

CMU 15-892

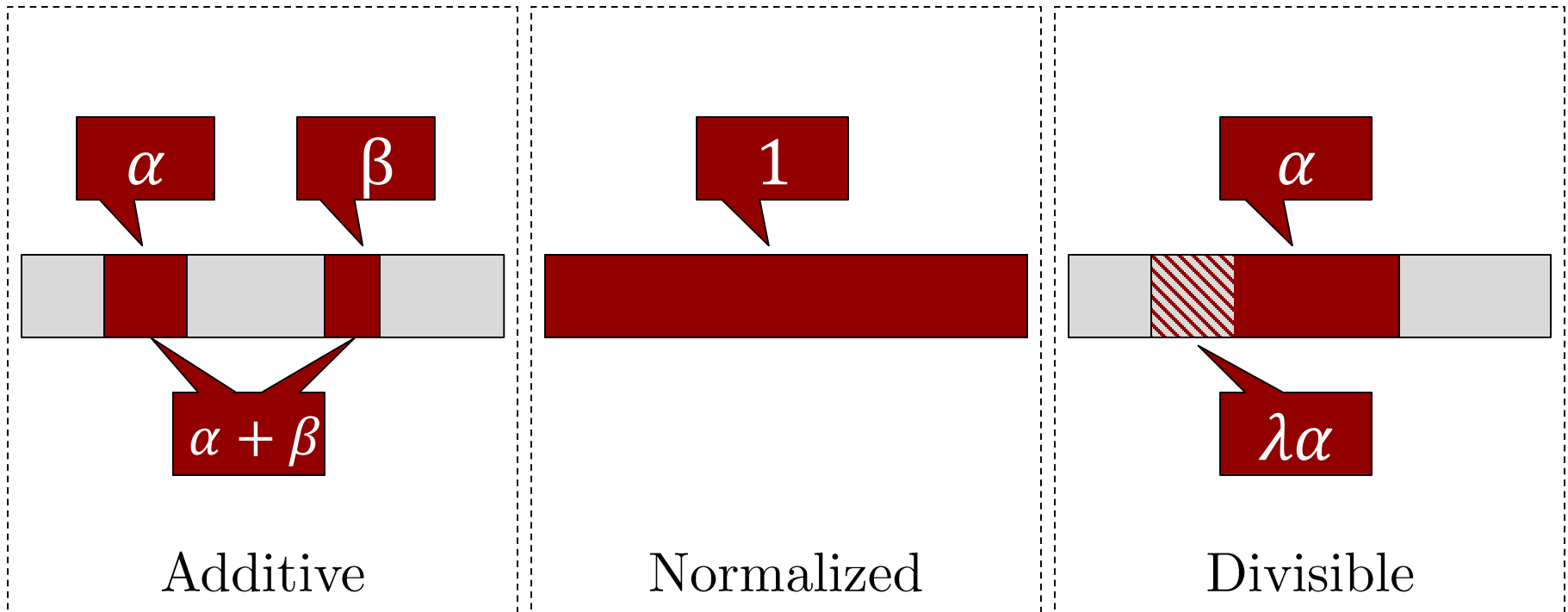
CAKE CUTTING

TEACHER:
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(ON BEHALF OF SANDHOLM)

- Single heterogeneous good, represented as $[0,1]$
- Set of players $N = \{1, \dots, n\}$
- Piece of cake $X \subseteq [0,1]$: finite union of disjoint intervals



Each player i has a valuation V_i that is:



FAIRNESS, FORMALIZED

- Our goal is to find an allocation A_1, \dots, A_n

- Proportionality:

$$\forall i \in N, V_i(A_i) \geq \frac{1}{n}$$

- Envy-Freeness (EF):

$$\forall i, j \in N, V_i(A_i) \geq V_i(A_j)$$



FAIRNESS, FORMALIZED

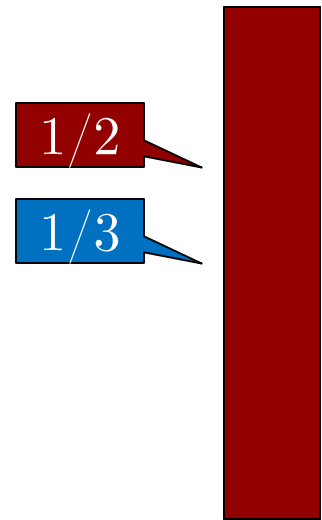
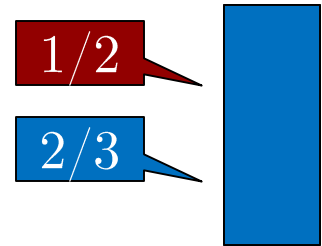
Poll 1: What is the relation between proportionality and EF?

