CS 15-892 Foundations of Electronic Marketplaces
Homework 1

Fall Semester 2015

Tuomas Sandholm & John Dickerson
Carnegie Mellon University
Computer Science Department

September 22, 2015

Due to Prof. Sandholm by email before the beginning of class on October 7th. In this homework, you may use any sources that you want but you must cite the sources that you use. Teamwork is not allowed.

1. (10 pts) Throughout this question, you may restrict your analysis to pure strategies.

   a. Draw the strategic (normal) form of the following game tree.

   b. Name the dominant strategy equilibria of this game, if there are any.

   c. Name the iterated dominance solutions of this game, if there are any.

   d. Name the Nash equilibria of this game, if there are any.

   e. Name the Nash equilibria, if there are any, which do not involve the play of weakly dominated strategies.

   f. Name the subgame perfect Nash equilibria of this game, if there are any.

   g. Name the Pareto efficient outcomes of this game, if there are any.

   h. Name the social welfare maximizing outcomes of this game, if there are any.

```
      3, 3
    /
   /
  /
1          4, 4
  \       /
   \     /
    \   /
     2
      \
       \
        \
           2, 7
```

2. **Game of chicken.** Let there be two teenagers who play the following risky game. They head toward each other in separate cars. Just before collision, each one has the choice of continuing straight, or avoiding collision by turning right. If both continue, they will die in the collision. If one continues and the other turns, both survive but the former becomes the hero and the latter is humiliated. If both turn, both survive but both are moderately humiliated. Let the game be represented by the following normal form:

```
                                   Agent 2

                            straight        turn
                         ----------------------------
                         |                          |
        straight         |   -3, -3          2, 0   |
Agent 1                  |                          |
        turn             |    0, 2           1, 1   |
                         |__________________________|
```

   a. (5 pts) Does this game have pure strategy Nash equilibria? If so, what are they?

   b. (15pts) What are the mixed strategy Nash equilibria of this game?

   c. (2pts) In each equilibrium, what is the probability that the youngsters will die?

3. **Iterated elimination of dominated strategies as a solution concept for games.** An agent's strategy is *strictly dominated* if that agent has another strategy that gives strictly higher payoff to the agent no matter what strategies other agents choose. An agent's strategy is *weakly dominated* if that agent has another strategy that gives at least equally high payoff to the agent no matter what strategies other agents choose, and strictly higher payoff to the agent for at least one choice of strategies by the others. Now, to try to solve a game, we can iteratively eliminate dominated strategies until all the remaining strategies are undominated.

   a. (10pts) Show that iterated elimination of weakly dominated strategies is path dependent, i.e., the order in which strategies are eliminated can affect the outcome.

   b. (10pts) Prove that iterated elimination of strictly dominated strategies is path independent, i.e., the order in which strategies are eliminated cannot affect the outcome. For simplicity, you do not have to consider mixed strategies.

   c. (10pts) Show that a pure strategy can be strictly dominated by a mixed strategy in a game where the pure strategy is not even weakly dominated by any pure strategy.

   d. (10pts) Show that an agent's mixed strategy can be strictly dominated in a game where none of the agent's pure strategies are even weakly dominated.

4. **Poker.** In three-card poker, two players are both dealt a card from a deck consisting of a king (K), a queen (Q), and a jack (J) (the final card is not dealt). Both players ante \$1, and player 1 (P1) is first to act. He has two choices: he can bet \$1 (B) or check (C). If P1 bets, player 2 (P2) can either call or fold. If P1 bets and P2 calls,

then whoever has the higher card (K > Q > J) wins the entire pot of \$4. If P1 bets and P2 folds, then P1 wins the pot of \$2. If P1 checks, then P2 can either bet or check. If P1 checks and P2 checks, then whoever has the higher card wins \$2. If P1 checks and P2 bets, then P1 can either call or fold. If P1 calls, then whoever has the higher card wins \$4. If P1 folds, then P2 wins \$2.

(a) Draw the extensive-form game tree of this game. [3 pts.]

(b) How large would the equivalent normal-form representation of this game be? Note that you do NOT need to explicitly write down the normal-form representation of this game. [3 pts.]

