^{*K}\}$; if there exists $k \in \{1, \ldots, K\}$ such that

$$v_j^* = v_j^{*k} \qquad \text{for all } j \neq i,$$

then the auction returns to step (d) using $(v_1^{*k}, \ldots, v_n^{*k})$ to determine the winner's payment; otherwise, the good is not awarded, and no buyer makes a payment.

Steps (a)–(c) exactly mirror steps (i)–(iii) of Proposition 1 in the case of two buyers. As for step (d), we are attempting, once again, to have the winner pay the lowest bid for which he would still have won the auction. For expositional purposes, let us suppose that the bid schedules $\hat{b}_j(\cdot)$ are single-valued. If buyer i is the winner, let us reduce v_i (starting from $v_i = v_i^{\circ}$) until it is equal to the second-highest bid. Note, however, that as we reduce v_i, the other buyers' bids change (because they are functions of v_i). We therefore seek a vector $(v_1^*, \ldots, v_n^*)$ for which

$$v_i^* = \max_{j \neq i} \hat{b}_j(v_{-j}^*)$$

and that is *consistent* in the sense that

$$v_k^* = \hat{b}_k(v_{-k}^*) \qquad \text{for all } k \neq i.$$

(These conditions generalize to (29) and (30) when the bid schedules need not be single-valued.)

Suppose that there are multiple fixed points $(v_1^\circ, \ldots, v_n^\circ)$ at step (b). Step (e) is intended to resolve the indeterminacy (readers uninterested in multiple fixed points may wish to skip directly to the discussion of truthful bidding below). After the multiplicity arises, a second stage is played in which buyers' reported bid correspondences are first made public, and then (simultaneously) each buyer chooses from among the fixed points that have arisen. If buyers have bid truthfully (we will make precise what this means below), then, provided that a weak condition on valuation functions (see (38)) is satisfied, they will have complete information at this second stage about other buyers' valuations and so, in particular, will be able to identify the fixed point corresponding to true valuations (as in the two-buyer case, the true valuations will always constitute a fixed point, although as noted above there could also be others). Thus, the auction will induce an equilibrium in which each buyer chooses his true valuation in this second stage if the rules stipulate that, should any buyer i choose a component v_i° that is not consistent with v_{-i}°, the good is not allocated at all.[23]

Finally, step (f) is meant to resolve the indeterminancy created by multiple fixed points $(v_1^*, \ldots, v_n^*)$ at step (d). Again, this is accomplished by having a second stage in which, after learning one another's bid schedules (and so, provided that condition (38) holds, obtaining complete information about one another's valuations), buyers choose among the fixed points

23. Of course, this threat would provide only a neutral incentive to be truthful to a buyer who (from the revelation of the others' bid functions) knows that he is not going to win (i.e., he would be indifferent between being truthful and untruthful). But, as in the case of eliminating multiple equilibria (see footnote 18), we could modify the auction somewhat to give all buyers a strict incentive to tell truth (at the cost of making the auction more complex). See also the discussion of dynamic auctions in Section IV.

satisfying (29) and (30). In particular, they can identify the "truthful" fixed point $(v_i(s'_i,s_{-i}),v_{-i}(s'_i,s_{-i}))$, where s'_i is sufficiently less than the true value s_i so that

$$v_i(s'_i,s_{-i}) = \max_{j\neq i} v_j(s'_i,s_{-i}).$$

What does it mean for buyer i to bid truthfully in this auction? By analogy with the two-buyer case, we shall say that buyer i's schedule $\hat{b}_i(\cdot)$ is *truthful* for signal value s_i if he sets $\hat{b}_i(\cdot) = b_i(\cdot)$, where

$$b_i : V_{-i} \longrightarrow \mathbb{R}_+$$

is such that

$$\text{(31)} \quad V_{-i} = \{v_{-i} | \text{there exists } s'_{-i} \in S_{-i} = \bigtimes_{j\neq i} [\underline{s}_j,\overline{s}_j] \text{ such that,}$$
$$\text{for all } j \neq i,\ v_j(s_i,s'_i) \geq v_j, \text{ with equality if } s'_j > \underline{s}_j\}$$

and, for all $v_{-i} \in V_{-i}$,

$$\text{(32)} \quad b_i(v_{-i}) = \{v_i | \text{there exists } s'_{-i} \in S_{-i} \text{ such that } v_i = v_i\,(s_i,s'_{-i})$$
$$\text{and, for all } j\neq i,\ v_j(s_i,s'_{-i}) \geq v_j,$$
$$\text{with equality if } s'_j > s_j\}.$$

In other words, buyer i sets the domain V_{-i} to consist of all possible valuation vectors v_{-i} that the other buyers could have, i.e., the valuation vectors that are consistent with *some* vector of signal values for those buyers (for completeness, V_{-i} also includes valuation values that are smaller than those consistent with any signal value). And $b_i(v_{-i})$ consists of those valuations for buyer i that are consistent with the others having valuations v_{-i}. It is easy to see that, if buyers are truthful in this sense, the true valuations $(v_1(s_1, \ldots , s_n), \ldots , v_n(s_1, \ldots , s_n))$ constitute a fixed point satisfying (27) and that, when buyer i is the winner, the payment $\max_{j\neq i} v_j(s'_i,s_{-i})$, where $v_i(s'_i,s_{-i}) = \max_{j\neq i} v_j(s'_i,s_{-i})$, satisfies (28) and (29). (Readers not interested in the issue of multiple fixed points may wish to skip to the statement of Proposition 2 at this point.)

If the auction moves to step (e), where buyers have to choose among different fixed points satisfying (27), we shall say that buyer i with signal s_i is *truthful* if, assuming that there exists a unique vector s'_{-i} such that, for all $j \neq i$ and all $v_{-j} \in \hat{V}_{-j}$,[24]

$$\text{(33)} \quad \hat{b}_j(v_{-j}) = \{v_j | \text{there exist } s''_{-j} \in S_{-j}, \text{ such that } v_j = v_j(s'_j, s''_{-j}) \text{ and, for all } k \neq j,\ v_k(s'_j, s''_{-j}) \geq v_k \text{ with equality if } s''_k \geq \underline{s}_k\},$$

then he chooses[25]

$$\text{(34)} \qquad v_i^\circ = v_i(s_i, s'_{-i}).$$

If there exist either no such vectors s'_{-i} or multiple such vectors, then buyer i randomizes uniformly over all fixed points satisfying (27).

If the auction moves to step (f), where, given that buyer i is the winner, each buyer $j \neq i$ has to choose among different vectors satisfying (29) and (30), we shall say that buyer j with signal s_j is *truthful* if, assuming that there exists a unique vector s'_{-j} such that for all $k \neq j,i$ and all $v_{-k} \in \hat{V}_{-k}$

$$\text{(35)} \quad \hat{b}_k(v_{-k}) = \{v_k | \text{ there exists } s''_{-k} \text{ such that } v_k = v_k(s'_k, s''_{-k}) \text{ and, for all } l \neq k,\ v_l(s'_k, s''_{-k}) \geq v_l \text{ with equality if } s''_l > \underline{s}_l\}$$

and

$$\text{(36)} \qquad v_i(s_j, s'_{-j}) \geq \max_{l \neq i} v_l(s_j, s'_j),$$

where (36) holds with equality if $s'_i > \underline{s}_i$, he chooses v_j^* such that

$$\text{(37)} \qquad v_j^* = v_j(s_j, s'_{-j}).$$

If there exists either no such vector s'_{-j} or multiple such vectors, then buyer j randomizes uniformly over all fixed points satisfying (29) and (30).

Steps (e) and (f) rely on the property that if buyer i bids truthfully, his bid function $b_i(\cdot)$ will be consistent with a unique signal value, namely, the true value s_i. In fact, this uniqueness property need not always hold.[26] Nevertheless, a fairly weak

24. Formula (33) implies that each buyer $j \neq i$ has bid as though his true signal were s'_j.

25. Notice that if buyer i is truthful and, for all $j \neq i$, $\hat{b}_j(\cdot)$ satisfies (33), then $(v_1(s_i, s'_{-i}), \ldots, v_n(s_i, s'_{-i}))$ is a fixed point satisfying (27).

26. For example, suppose that $v_1(s_1, s_2) = s_1 - 2s_2 + 5$ and $v_2(s_1, s_2) = s_2 - \frac{1}{2}s_1 + 5$. Then $b_1(v_2) = 15 - 2v_2$ *regardless* of the value of s_1.

condition, namely,

$$(38)\quad \det\begin{pmatrix} \frac{\partial v_1}{\partial s_1} & \cdots & \frac{\partial v_m}{\partial s_1} \\ \vdots & & \vdots \\ \frac{\partial v_1}{\partial s_m} & \cdots & \frac{\partial v_m}{\partial s_m} \end{pmatrix} \neq 0, \qquad \text{for all } m = 1, \ldots, n,$$

ensures that it *will* obtain. Notice that (38) automatically holds in the case of private values, since then only the main diagonal entries are nonzero.[27]

PROPOSITION 2. Assume that, for all $i = 1, \ldots, n$, buyer i's valuation function satisfies (1), (6), and (26) and that buyers' valuation functions collectively satisfy (38). Then, the n-buyer generalization of the Vickrey auction given by steps (a)–(f) above is *efficient*. Specifically, it is an equilibrium for each buyer i to bid truthfully, i.e., to set $\hat{b}_i(\cdot) = b_i(\cdot)$ satisfying (31) and (32) (and to choose, if need be, v_i° and v_i^* according to (34) and (37)). Moreover, if buyers are truthful, the auction results in an efficient allocation.

Proof. See the Appendix.

In the Appendix it is shown how the auction of Proposition 2 applies to the model of Example 3.