1. Proportionality \Rightarrow EF
2. EF \Rightarrow proportionality
3. Equivalent
4. Incomparable



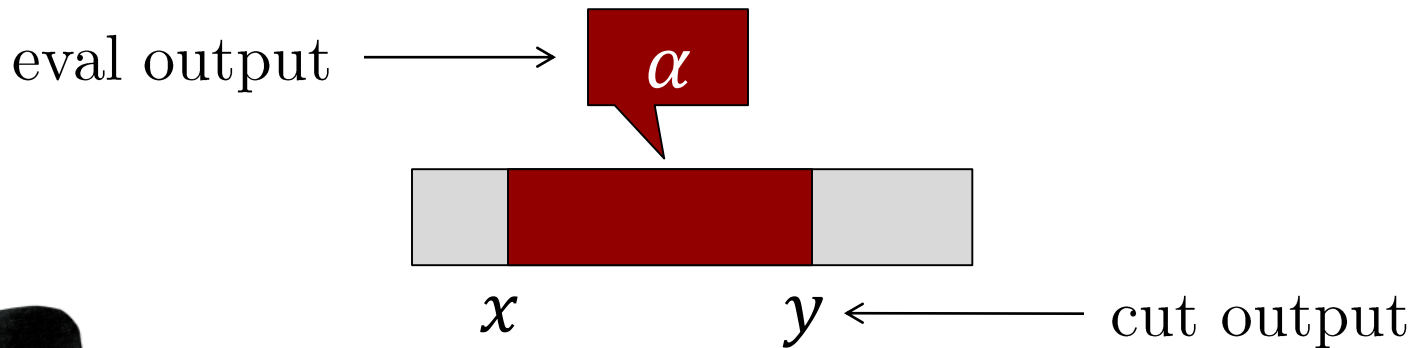
CUT-AND-CHOOSE

- Algorithm for $n = 2$ [Procaccia and Procaccia, circa 1985]
- Player 1 divides into two pieces X, Y s.t.
$$V_1(X) = 1/2, V_1(Y) = 1/2$$
- Player 2 chooses preferred piece
- This is EF and proportional



THE ROBERTSON-WEBB MODEL

- What is the time complexity of C&C?
- Input size is n
- Two types of queries
 - $\text{Eval}_i(x, y)$ returns $V_i([x, y])$
 - $\text{Cut}_i(x, \alpha)$ returns y such that $V_i([x, y]) = \alpha$



THE ROBERTSON-WEBB MODEL

- Two types of queries
 - $\text{Eval}_i(x, y) = V_i([x, y])$
 - $\text{Cut}_i(x, \alpha) = y$ s.t. $V_i([x, y]) = \alpha$

#queries needed to find an
EF allocation when $n = 2$?



DUBINS-SPANIER

- Referee continuously moves knife
- Repeat: when piece left of knife is worth $1/n$ to player, player shouts “stop” and gets piece
- That player is removed
- Last player gets remaining piece

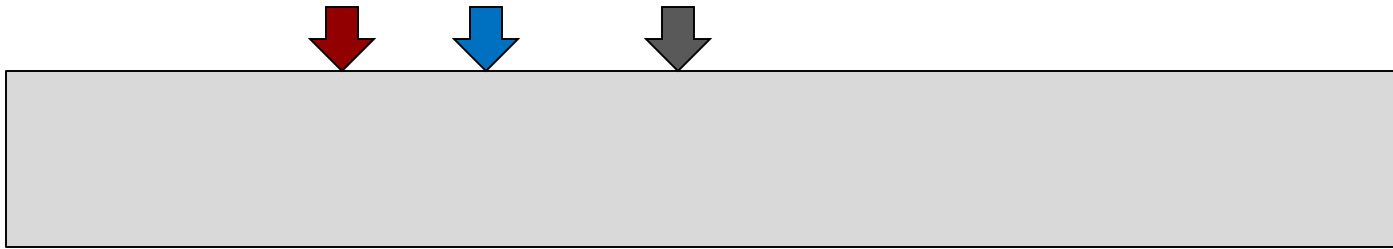
DUBINS-SPANIER

Poll 2: What is the complexity of DS in the RW model?