(c) Say that an action a for player i at information set s in an extensive-form game is weakly dominated if there exists another action b at the same information set, such that for all leaf nodes n_x reachable by playing action x at s, we have $u_i(n_b) \geq u_i(n_a)$. Based on your extensive-form representation, perform iterated elimination of weakly dominated actions and list all actions that end up being eliminated for each player. [3 pts.]

(d) Now reconstruct the extensive-form representations of the game from part (4a) with the dominated actions eliminated. [3 pts.]

(e) Write down the equivalent normal-form representation of the game you just constructed in part (4d). [3 pts.]

(f) What is one equilibrium of this game? What is the value of the game to P1? You can use any method you want (we recommend solving it manually from the extensive-form representation), but must show your work and justify your answer. [6 pts.]

(g) What are all of the equilibria of this game? You can use any method you want (again we recommend solving it manually from the extensive-form representation), but must show your work and justify your answer.

Hint: There are infinitely many equilibria, so you should provide as detailed a description as possible of the set of equilibria. For example, you could give a parameterized strategy for each player, and say what sets the parameters can range over. Of course, your parameterized strategies should include your answer to part (4f) as a special case. [7 pts.]

5. **MIP Nash.** Consider the game given in Figure 1

	L	M	R
U	$0, 0$	$4, 5$	$5, 4$
M	$5, 4$	$0, 0$	$4, 5$
D	$4, 5$	$5, 4$	$0, 0$

Figure 1: A two-player game.

(a) Write a mixed-integer program (MIP) for finding the *welfare-maximizing* Nash equilibrium in this game. [4 pts.]

(b) What is the optimal solution to the MIP? (You may use any software for solving this problem; however, it is straightforward to compute by hand.) [6 pts.]

6. **Price of anarchy and stability.** The *price of anarchy* in a game is a measure of how agents' selfish behavior impedes global system efficiency. Given N agents, strategy sets $S_i \in \mathcal{S}$ and utility functions $u_i(s_1, \ldots, s_N)\ \forall i \in [N]$, let $S = S_1 \times \ldots \times S_N$ be the set of all outcomes and $E \subseteq S$ be the set of outcomes that are in Nash equilibrium. Then, for any global welfare function $W : S \to \mathbb{R}$, the price of anarchy PoA of a game is defined as

$$\text{PoA} = \frac{\max_{s \in S} W(s)}{\min_{s \in E} W(s)},$$

that is, the relative global efficiency loss due to the worst equilibrium outcome relative to the best solution enforced by some centralized planner.

We will explore the price of anarchy in the following two slighty modified games from class, shown in Figure 2 as games G^1 and G^2.

	cooperate	defect
cooperate	$3, 3$	$0, 5$
defect	$5, 0$	$1, 1$

	Bach	Stravinsky
Bach	$3, 1$	$0, 0$
Stravinsky	$0, 0$	$1, 2$

Figure 2: The Prisoner's Dilemma game (G^1) and the Bach or Stravinsky game (G^2).

(a) Consider the *utilitarian objective* $W^U(s) = \sum_{i \in [N]} u_i(s)$, for any $s \in S$. What is PoA_{W^U} for games G^1 and G^2 above? [2 pts.]

(b) Consider the *egalitarian objective* $W^E(s) = \min_{i \in [N]} u_i(s)$, for any $s \in S$. What is PoA_{W^E} for both games? [2 pts.]

(c) Define your own (non-trivial) welfare function W^*. Motivate it; what does it mean? What is PoA_{W^*} for both games? [2 pts.]

The price of anarchy is perhaps too conservative in that it purposefully selects as a baseline the *worst* equilibrium outcome. Let the *price of stability* instead measure the global efficiency loss due to the *best* equilibrium outcome, with respect to the optimal solution enforced by some centralized planner. Formally, let the price of stability PoS of a game be

$$\text{PoS} = \frac{\max_{s \in S} W(s)}{\max_{s \in E} W(s)}.$$

(d) Again, consider the utilitarian objective $W^U(s) = \sum_{i \in [N]} u_i(s)$, for any $s \in S$. What is PoS_{W^U}? [2 pts.]

(e) Again, consider the egalitarian objective $W^E(s) = \min_{i \in [N]} u_i(s)$, for any $s \in S$. What is PoS_{W^E}? [2 pts.]

(f) Again, consider your own welfare function W^*. What is PoS_{W^*}? [2 pts.]