II.B. Multidimensional Signals

There are some circumstances in which it is natural to think of buyers' signals as being one-dimensional. The case of private values (Example 1) is one such instance.[28] Our "noisy signal" model (Example 2) is another. However, there are many other cases in which a buyer's information cannot be reduced to one dimension. Consider the following example.

Example 5. There are two wildcatters competing for the right to drill for oil on a tract of land consisting of an eastern and

27. Note that the example in the preceding footnote violates (38).

28. Even with private values, a buyer may receive many signals. But as long as his private signals are uncorrelated with those of other buyers, his *valuation* (which is one-dimensional) will be a sufficient statistic for all his information. Thus, the private-values case is inherently one-dimensional.

western region. Wildcatter 1 has a (fixed) cost of drilling c_1, which is *private* information. She also performs a private test that tells her that the expected quantity of oil in the eastern region is q_1. Similarly, wildcatter 2 has a private fixed cost c_2 and observes the expected quantity of oil, q_2, in the western region. All four signals c_1,c_2,q_1,q_2, are independently distributed. The price of oil is 1. Hence, wildcatter i's expected payoff conditional on all signals (and gross of any payment she must make) is

$$q_1 + q_2 - c_i. \tag{i}$$

Note that this model combines elements of pure private values (the signal c_i) with those of pure common values (the signal q_i). Note too, from (i), that wildcatter 1's information can be summarized, from her *own* standpoint, by the one-dimensional signal $t_1 = q_1 - c_1$. Indeed, for any Δ, her preferences are exactly the same when her signal values are (q'_1,c'_1) as when they are $(q'_1 + \Delta,c'_1 + \Delta)$. However, t_1 is *not* an adequate summary of 1's information from wildcatter 2's standpoint: 2 cares about q_1, but not about c_1 (so, in particular, he would bid more aggressively if he knew that signal values were $(q'_1 + \Delta,c'_1 + \Delta)$ rather than (q_1,c_1) for $\Delta > 0$). Hence, buyer 1's information cannot be summarized by a one-dimensional signal.

Furthermore, *no* auction can be fully efficient in this example. To show this, let us focus henceforth on equilibria having the property that if, regardless of other players' behavior, player i's decision problem is the same for two different signal values s'_i and s''_i, then his equilibrium behavior is the same for either value (call such equilibria *regular*). Consider c_1, c_2 and Δ such that

$$c_1 < c_2 < c_1 + \Delta. \tag{ii}$$

For full efficiency, (ii) implies that drilling rights should be awarded to wildcatter 1 if her costs are c_1 but not if they are $c_1 + \Delta$. However, her decision problem is the same for signal values (q_1,c_1) as for $(q_1 + \Delta,c_1 + \Delta)$, and so in any regular equilibrium she will behave the same way in either case. Thus, there is no way of devising an auction in which the outcome differs between (q_1,c_1) and $(q_1 + \Delta,c_1 + \Delta)$. That is, there is no way to attain full efficiency.

More generally, we can express the difficulty illustrated by Example 3 as follows (Maskin [1992] establishes a version of the following Proposition 3; see Jehiel and Moldovanu [1998] for a closely related impossibility result).

PROPOSITION 3. Suppose that the model of subsection II.A is generalized so that for at least one buyer i, s_i is multidimensional, i.e., $s_i = (s_i^1, \ldots, s_i^m) \in \mathbb{R}^m$ for $m > 1$. Suppose that s_i is distributed independently of s_{-i}. If there exist signal values s'_i, s''_i and s'_{-i} such that

$$v_i(s'_i,\cdot) = v_i(s''_i,\cdot), \tag{39}$$

but

$$\arg\max_j v_j(s'_i,s'_{-i}) \neq \arg\max_j v_j(s''_i,s'_{-i}), \tag{40}$$

then there is no efficient auction with regular equilibria.

Proof. In any regular equilibrium (39) and the fact that s_i is independent of s_{-i} imply that equilibrium play when the signal values are (s'_i,s'_{-i}) remains as equilibrium play when the signal values are (s''_i,s''_{-i}). But from (40) the winning buyer of an *efficient* auction cannot be the same for (s'_i,s'_{-i}) as for (s''_i,s''_{-i}). Hence, there is no efficient auction. QED

Notice that hypothesis (39) can readily be satisfied if, as in Example 5, buyer i's valuation is *separable* between his own signal and those of others. That is, there exist functions $\tau_i : \mathbb{R}^m \rightarrow \mathbb{R}$ and $\psi_i(\cdot)$ such that

$$v_i(s_i,s_{-i}) = \psi_i(\tau_i(s_i),s_{-i}) \qquad \text{for all } (s_i,s_{-i}). \tag{41}$$

Proposition 3 tells us that full efficiency is too much to expect in the multidimensional case.[29] Thus, all we can reasonably hope for is efficiency subject to the constraints of incentive-compatibility. It is straightforward to show, however, that our generalized Vickrey auction is efficient in this *constrained* sense, under fairly weak assumptions.

Let us assume that, for all i, s_i is multidimensional (i.e., $s_i \in S_i$, where $S_i \subseteq \mathbb{R}^m$ and $m > 1$) and that signals are independently distributed across buyers (the independence assumption can be

29. Jehiel and Moldovanu [1998] show that full efficiency may not be attainable even if, for each possible allocation of resources, only a one-dimensional component of each buyer's signal is pertinent to buyers' payoffs from that allocation. By contrast, in Example 5, both c_1 and q_1 are pertinent to wildcatter 1's payoff if she wins the drilling rights.

relaxed considerably; see footnote).[30] Also suppose that, for all i, buyer i's valuation function is separable in the sense of (41). For each vector $(t_1, \ldots, t_n)$ in the range of $(\tau_1(\cdot), \ldots, \tau_n(\cdot))$, define

$$(42) \quad w_i(t_1, \ldots, t_n) = E_{s_{-i}}[\psi_i(t_i, s_{-i}) | \tau_j(s_j) = t_j, \text{ for all } j \neq i].$$

Thus, $w_i(t_1, \ldots, t_n)$ is buyer i's expected valuation when his own signal value s_i is such that $\tau_i(s_i) = t_i$ (notice that, given t_i, buyer i is indifferent between all signal values s_i for which $\tau_i(s_i) = \tau_i$, and so he will behave in the same way for any of them) and each other buyer j's signal value s_j is such that $\tau_j(s_j) = t_j$ (independence implies that the expectation does not depend on s_i).

Because, in any regular equilibrium, each buyer i will behave the same way for all signal values leading to the same t_i, *constrained efficiency* means awarding the good to the buyer for whom $w_i(t_1, \ldots, t_n)$ is highest. But as long as the derived valuation functions $w_i(\cdot)$ satisfy conditions (1) and (6), all conclusions from the one-dimensional case go through.[30] We have

PROPOSITION 4. Suppose that, for all i, buyer i's signal s_i is multidimensional and that signals are independently distributed across buyers.[31] Assume that each buyer i's valuation function is separable in the sense of (41), and define the derived valuation function $w_i(t_i, \ldots, t_n)$ as in (42). Then, restricting to regular equilibria, the generalized Vickrey auction of Proposition 2 will be constrained efficient if, for all i, $w_i(\cdot)$ satisfies (1), (6), (26), and (38).

In the Appendix we show how Proposition 4 applies to the model of Example 5.

III. MULTIPLE GOODS

III.A. Formulation

The efficient auction of Propositions 2 and 4 can be extended to multiple goods through an appropriate generalization of the

30. Jehiel, Moldovanu, and Stacchetti [1996] give conditions under which one can reduce a multidimensional problem to one dimension for the purpose of revenue maximization.

31. Actually, independence is much stronger than necessary. All that is needed in that, for all t_i and $s_i', s_i'' \in S_i$ such that $\tau_i(s_i') = \tau_i(s_i'') = t_i$, the distribution over s_{-i} conditional on s_i' is the same as that conditional on s_i''. Notice that this weaker condition is automatically satisfied when buyers' signals are one-dimensional.

Clarke [1971]-Groves [1973] logic. Let us continue to assume that, for all $i = 1, \ldots, n$, buyer i observes a multidimensional signal $s_i \in S_i$, where S_i is a convex subset of $\mathbb{R}^m$. Assume too that there are l (indivisible) goods, indexed by $j = 1, \ldots, l$. For each $H \subseteq \{1, \ldots, l\}$, let $v_{i,H}(s_1, \ldots, s_n)$ denote buyer i's valuation for the set of goods H, conditional on the signals $(s_1, \ldots, s_n)$. Denote by $\{v_{i,H}\}_H$ the collection of buyer i's valuation functions for all possible sets of goods H.

As in the case of a single good, we assume separability of valuation functions and (conditional) independence of signals, which allows us in effect to reduce signal spaces to one dimension. Specifically, for all i and $k \neq i$, assume that there exist functions $\tau_i : S_i \to \mathbb{R}$, $\rho_k : S_i \to \mathbb{R}$, $\{\psi_{i,H}\}_H$ and $\{\Gamma_{k,H}\}_H$ such that, for all $H \subseteq \{1, \ldots l\}$,

$$v_{i,H}(s_i,s_{-i}) = \psi_{i,H}(\tau_i(s_i), s_{-i}) \qquad \text{for all } (s_i,s_{-i}). \tag{43a}$$

and

$$v_{k,H}(s_i,s_{-i}) = \Gamma_{k,H}(\rho_k(s_i),s_{-i}) \qquad \text{for all } (s_i,s_{-i}). \tag{43b}$$

Assume also that, for all t_i and all s'_i and s''_i such that $\tau_i(s'_i) = \tau_i(s''_i) = t_i$, the probability distribution over s_{-i} conditional on s'_i is the same as that conditional on s''_i (this is an assumption of *conditional independence*). Then, for all H, t_i and t_{-i},

$$w_{i,H}(t_i,t_{-i}) = E_{s_{-i}}[\psi_{i,H}(t,s_{-i}) | \tau_j(s_j) = t_j, \quad \text{for all } j \neq i] \tag{44}$$

represents buyer i's expected valuation for set H when his signal s_i is such that $\tau_i(s_i) = t_i$ and other buyers' signals s_{-i} are such that $\tau_j(s_j) = t_j$ for all $j \neq i$. Henceforth, we will work with the "summary" signals t_i and the "derived" valuation functions $w_{i,H}(t_1, \ldots, t_n)$ (where, for all j, $t_j \in T_j = [\underline{t}_j,\bar{t}_j]$), rather than with the primitive functions $v_{i,H}(s_1, \ldots, s_n)$.