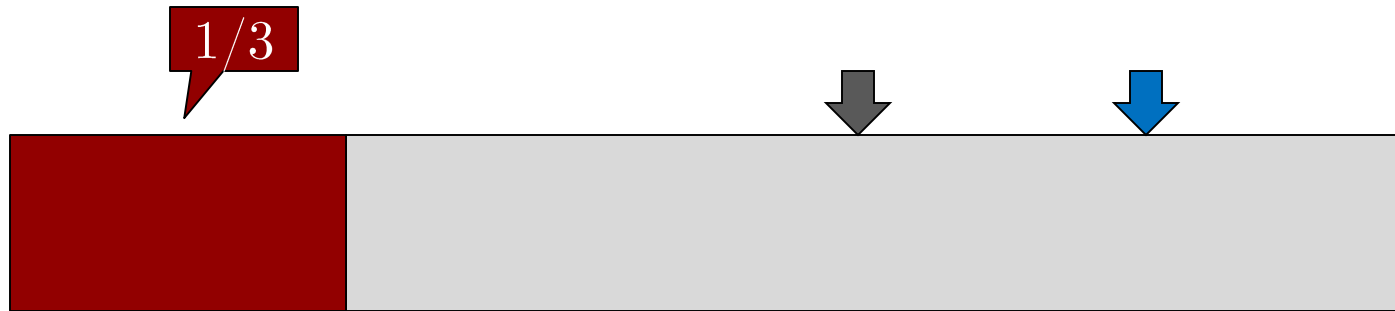
1. $\Theta(n)$
2. $\Theta(n \log n)$
3. $\Theta(n^2)$
4. $\Theta(n^2 \log n)$