Because the same function τ_i must satisfy (43a) for all combinations of goods H, our assumptions on preferences are considerably more restrictive in the multigood then in the one-good case.[32] Nevertheless, there are natural settings in which they are satisfied.[33]

32. See Theorem 6.5 in Jehiel and Moldovanu [1998].

33. Even when the separability assumption fails, there is a broad range of settings for which the generalized Vickrey auction is constrained efficient (see Example 8 below).

Example 6. Consider additive preferences (the case in which a buyer's utility for set of goods is the sum of his utilities for the individual goods). In this case, provided that a buyer's utility for a given good depends only on the signals specific to that good and that the signals for different goods are distributed independently, we can auction off each good separately, and so the problem becomes one-dimensional under the same assumptions as in the one-good case.

Example 7. Imagine that there are a number of electricity-generating plants available for auction to electricity-producing firms. Each firm i can produce q_i units of electricity from a given plant at fixed cost c_i, where q_i and c_i might depend on how many other plants the firm owns. Let us suppose that the q_i's and c_i's are common knowledge but that each firm i observes a private signal s_i that gives it information about the expected price p of electricity. Hence firm i's profit from a given plant is

$$p(s_1, \ldots, s_n)q_i - c_i.$$

If p is separable in the s_i's, then, once again, our separability assumption holds.

Let $(H_1, \ldots, H_n)$ be a partition of $\{1, \ldots, l\}$ (i.e., $\cup_{i=1}^{n} H_i = \{1, \ldots, l\}$ and, for all $i \neq j$, $H_i \cap H_j = \emptyset$). We will call such a partition an *allocation* of goods across buyers, where, for all i, H_i corresponds to the set of goods allocated to buyer i. We will call an allocation constrained efficient with respect to $(t_1, \ldots, t_n)$ if it solves

$$\max_{(H_1, \ldots, H_n)} \sum_{i=1}^{n} w_{i,H_i}(t_1, \ldots, t_n)$$

(efficiency is constrained since the maximization is conditional on $(t_1, \ldots, t_n)$ rather than on the finer information $(s_1, \ldots, s_n)$). Finally, an auction is *constrained efficient* if, for all $(t_1, \ldots, t_n)$, the corresponding equilibrium allocation is constrained efficient, and a buyer's allocation of goods does not depend on which best response he chooses against the other buyers' equilibrium strategies.

The counterpart to condition (1) in our multigood framework is[34]

$$
(45)\qquad \begin{cases} \text{for all } i \text{ and } (t_1, \ldots, t_n) \text{ if } H \text{ and } H' \\ \quad \text{are two sets of goods such that} \\ w_{i,H}(t_1, \ldots, t_n) - w_{i,H'}(t_1, \ldots, t_n) > 0 \\ \text{then} \\ \dfrac{\partial}{\partial t_i}(w_{i,H}(t_1, \ldots, t_n) - w_{i,H'}(t_1, \ldots, t_n)) > 0, \end{cases}
$$

i.e., if buyer i prefers H to H', then an increase in t_i makes his preference even stronger. The generalization of (6) is

$$
(46)\qquad \begin{cases} \text{if, for buyer } i, \text{ signal } (t_1, \ldots, t_n), \\ \quad \text{and allocations } (H_1, \ldots, H_n), (H'_1, \ldots, H'_n), \\ \quad w_{i,H_i}(t_1, \ldots, t_n) > w_{i,H'_i}(t_1, \ldots, t_n) \\ \text{and} \\ \quad \displaystyle\sum_{j=1}^{n} w_{j,H_j}(t_1, \ldots, t_n) = \sum_{j=1}^{n} w_{j,H'_j}(t_1, \ldots, t_n) \\ \quad \displaystyle = \max_{\{\hat{H}_j\}} \sum_{j=1}^{n} w_{j,\hat{H}_j}(t_1, \ldots, t_n) \\ \text{then} \\ \quad \displaystyle\frac{\partial}{\partial t_i}\sum_{j=1}^{n} w_{j,H_j}(t_1, \ldots, t_n) > \frac{\partial}{\partial t_i}\sum_{j=1}^{n} w_{j,H'_j}(t_1, \ldots, t_n). \end{cases}
$$

Condition (46) says that, for any two allocations, buyer i's "summary" signal t_i has a greater marginal effect on the total surplus of the allocation that he prefers (at any point where the total surpluses are equal and maximal). Note that the last

34. For efficiency, we must compare buyer i's valuation for H with that for H'. In the one-good case, this amounts to comparing $v_i(s_1, \ldots, s_n)$ with 0, which is why (1) and (6) (perhaps misleadingly) appear not to be expressed in terms of differences of valuations.

inequality in (46) can be rewritten as

$$\text{(47)}\quad \frac{\partial}{\partial t_i}(w_{i,H_i}(t_i,t_{-i}) - w_{i,H'_i}(t_i,t_{-i})) > \frac{\partial}{\partial t_i}\sum_{j\neq i}(w_{j,H'_j}(t_i,t_{-i}) - w_{j,H_j}(t_i,t_{-i})).$$

The counterpart of (26) is the requirement that

$$\text{(48a)}\quad \begin{cases} \text{for all } i,\ (t_i,t_{-i}),\ (H_1,\ldots,H_n) \text{ and } (H'_1,\ldots,H'_n), \\ \text{if } w_{i,H_i}(t_i,t_{-i}) - w_{i,H'_i}(t_i,t_{-i}) > 0 \\ \quad \text{then there exists } t'_i > t_i \text{ such that} \\ w_{i,H_i}(t'_i,t_{-i}) - w_{i,H'_i}(t'_i,t_{-i}) \\ \qquad > \sum_{j\neq i}(w_{j,H'_j}(t'_i,t_{-i}) - w_{j,H_j}(t'_i,t_{-i})), \end{cases}$$

i.e., if buyer i prefers H_i to H'_i then, for a high enough value of t_i, $(H_1,\ldots,H_n)$ is socially preferred to $(H'_1,\ldots,H'_n)$.

Finally, for the case of three or more buyers, the counterpart to (38) is the condition there exist $H_1,\ldots,H_n$ such that

$$\text{(48b)}\qquad \det\begin{pmatrix} \frac{\partial w_{1,H_1}}{\partial t_1} & \cdots & \frac{\partial w_{m,H_m}}{\partial t_1} \\ \vdots & & \vdots \\ \frac{\partial w_{1,H_1}}{\partial t_m} & \cdots & \frac{\partial w_{m,H_m}}{\partial t_m} \end{pmatrix} \neq 0, \qquad \text{for all } m = 1,\ldots,n.$$

We will exhibit a generalized Vickrey-Clarke-Groves auction that is constrained efficient, as long as the derived valuation functions satisfy (45), (46), (48a), and (48b). As in Section II we first consider the case of two buyers (subsection III.B). The case of more than two buyers then follows (subsection III.C).

III.B. Two Buyers

In this case buyer i's ($i = 1,2$) preferences can be represented by the derived valuations functions for each $H \subseteq \{1,\ldots,l\}$ and an allocation is a pair (H_1,H_2). To extend the auction of Proposition 2 to this setting, consider the auction in which

(A) for each subset of goods $H \subseteq \{1, \ldots, l\}$, buyer 1 reports bid function,

$$\hat{b}_{1,H} : \hat{W}_2 \to \mathbb{R}_+,$$

where, for each $\{w_{2,H'}\}_{H'}$ (the set $\{w_{2,H'}\}_{H'}$ specifies buyer 2's valuation $w_{2,H'}$ for each possible subset of goods H'), $\hat{b}_{1,H}(\{w_{2,H'}\}_{H'})$ is buyer 1's corresponding bid for subset H, and $\hat{W}_2$ is a subset of $\mathbb{R}^{2^l-1}$; similarly buyer 2 reports $\hat{b}_{2,H}(\cdot)$ for each $H \subseteq \{1, \ldots, l\}$;

(B) a fixed point $(\{w^{\circ}_{1,H'}\},\{w^{\circ}_{2,H'}\}_{H'})$ is calculated: for all H,

$$w^{\circ}_{1,H} = \hat{b}_{1,H}(\{w^{\circ}_{2,H'}\}_{H'})$$

and

$$w^{\circ}_{2,H} = \hat{b}_{2,H}(\{w^{\circ}_{1,H'}\}_{H'})$$

(if no fixed point exists then no goods are allocated; if there are multiple fixed points, then buyers choose among them as in stage (e) of the auction of Proposition 2);

(C) consider the allocation $(H^{\circ}_1,H^{\circ}_2)$ that solves

$$\max_{(H_1,H_2)} (w^{\circ}_{1,H_1} + w^{\circ}_{2,H_2}) \tag{49}$$

(thus, $(H^{\circ}_1,H^{\circ}_2)$ is the allocation that maximizes *apparent surplus*—the surplus that would result if $\{(w^{\circ}_{1,H'},w^{\circ}_{2,H'})\}_{H'}$ were the true valuations); let H°_i be the set of goods assigned to buyer i;