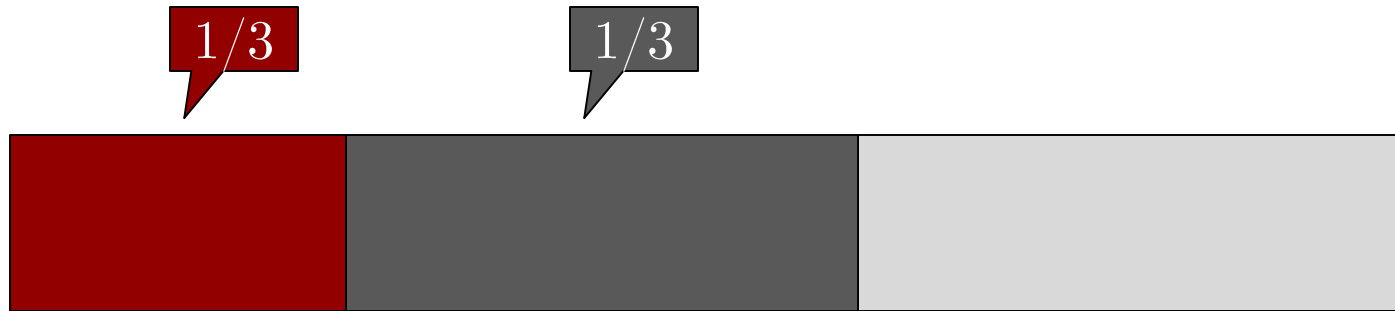
DUBINS-SPANIER



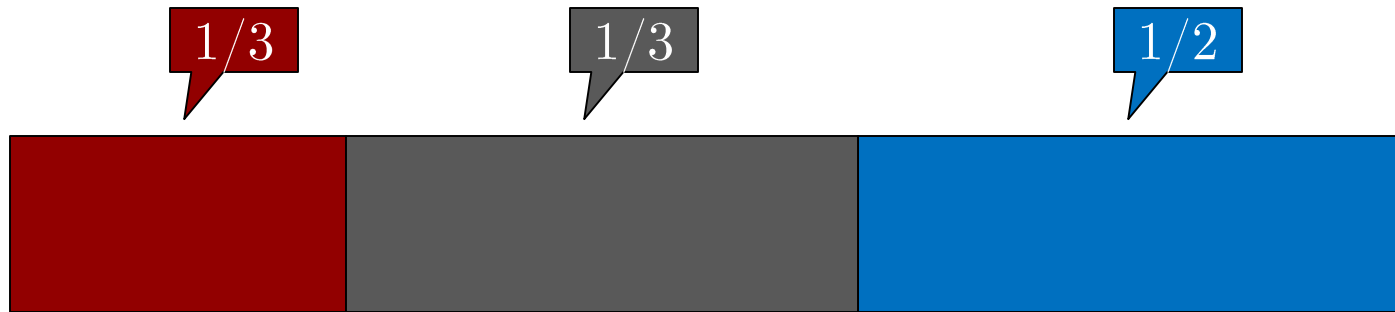
DUBINS-SPANIER



DUBINS-SPANIER



DUBINS-SPANIER



EVEN-PAZ

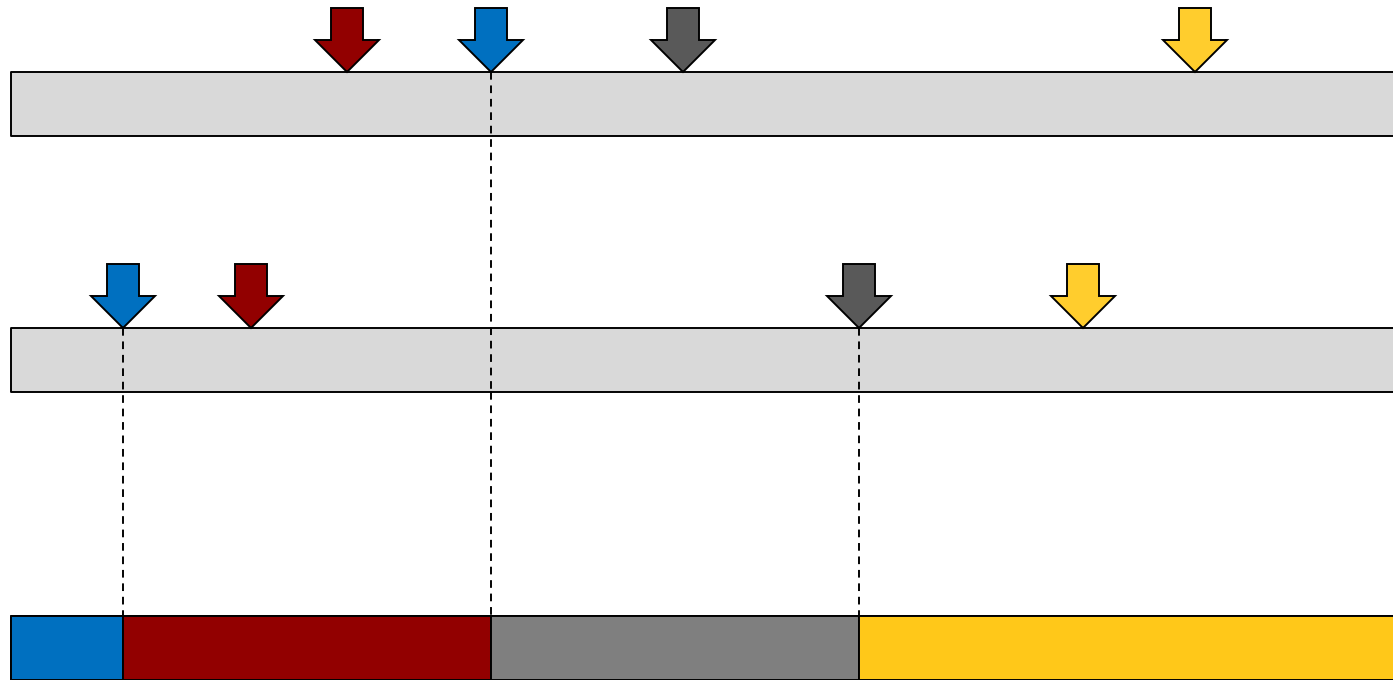
- Given $[x, y]$, assume $n = 2^k$
- If $n = 1$, give $[x, y]$ to the single player
- Otherwise, each player i makes a mark z s.t.

$$V_i([x, z]) = \frac{1}{2} V_i([x, y])$$

- Let z^* be the $n/2$ mark from the left
- Recurse on $[x, z^*]$ with the left $n/2$ players, and on $[z^*, y]$ with the right $n/2$ players



EVEN-PAZ

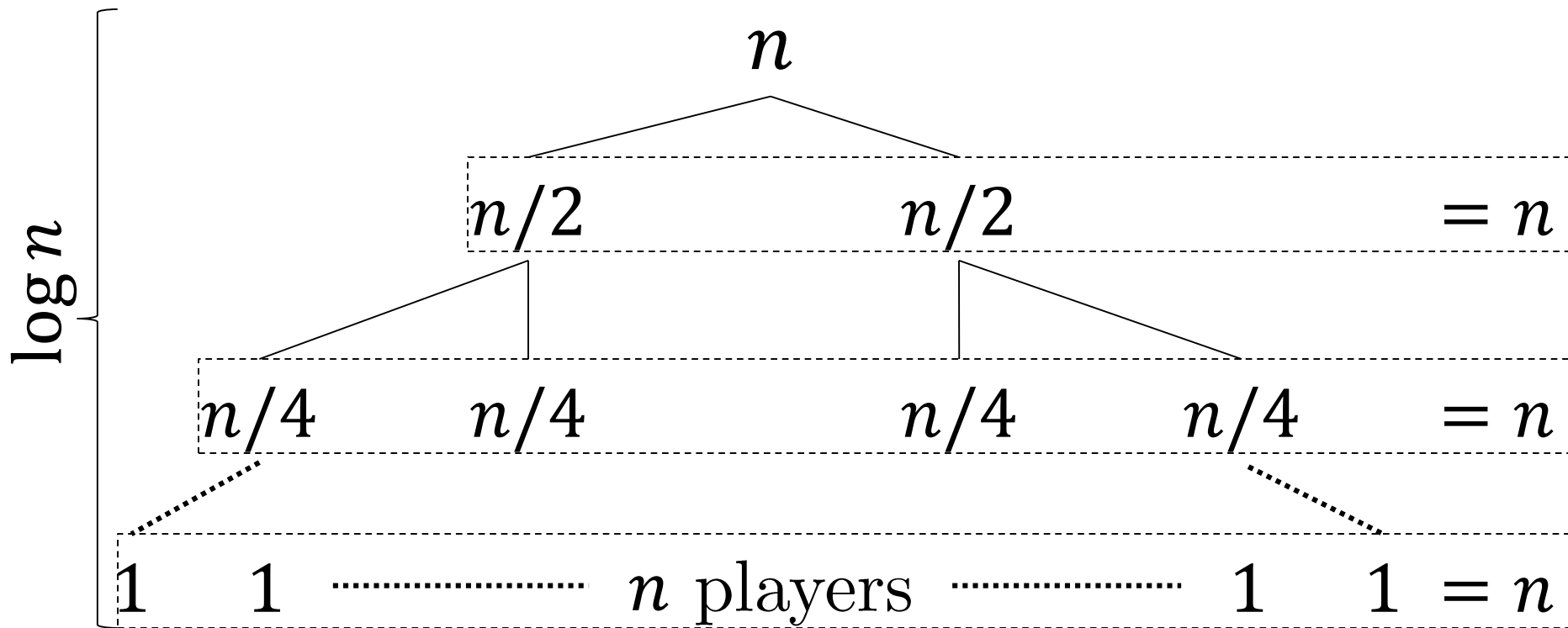


EVEN-PAZ: PROPOTIONALITY

- **Claim:** The Even-Paz protocol produces a proportional allocation
- **Proof:**
 - At stage 0, each of the n players values the whole cake at 1
 - At each stage the players who share a piece of cake value it at least at $V_i([x, y])/2$
 - Hence, if at stage k each player has value at least $1/2^k$ for the piece he's sharing, then at stage $k + 1$ each player has value at least $\frac{1}{2^{k+1}}$
 - The number of stages is $\log n$ ■