(D) if buyer 1 is assigned H°_1, consider a sequence $[\{w^1_{1,H'}\}_{H'}, (H^1_1,H^1_2)], \ldots, [\{w^R_{1,H'}\}_{H'},(H^R_1,H^R_2)]$, where $\{w^r_{1,H'}\}_{H'} \in \hat{W}_1$, $r = 1, \ldots, R$, such that

$$(H^r_1,H^r_2) \text{ solves } \max_{(H_1,H_2)} (w^r_{1,H_1} + \hat{b}_{2,H_2}(\{w^r_{1,H'}\}_{\hat{H}'}), \quad r = 1, \ldots, R \tag{50}$$

$$w^{\circ}_{1,H^{\circ}_1} > w^1_{1,H^{\circ}_1} \quad \text{and} \quad w^{r-1}_{1,H^{r-1}_1} > w^r_{1,H^{r-1}_1}, \qquad r = 2, \ldots, R, \tag{51}$$

$$w^1_{1,H^{\circ}_1} + \hat{b}_{2,H^{\circ}_2}(\{w^1_{1,H'}\}_{H'}) = w^1_{1,H^1_1} + \hat{b}_{2,H^1_2}(\{w^1_{1,H'}\}_{H'}) \tag{52}$$

and, for all $r = 2, \ldots, R$,

$$w^r_{1,H^{r-1}_1} + \hat{b}_{2,H^{r-1}_2}(\{w^r_{1,H'}\}_{H'}) = w^r_{1,H^r_1} + \hat{b}_{2,H^r_2}(\{w^r_{1,H'}\}_{H'})$$

and either

$$H^R_1 = \varnothing \tag{53}$$

or

(54) for all $\{w_{1,H'}\}_{H'} \in \hat{W}_1$ and all H, $w_{1,H} \geq w^R_{1,H}$ and $H^R_1 = H_1{}^{R-1}$;

then buyer 1 pays

$$(55)\quad \sum_{r=1}^{R} (\hat{b}_{2,H^r_2}(\{w^r_{1,H'}\}_{H'}) - \hat{b}_{2,H^{r-1}_2}(\{w^r_{1,H'}\}_{H'}) + \hat{b}_{2,\{1,\ldots,I\}}(\{w^R_{1,H'}\}_{H'}) - \hat{b}_{2,H^R_2}(\{w^R_{1,H'}\}_{H'})$$

(in equilibrium there can be multiple sequences satisfying (50)–(54); if so, then buyer 2 chooses among them as in stage (f) of the auction of Proposition 2); buyer 2's payment is determined symmetrically.

To informally understand the rationale for buyer 1's payment, think of starting at $\{w_{1,H'}\}_{H'} = \{w^\circ_{1,H'}\}_{H'}$ and reducing $\{w_{1,H'}\}_{H'}$ continuously so that it remains in $\hat{W}_1$ (in equilibrium there will be a unique way that this can be done—see (58)—but if there are multiple ways, choose one arbitrarily). Let $\{w_{1,H'}\}_{H'} = \{w^1_{1,H'}\}_{H'}$ be the first point at which the surplus-maximizing allocation switches from (H°_1, H°_2) to some other allocation (H^1_1, H^1_2) (in the one-good case this is the point v^*_1, where the good is reallocated from buyer 1 to buyer 2). Now, according to the Clarke [1971]-Groves [1973] mechanism, buyer 1 should pay his marginal impact on buyer 2. At $\{w_{1,H'}\}_{H'} = \{w^1_{1,H'}\}_{H'}$, buyer 2 is just on the verge of being allocated H^1_2 rather than H°_2. Hence, the marginal impact on him (as measured by his own bid function) of raising $\{w_{1,H'}\}_{H'}$ from just below to just above $\{w^1_{1,H'}\}_{H'}$ is

$$(56)\quad \hat{b}_{2,H^1_2}(\{w^1_{1,H'}\}_{H'}) - \hat{b}_{2,H^\circ_2}(\{w^1_{1,H'}\}_{H'})$$

(in the one-good case, (56) reduces to $\hat{b}_2(v^*_1) - \hat{b}_2(\emptyset) = \hat{b}_2(v^*_1)$). And so, buyer 1 should pay (56) for this marginal impact. Continuing to reduce the valuations $\{w_{1,H'}\}_{H'}$, we find that, for each $r = 2, \ldots, R$, $\{w_{1,H'}\}_{H'} = \{w^r_{1,H'}\}_{H'}$ is the point at which the surplus-maximizing allocation switches from (H^{r-1}_1, H^{r-1}_2) to (H^r_1, H^r_2). The reductions continue until ultimately buyer 1 is either allocated no goods (i.e., (53) holds), or we reach the lower end-point of $\hat{W}_1$ (i.e., (54) holds). At $\{w_{1,H'}\}_{H'} = \{w^r_{1,H'}\}_{H'}$, buyer 1's marginal impact on buyer 2 is

$$(57)\quad \hat{b}_{2,H^r_2}(\{w^r_{1,H'}\}_{H'}) - \hat{b}_{2,H^{r-1}_2}(\{w^r_{1,H'}\}_{H'}).$$

Summing (57) (and adding one more term, $\hat{b}_{2,\{1,\ldots,I\}}(\{w^R_{1,H'}\}_{H'})$ −

$\hat{b}_{2,H_2^R}(\{w^R_{1,H'}\}_{H'})$, if (53) does not hold), we find that buyer 1's total marginal impact on buyer 2 is given by (55). Notice that in the case of private values—where buyer 2's bid function does not depend on $\{w_{1,H'}\}_{H'}$—(55) collapses to

$$\hat{b}_{2,\{1,\ldots,l\}} - \hat{b}_{2,H_2^\circ},$$

which is the standard Clarke-Groves payment.

We will show that, in the auction given by (A)–(D) (strictly speaking, there should be extra steps to deal with multiple fixed points in step (B) and multiple sequences in step (D), but we shall skip over these), it is an equilibrium for buyers to bid truthfully and that such behavior results in an efficient allocation. Truthful bidding by buyer 1 with summary signal t_1 entails setting

$$(58) \qquad \hat{W}_2 = W_2 = \{\{w_{2,H'}\}_{H'} \mid \text{there exists } t'_2 \text{ such that } w_{2,H'} = w_{2,H'}(t_1,t'_2) \text{ for all } H'\}$$

and

$$\hat{b}_{1,H}(\cdot) = b_{1,H}(\cdot),$$

such that, for all $H \subseteq \{1,\ldots,l\}$ and $\{w_{2,H'}\}_{H'} \in W_2$,

$$(59) \qquad b_{1,H}(\{w_{2,H'}\}_{H'}) = w_{1,H}(t_1,t'_2),$$

where t'_2 is such that $w_{2,H'} = w_{2,H'}(t_1,t'_2)$ for all H'. Formula (59) says that, given $\{w_{2,H'}\}_{H'}$, buyer 1 bids his true valuation $w_{1,H}(t_1,t'_2)$ for the subset of goods H, where t'_2 is the signal value consistent with the valuations $\{w_{2,H'}\}_{H'}$.

PROPOSITION 5. Assume that the derived valuation functions satisfy (45)–(48b). The two-buyer auction defined by (A)–(D) is constrained efficient. That is, truthful bidding (as defined by (58) and (59)) constitutes an equilibrium and results in a constrained efficient allocation. Moreover, unilateral deviation by any buyer to a nontruthful best response does not change his allocation.[35]

Proof. See Appendix.

In the Appendix we gave an explicit illustration of Proposition 5.

35. Remarks 1 and 2 from Proposition 1 carry over to Proposition 5. In particular, buyer 2 need bid "correctly" only for the values $\{w^r_{1,H'}\}_{H'}$.

III.C. Many Buyers

As we observed in Section II, the major difference between this and the two-buyer case is that we must now allow buyers to submit bid correspondences. Consider the following auction defined in four steps:

(a) for each nonempty set $H \subseteq \{1, \ldots, l\}$, each buyer i ($i = 1, \ldots, n$) submits a bid correspondence

$$\hat{b}_{i,H} : \hat{W}_{-i} \longrightarrow\!\!\!\!\rightarrow \mathbb{R}_+,$$

where $\{w_{-i,H'}\}_{H'}$ is a typical element of $\hat{W}_{-i}(\{w_{-i,H'}\}_{H'}$ specifies the other buyers' valuations for each possible set of goods H')

(b) a fixed point $\{(w^{\circ}_{1,H'}, \ldots, w^{\circ}_{n,H'})\}_{H'}$ is calculated so that

$$w^{\circ}_{i,H} \in \hat{b}_{i,H}(\{w^{\circ}_{-i,H'}\}_{H'}) \qquad \text{for all } i \text{ and } H$$

(if there is no fixed point, the good is not allocated, and no buyer makes a payment; if there are multiple fixed points, proceed as in step (e) of the auction of Proposition 2);

(c) choose the allocation $(H^{\circ}_1, \ldots, H^{\circ}_n)$ that solves

$$\max_{(H_1, \ldots, H_n)} \sum_{i=1}^{n} w^{\circ}_{i,H_i};$$

(d) to calculate the payment that buyer i makes for H°_i, consider a sequence

$$[\{(w^1_{1,H'}, \ldots, w^1_{n,H'})\}_{H'}, (H^1_1, \ldots, H^1_n)], \ldots,$$
$$[\{(w^R_{1,H'}, \ldots, w^R_{n,H'}\}_{H'}, (H^R_1, \ldots, H^R_n)],$$
$$\text{where } \{w^r_{-j,H'}\}_{H'} \in \hat{W}_{-j} \qquad \text{for all } j \text{ and } r = 1, \ldots, R,$$

such that

$$(60) \qquad w^r_{j,H} \in \hat{b}_{j,H}(\{w^r_{-j,H'}\}_{H'}) \qquad \text{for all } j \neq i, H, \text{ and } r = 1, \ldots, R,$$

$$(61) \qquad (H^r_1, \ldots, H^r_n) \text{ solves } \max_{(H_1, \ldots, H_n)} \sum_{j=1}^{n} w^r_{j,H_j}, \qquad r = 1, \ldots, R,$$

$$(62) \qquad w^{r-1}_{i,H^{r-1}_i} > w^r_{i,H^{r-1}_i}, \qquad r = 2, \ldots, R,$$

$$(63) \qquad \sum_{k=1}^{n} w^r_{k,H^{r-1}_k} = \sum_{k=1}^{n} w^r_{k,H^r_k}, \qquad r = 2, \ldots, R,$$

and either

$$H_i^R = \varnothing, \tag{64}$$

or

$$\text{for all } j \neq \text{i, all } \{w_{-j,H'}\}_{H'} \in \hat{W}_{-j} \text{ and all } H, \quad w_{i,H} \geq w_{i,H}^R \text{ and } H_i^R = H_i^{R-1}; \tag{65}$$

then buyer i pays

$$\sum_{r=1}^{R} \sum_{j \neq \text{i}} (w_{j,H_j^r}^r - w_{j,H_j^{r-1}}^r) + \sum_{j \neq \text{i}} (w_{j,H_j^{R+1}}^R - w_{j,H_j^R}^R), \tag{66}$$

where H_{-i}^{R+1} solves $\max_{H_{-i}} \Sigma_{j \neq i} w_{j,H_j}^R$ (if there is no sequence satisfying (60)–(65), then no goods are allocated, and no buyer makes a payment; if there are multiple sequences, then we proceed as in step (f) of auction of Proposition 2).

We claim that, in the auction given by (a)–(d), it is an equilibrium for buyers to bid truthfully and that such behavior results in an efficient allocation, provided that (45)–(48b) hold. Again, we must first make precise what constitutes truthful bidding.

For all i and H, we shall say that buyer i's bid correspondence

$$\hat{b}_{i,H} : \hat{W}_{-i} \longrightarrow \mathbb{R}_+$$

is *truthful* for summary signal t_i if $\hat{W}_{-i} = W_{-i}$, where

$$W_{-i} = \{\{w_{-i,H'}\}_{H'} \mid \text{there exists } t'_{-i} \in T_{-i} = \times_{j \neq i} [\underline{t}_j, \bar{t}_j] \text{ such that for all } j \neq i \text{ and } H', w_{j,H'}(t_i, t'_{-i}) = w_{j,H'}\}, \tag{67}$$

and $\hat{b}_{i,H}(\cdot) = b_{i,H}(\cdot)$, where, for all $\{w_{-i,H'}\}_{H'} \in W_{-i}$ and H,

$$b_{i,H}(\{w_{-i,H'}\}_{H'}) = \{w_{i,H}(t_i, t'_{-i}) \mid \text{there exists } t'_{-i} \text{ such that, for all } j \neq i \text{ and } H', w_{j,H'}(t_i, t'_{-i}) = w_{j,H'}\}. \tag{68}$$

If there are multiple fixed points in step (b)[36]—so that the auction moves to an additional stage as in step (e) of Proposition 2—buyer i with signal value t_i is *truthful* if, assuming that there exists a

36. Readers not interested in the issue of multiple fixed points can skip directly to the statement of Proposition 6.

unique vector t'_{-i} such that, for all $j \neq i$, all H, and all $\{w_{-j,H'}\}_{H'} \in \hat{W}_{-j}$,

$$\hat{b}_{j,H}(\{w_{-j,H'}\}_{H'}) = \{w_{j,H} \mid \text{there exists } t''_{-j} \text{ such that } w_{j,H} = w_{j,H}\,(t'_j,t''_{-j}) \text{ and, for all } H',\ w_{-j,H'}(t'_j,t''_{-j}) = w_{-j,H'}\},$$

then he chooses

$$\{w^{\circ}_{i,H'}\}_{H'} = \{w_{i,H'}(t_i,t'_{-i})\}.$$

If there exists either no such vector t'_{-i} or multiple such vectors, then buyer i randomizes uniformly over all fixed points in step (b).

If, in determining buyer i's payment, there are multiple sequences satisfying (60)–(65)—so that the auction moves to an additional stage as in step (f) of Proposition 2—buyer j with signal t_j is truthful if, assuming that there exists a unique vector t'_{-j} such that for all $k \neq j,i$, all H, and all $\{w_{-k,H'}\}_{H'} \in \hat{W}_{-k}$

$$\hat{b}_{k,H}(\{w_{-k,H'}\}_{H'} = \{w_{k,H} \mid \text{there exists } t''_{-k} \text{ such that } w_{k,H} = w_k(t'_k,t''_{-k}) \text{ and, for all } H',\ w_{-k,H'} = w_{-k,H'}(t'_k,t''_k)\},$$

he chooses the sequence $[\{(w^1_{1,H'}, \ldots, w^1_{n,H'})\}_{H'},(H^1_1, \ldots, H^1_n)], \ldots, [\{(w^R_{1,H'}, \ldots, w^R_{n,H'})\}_{H'},(H^R_1, \ldots, H^R_n)]$ satisfying (60)–(65) and such that there exist $t^1_j > t^2_j > \cdots > t^R_j$ for which $w^r_{j,H} = w_{j,H}(t^r_j,t'_{-j})$, $r = 1, \ldots, R$. If either there exists no such vector t'_{-j} or there exist multiple such vectors, then buyer j randomizes uniformly over all sequences satisfying (60)–(65).

PROPOSITION 6. Assume that, for all i and H, the derived valuation functions $w_{i,H}(\cdot)$ satisfy (45)–(48b). The generalized Vickrey-Clarke-Groves auction defined by (a)–(d) is constrained efficient.

The proof of Proposition 6 is essentially the same as that of Proposition 5, and so we omit it.

IV. AN OPEN QUESTION

We have provided conditions under which a generalized Vickrey auction is constrained efficient. Vickrey auctions for multiple goods are sometimes criticized as demanding too much information of a buyer: he is asked to submit a bid for each possible combination of goods, i.e., $2^l - 1$ bids in all. Furthermore, in our common-values setting, these bids must be made contingent on all other buyers' valuations.

In our view, these criticisms are overblown. A buyer could be permitted to submit bids only on those combinations of goods he is potentially interested in (with default values of zero, say, being assigned to all other combinations). Furthermore, he could choose to make his bids contingent only on those other buyers' valuations that, he believes, share a significant common component with his own valuation.

Nevertheless, there are at least two important advantages that a (suitably formulated) English auction could have over a generalized Vickrey auction. First, at any instant, a buyer in an English auction need make only a binary decision: whether or not to drop out. In this sense, it is markedly simpler than our generalized Vickrey auction, in which the buyer must submit a fully contingent bid in advance.

To appreciate the second advantage, recall Example 3, in which for signal values (s_1,s_2,s_3) in the neighborhood (1,1,1), it is not efficient for buyer 3 to win and yet his signal value is pertinent in determining which of buyer 1 or 2 should win. In the English auction, buyer 3's true signal value can be inferred even though he does not win, because he will drop out when the price reaches s_3. In the generalized Vickrey auction, by contrast, buyer 3 must truthfully bid $b_3(v_1,v_2)$ in order for s_3 to be revealed. Although truthful bidding is optimal, buyer 3 has only a neutral incentive to do so if $b_3(v_1,v_2)$ is less than v_1 and v_2 (as in footnote 23, we could provide a strict incentive but only at the cost of a more elaborate set of rules).[37]

Thus, on both counts, we regard finding an appropriate "English" auction (i.e., a dynamic auction with binary decisions at each instant) counterpart to our Vickrey auction with multiple goods as a leading topic for further research.

Appendix

Proof of Proposition 2. Suppose that all buyers other than buyer i are truthful. We must first show that buyer i finds it optimal to be truthful as well. Suppose that buyers' signals are $(s_1, \ldots, s_n)$. If buyer i wins the auction, then his payment, from

37. By contrast, if buyer 3 receives a multidimensional signal, there will be *no* way to give him a strict incentive to distinguish between two signal values for which his preferences are identical; see Example 5. That is one reason why full efficiency is, in general, impossible with multidimensional signals.

(28)–(30), is

$$\max_{j \neq i} v_j^*, \tag{A.1}$$

where $(v_1^*, \ldots, v_n^*)$ is a vector such that

$$v_i^* = \max_{j \neq i} v_j^* \tag{A.2}$$

and

$$v_k^* \in b_k(v_{-k}^*) \qquad \text{for all } k \neq i, \tag{A.3}$$

and where $b_k(\cdot)$ is given by (32). Now, from (1) and (26), there exists a unique vector $(\hat{v}_1^*, \ldots, \hat{v}_n^*)$ and signal value $s'_i \in S_i$ such that

$$\hat{v}_i^* = \max_{j \neq i} \hat{v}_j^*, \tag{A.4}$$

$$\hat{v}_k^* = v_k(s'_i, s_{-i}) \qquad \text{for all } k \neq i, \tag{A.5}$$

and

$$\hat{v}_i^* \leq v_i(s'_i, s_{-i}), \qquad \text{with equality if } s'_i > \underline{s}_i. \tag{A.6}$$

From (31), (32), (A.5), and (A.6), $\hat{v}_{-j}^* \in V_{-j}$ for all j and

$$\hat{v}_k^* \in b_k(\hat{v}_{-k}^*) \qquad \text{for all } k \neq i. \tag{A.7}$$

Formulas (A.4) and (A.7) imply that the vector $(\hat{v}_1^*, \ldots, \hat{v}_n^*)$ satisfies (A.2) and (A.3). Let us suppose for now that $(\hat{v}_1^*, \ldots, \hat{v}_n^*)$ is the unique such vector (we shall consider the case of multiple vectors satisfying (A.2) and (A.3) below). Then, from (A.1) and (A.2), player i's net payoff from winning is

$$v_i(s_i, s_{-i}) - \hat{v}_i^*. \tag{A.8}$$

It suffices, therefore, to show that, if buyer i is truthful, he wins the auction if and only if (A.8) is positive.