EVEN-PAZ: COMPLEXITY



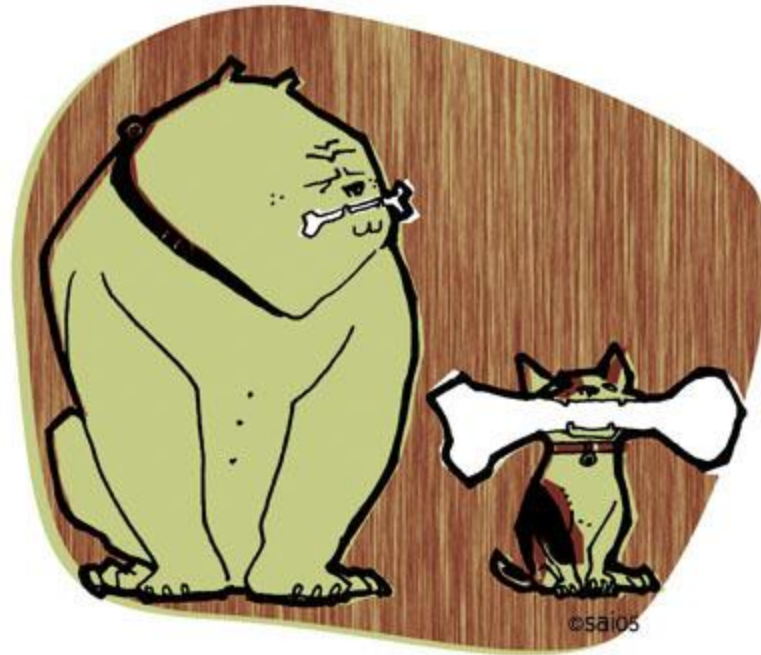
Overall: $n \log n$

COMPLEXITY OF PROPORTIONALITY

- **Theorem [Edmonds and Pruhs 2006]:** Any proportional protocol needs $\Omega(n \log n)$ operations in the RW model
- We will prove the theorem on Tuesday
- The Even-Paz protocol is provably optimal!



WHAT ABOUT ENVY?



SELFRRIDGE-CONWAY

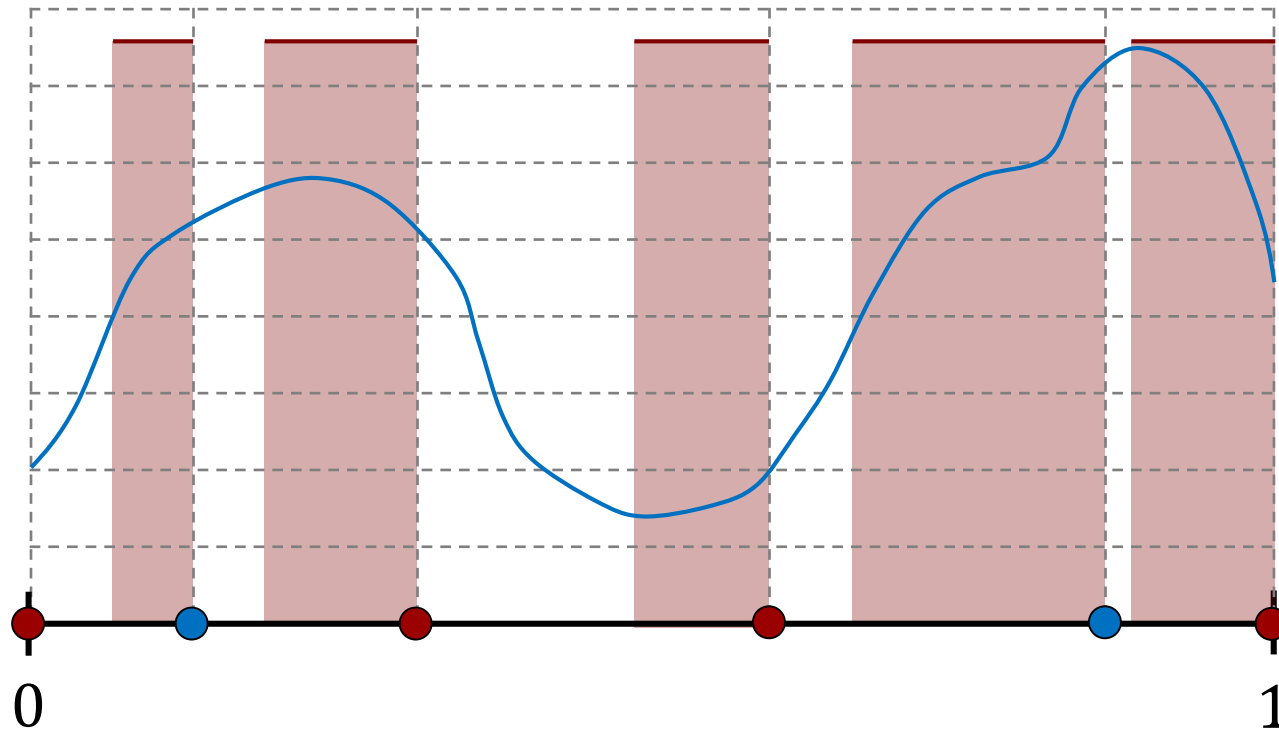
- **Stage 0**
 - Player 1 divides the cake into three equal pieces according to V_1
 - Player 2 trims the largest piece s.t. there is a tie between the two largest pieces according to V_2
 - Cake 1 = cake w/o trimmings, Cake 2 = trimmings
- **Stage 1 (division of Cake 1)**
 - Player 3 chooses one of the three pieces of Cake 1
 - If player 3 did not choose the trimmed piece, player 2 is allocated the trimmed piece
 - Otherwise, player 2 chooses one of the two remaining pieces
 - Player 1 gets the remaining piece
 - Denote the player $i \in \{2, 3\}$ that received the trimmed piece by T , and the other by T'
- **Stage 2 (division of Cake 2)**
 - T' divides Cake 2 into three equal pieces according to $V_{T'}$
 - Players T , 1, and T' choose the pieces of Cake 2, in that order