Now, from (A.6), (A.8) is positive if and only if either

$$v_i(s_i, s_{-i}) > v_i(s'_i, s_{-i}) \tag{A.9a}$$

or

$$s_i = s'_i \qquad \text{and} \qquad v_i(s'_i, s_{-i}) > \hat{v}_i^*. \tag{A.9b}$$

But, from (1), (A.9a) holds if and only if

$$s_i > s'_i. \tag{A.10}$$

Now from (A.4)–(A.6)

$$v_i(s'_i,s_{-i}) \geq \max_{j \neq i} v_j(s'_i,s_{-i}). \tag{A.11}$$

Hence, from (6) and (A.11), (A.10) holds if and only if

$$v_i(s_i,s_{-i}) > \max_{j \neq i} v_j(s_i,s_{-i}). \tag{A.12}$$

But from (31) and (32)

$$v_j(s_i,s_{-i}) \in b_j(v_{-j}(s_i,s_{-i})) \qquad \text{for all } j.$$

That is, $(v_1(s_i,s_{-i}), \ldots, v_n(s_i,s_{-i}))$ is a fixed point of the correspondence

$$(v_1, \ldots, v_n) \mapsto (b_1(v_{-1}), \ldots, b_n(v_{-n})). \tag{A.13}$$

Suppose for the moment that it is the *unique* fixed point. Then, if buyer i is truthful, the rules of the auction ((A.9a) and (A.9b)), (A.11), and (A.12) ensure that buyer i wins if and only if (A.8) is positive, as required. Finally, observe that (A.12) ensures that the allocation resulting from truthful bidding is efficient.

If there are multiple fixed points of (A.13), then the auction has an additional step (e) in which the players choose among these. We claim that, from (38), the buyers can infer one another's signal values from the bid functions. To see this, suppose for simplicity that $n = 3$. Let us focus on buyer 3. If another buyer cannot not infer the value of s_3 from $b_3(\cdot)$, then there must exist a pair of argument values (v_1,v_2) for which the corresponding truthful bid by buyer 3 is the same for two different values of s_3. That is, there must exist (s'_1,s'_2,s'_3) and (s''_1,s''_2,s''_3) such that

$$\begin{aligned} v_1 &= v_1(s'_1,s'_2,s'_3) = v_1(s''_1,s''_2,s''_3) \\ v_2 &= v_2(s'_1,s'_2,s'_3) = v_2(s''_1, s''_2,s''_3) \\ v_3&(s'_1,s'_2,s'_3) = v_3(s''_1,s''_2,s''_3). \end{aligned} \tag{A.14}$$

For each $s_3 \in [s'_3, s''_3]$ consider the pair of differential equations

$$\frac{\partial v_1}{\partial s_1}(s_1(s_3),s_2(s_3),s_3)\frac{ds_1}{ds_3}(s_3) + \frac{\partial v_1}{\partial s_2}(s_1(s_3),s_2(s_3),s_3)\frac{ds_2}{ds_3}(s_3) + \frac{\partial v_1}{\partial s_3}(s_1(s_3),s_2(s_3),s_3) = 0,$$

(A.15)

$$\frac{\partial v_2}{\partial s_1}(s_1(s_3),s_2(s_3),s_3)\frac{ds_1}{ds_3}(s_3) + \frac{\partial v_2}{\partial s_2}(s_1(s_3),s_2(s_3),s_3)\frac{ds_2}{ds_3}(s_3) + \frac{\partial v_2}{\partial s_3}(s_1(s_3),s_2(s_3),s_3) = 0,$$

with boundary conditions $s_1(s'_3) = s'_1$ and $s_2(s'_3) = s'_2$. From (38), we can solve uniquely for ds_1/ds_3 and ds_2/ds_3. Hence, there exist locally unique functions functions $s_1(\cdot)$ and $s_2(\cdot)$ such that, for all $s_3 \in [s'_3, s''_3]$,

$$v_1(s_1(s_3),s_2(s_3),s_3) = v_1(s'_1,s'_2,s'_3)$$

(A.16) and

$$v_2(s_1(s_3),s_2(s_3),s_3) = v_2(s'_1,s'_2,s'_3).$$

Suppose first that

(A.17) $$(s_1(s''_3),s_2(s''_3)) = (s''_1,s''_2).$$

Then from (A.14) and (A.17), we have

(A.18)

$$\begin{aligned} 0 &= v_3(s''_1,s''_2,s''_3) - v_3(s'_1,s'_2,s'_3) \\ &= (s''_3 - s'_3)\left[\frac{\partial v_3}{\partial s_1}(s_1(\hat{s}_3),s_2(\hat{s}_3),\hat{s}_3)\frac{ds_1}{ds_3}(\hat{s}_3)\right. \\ &\quad + \frac{\partial v_3}{\partial v_2}(s_1(\hat{s}_3),s_2(\hat{s}_3),\hat{s}_3)\frac{ds_2}{ds_3}(\hat{s}_3) \\ &\quad \left. + \frac{\partial v_3}{\partial s_3}(s_1(\hat{s}_3),s_2(\hat{s}_3),\hat{s}_3)\right], \end{aligned}$$

for some $\hat{s}_3 \in [s'_3, s''_3]$. From (38) and (A.15) we can rewrite the right-hand side of (A.18) as

$$(s_3'' - s'_3)\left[\frac{\partial v_3}{\partial s_1}\left(-\frac{\partial v_1}{\partial s_3}\frac{\partial v_2}{\partial s_2} + \frac{\partial v_1}{\partial s_2}\frac{\partial v_2}{\partial s_3}\right) + \frac{\partial v_3}{\partial s_2}\left(-\frac{\partial v_1}{\partial s_1}\frac{\partial v_2}{\partial s_3} + \frac{\partial v_2}{\partial s_1}\frac{\partial v_1}{\partial s_3}\right) + \frac{\partial v_3}{\partial s_3}\left(\frac{\partial v_1}{\partial s_1}\frac{\partial v_2}{\partial s_2} - \frac{\partial v_1}{\partial s_2}\frac{\partial v_2}{\partial s_1}\right)\right] \Big/ \left(\frac{\partial v_1}{\partial s_1}\frac{\partial v_2}{\partial s_2} - \frac{\partial v_1}{\partial s_2}\frac{\partial v_2}{\partial s_1}\right), \tag{A.19}$$

where all derivatives are evaluated at $s_3 = \hat{s}_3$. But the expression in (A.19) in square brackets is

$$\det\begin{pmatrix} \frac{\partial v_1}{\partial s_1} & \frac{\partial v_2}{\partial s_1} & \frac{\partial v_3}{\partial s_1} \\ \frac{\partial v_1}{\partial s_2} & \frac{\partial v_2}{\partial s_2} & \frac{\partial v_3}{\partial s_2} \\ \frac{\partial v_1}{\partial s_3} & \frac{\partial v_2}{\partial s_3} & \frac{\partial v_3}{\partial s_3} \end{pmatrix},$$

whereas the denominator is

$$\det\begin{pmatrix} \frac{\partial v_1}{\partial s_1} & \frac{\partial v_2}{\partial s_1} \\ \frac{\partial v_1}{\partial s_2} & \frac{\partial v_2}{\partial s_2} \end{pmatrix}.$$

Hence, from (38), (A.19) is nonzero, which contradicts (A.18). Suppose therefore that

$$(s_1(s_3''), s_2(s_3'')) = (\hat{s}_1'', \hat{s}_2'') \neq (s_1'', s_2''). \tag{A.20}$$

For $s_2 \in [s_2'', \hat{s}_2'']$ consider the differential equation

$$\frac{\partial v_1}{\partial s_1}(\hat{s}_1(s_2), s_2, s_3'')\frac{d\hat{s}_1}{ds_2}(s_2) + \frac{\partial v_1}{\partial s_2}(\hat{s}_1(s_2), s_2, s_3'') = 0 \tag{A.21}$$

with boundary condition $\hat{s}_1(s_2'') = s_1''$. From (1) there exists a unique local solution $\hat{s}_1(s_2)$:

$$v_1(\hat{s}_1(s_2),s_2,s''_3) = v_1(s''_1,s''_2,s''_3) \qquad \text{for all } s_2 \in [s''_2,\hat{s}''_2].$$

In particular,

$$v_1(\hat{s}_1(\hat{s}''_2),\hat{s}''_2,s''_3) = v_1(s''_1,s''_2,s''_3). \tag{A.22}$$

From (1), (A.16), (A.20), and (A.22), we have

$$\hat{s}_1(\hat{s}''_2) = \hat{s}''_1.$$

Hence,

$$\begin{aligned} 0 &= v_2(\hat{s}''_1,\hat{s}''_2,\hat{s}''_3) - v_2(s''_1,s''_2,s''_3) \\ &= (\hat{s}''_2 - s''_2)\left[\frac{\partial v_2}{\partial s_1}(\hat{s}_1(s^*_2),s^*_2,s''_3)\,\frac{d\hat{s}_1}{ds_2}(s^*_2) + \frac{\partial v_2}{\partial s_2}(\hat{s}_1(s^*_2),s^*_2,s''_3)\right] \end{aligned} \tag{A.23}$$

for some $s^*_2 \in [s''_2,\hat{s}''_2]$. From (A.21) we can rewrite the right-hand side of (A.23) as

$$(\hat{s}''_2 - s''_2)\left[-\frac{\partial v_2}{\partial s_1}\frac{\partial v_1}{\partial s_2} + \frac{\partial v_1}{\partial s_1}\frac{\partial v_2}{\partial s_2}\right]\Bigg/\frac{\partial v_1}{\partial s_1}, \tag{A.24}$$

where all derivatives are evaluated at $s_2 = s^*_2$. But from (38), (A.24) is nonzero, which contradicts (A.23). We conclude that (A.14) cannot hold after all, and so buyers can indeed infer one another's signal values from their bid functions. Hence if buyer i bids truthfully, the definition of truthful behavior (34) in step (e) ensures that buyers will choose the true valuations $(v_1(s_i,s_{-i}), \ldots, v_n(s_i,s_{-i}))$ in step (e), and so the previous analysis implies that buyer i has the incentive to bid truthfully and that doing so leads to an efficient outcome.

Finally, suppose that there are multiple solutions $(v^*_1, \ldots, v^*_n)$ to (A.2) and (A.3). In that case, the auction moves to step (f). But (38) and the definition of truthful behavior (35)–(37) ensure that buyers $j \neq i$ will together choose $\hat{v}^*_{-i}$ such that there exists $\hat{v}^*_i$ satisfying (A.4)–(A.7), and so the previous analysis implies again that truthful behavior pays for buyer i.‖

Application of Proposition 2

It may be helpful to see how Proposition 2 applies to the three-buyer model from Example 3. Recall from that example that

$$\text{(i)} \qquad v_1(s_1,s_2,s_3) = s_1 + \tfrac{1}{2}s_2 + \tfrac{1}{4}s_3$$

$$\text{(ii)} \qquad v_2(s_1,s_2,s_3) = s_2 + \tfrac{1}{4}s_1 + \tfrac{1}{2}s_3$$

$$\text{(iii)} \qquad v_3(s_1,s_2,s_3) = s_3.$$

Note, from (iii), that buyer 3's valuation does not depend on s_1 and s_2 and so, given s_3, his truthful bid function (in this example, the bid schedules are functions rather than correspondences) is a constant:

$$\text{(iv)} \qquad b_3(v_1,v_2) = s_3.$$

As for buyer 1, note from (ii), that we can rewrite s_2 as

$$s_2 = v_2 - \tfrac{1}{4}s_1 - \tfrac{1}{2}v_3,$$

and so, given s_1, buyer 1's truthful bid function is

$$\text{(v)} \qquad \begin{aligned} b_1(v_2,v_3) &= s_1 + \tfrac{1}{2}(v_2 - \tfrac{1}{4}s_1 - \tfrac{1}{2}v_3) + \tfrac{1}{4}v_3 \\ &= \tfrac{7}{8}s_1 + \tfrac{1}{2}v_2. \end{aligned}$$

Similarly, given s_2, buyer 2's truthful bid function is

$$\text{(vi)} \qquad \begin{aligned} b_2(v_1,v_3) &= s_2 + \tfrac{1}{4}(v_1 - \tfrac{1}{2}s_2 - \tfrac{1}{4}v_3) + \tfrac{1}{2}v_3 \\ &= \tfrac{7}{8}s_2 + \tfrac{1}{4}v_1 + \tfrac{7}{16}v_3. \end{aligned}$$

From (iv)–(vi), we obtain the (unique) fixed point

$$v_1^\circ = \tfrac{7}{8}s_1 + \tfrac{1}{2}v_2^\circ$$

$$v_2^\circ = \tfrac{7}{8}s_2 + \tfrac{1}{4}v_1^\circ + \tfrac{7}{16}v_3^\circ$$

$$v_3^\circ = s_3;$$

that is, the fixed point is just the vector of actual valuations (i)–(iii).

Thus, we see that, if (a) buyers bid truthfully, (b) a fixed point $(v_1^\circ,v_2^\circ,v_3^\circ)$ is calculated, and—as our auction requires—(c) the good is allocated to the buyer i for whom v_i° is biggest, the good will be allocated efficiently. Suppose, for example, that $s_1 = s_2 = 1$ and that s_3 is either slightly less or slightly more than 1. If buyer 1 is the winner, then from (28)–(30), he should pay

$$v_1^* = \max\{v_2^*, v_3^*\},$$

where (since $v_2^* > v_3^*$)

$$v_1^* = b_2(v_1^*, v_3^*)$$

$$= 7/8 + 1/4 v_1^* + 7/16 v_3^*;$$

i.e.,

$$v_1^* = 7/6 + 7/12 v_3^* = 7/6 + 7/12 s_3.$$

Hence, buyer 1's net payoff is $(1 + 1/2 + 1/4 s_3) - (7/6 + 7/12 s_3)$. But buyer 1 wins if and only if $s_3 < 1$. We conclude that buyer 1 wins the auction precisely in those cases where his payoff from winning is positive, namely, when $s_3 < 1$.

Application of Proposition 4 to Example 5

Let us see how Proposition 4 applies to the model of Example 5. To complete the specification of that model, let us suppose that c_1 is uniformly distributed on [0,1], q_1 is uniformly distributed on [1,2], c_2 is uniformly distributed on [0,2], and q_2 is uniformly distributed on [2,3]. Then

$$t_i = q_i - c_i, \qquad i = 1,2$$

$$w_1(t_1,t_2) = E_{q_2,c_2}[t_1 + q_2 | q_2 - c_2 = t_2]$$

$$= \begin{cases} t_1 + 2 + \frac{t_2}{2}, & 0 \le t_2 \le 1 \\ t_1 + \frac{5}{2}, & 1 \le t_2 \le 2 \\ t_1 + \frac{3}{2} + \frac{t_2}{2}, & 2 \le t_2 \le 3 \end{cases}$$

and

$$w_2(t_1,t_2) = E_{q_1,c_1}[t_2 + q_1 | q_1 - c_1 = t_1]$$

$$= t_2 + 1 + \frac{t_1}{2}.$$

Notice that

$$1 = \frac{\partial w_i}{\partial t_i} > \frac{1}{2} \geq \frac{\partial w_j}{\partial t_i} \qquad \text{for all } i \text{ and } j \neq i.$$

Hence conditions (1) and (6) are satisfied, and so, from Proposition 1, our generalized Vickrey auction attains the constrained efficient outcome, i.e., the winner in equilibrium will be the wildcatter i for whom

$$w_i(t_1,t_2) > w_j(t_1,t_2).$$

Proof of Proposition 5. Suppose that buyer 2 is truthful. Let buyers' summary signals be (t_1,t_2). If buyer 1 also bids truthfully, then, from (58)–(59), $\{(w_{1,H}(t_1,t_2),w_{2,H}(t_1,t_2))\}_H$ is a fixed point satisfying step (B) of the auction (if there are other fixed points, the multiplicity can be resolved as in the auction of Proposition 2). Hence, from step (C), buyer 1 is allocated goods H_1° where

$$\text{(A.25)} \qquad (H_1^\circ,H_2^\circ) \text{ solves } \max_{(H_1,H_2)} \sum_{j=1}^{2} w_{j,H_j}(t_1,t_2).$$

Thus, the outcome of the auction is efficient, and it remains to show that buyer 1 will choose to bid truthfully.

We claim that if buyer i is truthful, then the sequence $[\{w_{1,H'}^r\}_{H'},(H_1^r,H_2^r)]$, $r = 1, \ldots, R$, determining buyer 1's payment satisfies

$$\text{(A.26)} \qquad w_{1,H'}^r = w_{1,H'}(t_1^r,t_2) \qquad \text{for all } H',$$

where $\{t_1^1, \ldots, t_1^R\}$ are signal values such that

$$\text{(A.27)} \qquad t_1 > t_1^1 > \cdots > t_1^R.$$

Hence (55) can be rewritten as

$$\sum_{r=1}^{R} (b_{2,\mathrm{H}_2^r}(\{w_{1,H'}(t_1^r,t_2)\}_{H'}) - b_{2,H_2^{r-1}}(\{w_{1,H'}(t_1^r,t_2)\}_{H'}))$$

$$+ b_{2,\{1,\ldots,l\}}(\{w_{1,H'}(t_1^R,t_2)\}_{H'}) - b_{2,H_2^R}(\{w_{1,H'}(t_1^R,t_2)\}_{H'}),$$

and thus, from (59), as

$$\text{(A.28)} \quad \sum_{r=1}^{R} (w_{2,H_2^r}(t_1^r,t_2) - w_{2,H_2^{r-1}}(t_1^r,t_2)) + w_{2,\{1,\ldots,I\}}(t_1^R,t_2) - w_{2,H_2^R}(t_1^R,t_2).$$

To see this, let t_1^1 be the lowest signal value for buyer 1 such that (H_1°,H_2°) remains the surplus-maximizing allocation for signal values for (t_1^1,t_2) (if t_1^1 does not exist—i.e., if (H_1°,H_2°) remains optimal for all signal values less than t_1—go to the next paragraph). And for a signal value slightly less than t_1^1 let (H_1^1,H_2^1) be the corresponding surplus-maximizing allocation. Note from (45), (59) and the definition of (H_1^1,H_2^1) that $[\{w_{1,H'}(t_1^1,t_2)\}_{H'},(H_1^1,H_2^1)]$ satisfies (50)–(52), as claimed (if there are other solutions to (50)–(52), proceed as in Proposition 2).