THE COMPLEXITY OF EF

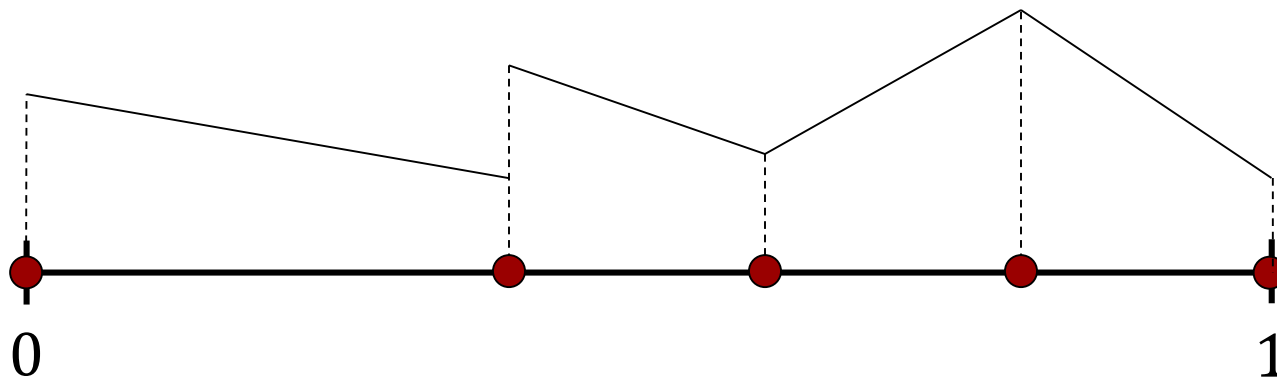
- Theorem [Brams and Taylor 1995]: There is an **unbounded** EF cake cutting algorithm in the RW model
- Theorem [P 2009]: Any EF algorithm requires $\Omega(n^2)$ queries in the RW model
- Theorem [Kurokawa, Lai, P, 2013]: EF cake cutting with **piecewise uniform valuations** is as hard as general case

THE COMPLEXITY OF EF



THE COMPLEXITY OF EF

- Theorem [Kurokawa, Lai, P, 2013]:
EF cake cutting with **piecewise linear valuations** is polynomial in the number of breakpoints



RW IS FOR HONEST KIDS

- EF protocol that uses n queries
- $f = 1-1$ mapping from valuation functions to $[0,1]$
- The protocol asks each player $\text{cut}_i(0, 1/2)$
- Player i replies with $y_i = f(V_i)$
- The protocol computes $V_i = f^{-1}(y_i)$
- We therefore need to assume that players are “honest”