Continuing similarly with $t_1^2 > t_1^3 > \cdots$ (from (46), only finitely many "switchpoints" t_1^r are possible), we reach t_1^R such that either $H_1^R = \varnothing$ (i.e., (53) holds) or $t_1^R = \underline{t}_1$ (in which case (54) holds). Hence, the claim is established, and buyer 1's net payoff is

$$\text{(A.29)} \quad w_{1,H_1^\circ}(t_1,t_2) - \sum_{r=1}^{R} (w_{2,H_2^r}(t_1^r,t_2) - w_{2,H_2^{r-1}}(t_1^r,t_2)) - w_{2,\{1,\ldots,I\}}(\underline{t}_1,t_2) - w_{2,H_2^R}(\underline{t}_1,t_2).$$

Suppose instead that buyer 1 does not bid truthfully. If $\{(w_{1,H}^{\circ\circ},w_{2,H}^{\circ\circ})\}_H$ is the resulting fixed point, then from (58) and (59) there exists t'_1 such that

$$w_{1,H}^{\circ\circ} = w_{1,H}(t'_1,t_2) \qquad \text{for all } H.$$

That is, buyer 1 is, in effect, bidding as though his signal value is t'_1. Assume for now that $t'_1 < t_1$. Then if, for some $P \in \{1, \ldots, R\}$, $t_1^P \geq t'_1 > t_1^{P+1}$, $H^{\circ\circ}_1 = H_1^P$ and buyer 1's net payoff is

$$\text{(A.30)} \quad w_{1,H_1^P}(t_1,t_2) - \sum_{r=P+1}^{R} (w_{2,H_2^r}(t_1^r,t_2) - w_{2,H_2^{r-1}}(t_1^r,t_2)) - (w_{2,\{1,\ldots,I\}}(\underline{t}_1,t_2) - w_{2,H_2^R}(\underline{t}_1,t_2)).$$

Subtracting (A.30) from (A.29), we obtain

$$\text{(A.31)}\quad w_{1,H_1^\circ}(t_1,t_2) - w_{1,H_1^P}(t_1,t_2) - \sum_{r=1}^{P} (w_{2,H_2^r}(t_1^r,t_2) - w_{2,H_2^{r-1}}(t_1^r,t_2)).$$

We must show that (A.31) is positive. From (52) and (A.26),

$$\text{(A.32)}\quad w_{1,H_1^{r-1}}(t_1^r,t_2) + w_{2,H_2^{r-1}}(t_1^r,t_2) = w_{1,H_1^r}(t_1^r,t_2) + w_{2,H_2^r}(t_1^r,t_2).$$

If, for some $r = 1, \ldots, P$, $w_{1,H_1^{r-1}}(t_1^r,t_2) < w_{1,H_1^r}(t_1^r,t_2)$ then, from (46) and (A.32),

$$w_{1,H_1^{r-1}}(t_1^{r-1},t_2) + w_{2,H_2^{r-1}},(t_1^{r-1},t_2) < w_{1,H_1^r}(t_1^{r-1},t_2) + w_{2,H_2^r}(t_1^{r-1},t_2),$$

a contradiction of (50). Hence

$$\text{(A.33)}\quad w_{1,H_1^{r-1}}(t_1^r,t_2) > w_{1,H_1^r}(t_1^r,t_2), \qquad r = 1, \ldots, P.$$

From (45) and (A.33) we have

$$\text{(A.34)}\quad w_{1,H_1^{r-1}}(t_1,t_2) - w_{1,H_1^r}(t_1,t_2) - (w_{1,H_1^{r-1}}(t_1^r,t_2) - w_{1,H_1^r}(t_1^r,t_2)) > 0, \qquad r = 1, \ldots, P.$$

Using (A.32) and (A.34), we obtain

$$\text{(A.35)}\quad w_{1,H_1^{r-1}}(t_1,t_2) - w_{1,H_1^r}(t_1,t_2) - (w_{2,H_2^r}(t_1^r,t_2) - w_{2,H_2^{r-1}}(t_1^r,t_2)) > 0.$$

But summing (A.35) over $r = 1, \ldots, P$, we find that (A.31) is positive. This also shows that buyer 1's allocation must still be H_1° if he deviates to a nontruthful best response. The argument is symmetric if $t_1 < t'_1$.‖

Illustration of Proposition 5

There are two buyers, 1 and 2, and two goods, A and B. The valuations are as follows:

$$w_{1,A}(t_1,t_2) = 3t_1 + t_2$$
$$w_{1,B}(t_1,t_2) = 2t_1$$
$$w_{1,AB}(t_1,t_2) = 5t_1 + 2t_2$$
$$w_{2,A}(t_1,t_2) = 2t_2 + \tfrac{1}{2}t_1$$

$$w_{2,B}(t_1,t_2) = 2t_2 + t_1$$

$$w_{2,AB}(t_1,t_2) = 4t_2 + 2t_1.$$

It is readily shown that the surplus-maximizing allocation is

$$(H_1,H_2) = \begin{cases} (\{A,B\},\varnothing), & \text{if } t_1 > t_2 \\ (A,B), & \text{if } t_1 < t_2 < 2t_1 \\ (\varnothing,\{A,B\}), & \text{if } t_2 > 2t_1. \end{cases}$$

From (58) and (59) truthful bidding for buyer 1 with signal value t_1 consists of setting

$$W_2 = \{(w_{2,A},w_{2,B},w_{2,AB}) \mid \text{there exists } t'_2 \text{ such that}$$

$$(w_{2,A},w_{2,B},w_{2,AB}) = (2t'_2 + \tfrac{1}{2}t_1, 2t'_2 + t_1, 4t'_2 + 2t_1)\}$$

and, for all $(w_{2,A},w_{2,B},w_{2,AB}) \in W_2$,

$$b_{1,A}(w_{2,A},w_{2,B},w_{2,AB}) = 3t_1 + w_{2,A} - \tfrac{1}{2}w_{2,B}$$

$$b_{1,B}(w_{2,A},w_{2,B},w_{2,AB}) = 2t_1$$

and

$$b_{1,AB}(w_{2,A},w_{2,B},w_{2,AB}) = 5t_1 + 2w_{2,A} - \tfrac{1}{2}w_{2,AB}.$$

Similarly, truthful bidding for buyer 2 with signal value t_2 amounts to taking $W_1 = \{(w_{1,A},w_{1,B},w_{1,AB}) \mid \text{there exists } t'_1 \text{ such that } (w_{1,A},w_{1,B},w_{1,AB}) = (t_2 + 3t'_1, 2t'_1, 2t_2 + 5t'_1)\}$ and, for all $(w_{1,A},w_{1,B},w_{1,AB}) \in W_1$,

$$b_{2,A}(w_{1,A},w_{1,B},w_{1,AB}) = 2t_2 + w_{1,A} - \tfrac{1}{2}w_{1,AB}$$

$$b_{2,B}(w_{1,A},w_{1,B},w_{1,AB}) = 2t_2 + \tfrac{1}{2}w_{1,B}$$

$$b_{2,AB}(w_{1,A},w_{1,B},w_{1,AB}) = 4t_2 + w_{1,B}.$$

Suppose that $(t_1,t_2) = (3,2)$. Then truthful bidding gives rise to the (unique) fixed point:

$$(w^\circ_{1,A},w^\circ_{1,B},w^\circ_{1,AB},w^\circ_{2,A},w^\circ_{2,B},w^\circ_{2,AB}) = (11,6,19,5\tfrac{1}{2},7,14).$$

Notice that $(H^\circ_1,H^\circ_2) = (\{A,B\},\varnothing)$ solves

$$\max_{(H_1,H_2)} w^\circ_{1,H_1} + w^\circ_{2,H_2},$$

which is the efficient allocation. Now, if we reduce buyer 1's valuations from (11,6,19) to

$$(w^1_{1,A}, w^1_{1,B}, w^1_{1,AB}) = (8,4,14),$$

then we are on the boundary between $(H^\circ_1, H^\circ_2) = (\{A,B\}, \varnothing)$ and

$$(H^1_1, H^1_2) = (A, B)$$

being the surplus-maximizing allocations, i.e., both allocations solve

$$\max_{(H_1,H_2)} w^1_{1,H_1} + b_{2,H_2}(8,4,14).$$

If we reduce buyer 1's valuations further to

$$(w^2_{1,A}, w^2_{1,B}, w^2_{1,AB}) = (5,2,9),$$

then we are on the boundary between $(H^1_1, H^1_2) = (A,B)$ and $(H^2_1, H^2_2) = (\varnothing, \{A,B\})$ being the surplus-maximizing allocations. That is, both allocations solve

$$\max_{(H_1,H_2)} w^2_{1,H_1} + b_{2,H_2}(5,2,9).$$

Hence, in the truthful equilibrium, buyer 1 pays

$$b_{2,H^1_2}(8,4,14) - b_{2,H^\circ_2}(8,4,14) + b_{2,H^2_2}(5,2,9) - b_{2,H^1_2}(5,2,9)$$
$$= 6 - 0 + 10 - 5 = 11.$$

And so buyer 1's equilibrium net payoff is

$$w_{1,AB}(3,2) - 11 = 8.$$

Because this is positive, buyer 1 is better off bidding truthfully than bidding in a way such that he is allocated no goods. If instead buyer 1 behaves in such a way that he is allocated only good A, his net payoff is

$$w_{1,A}(3,2) - 5 = 6.$$

Thus, buyer 1 is indeed best off bidding truthfully.

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