Limitations of VCG-Based Mechanisms

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Abstract

We consider computationally-efficient incentive-compatible mechanisms that use the VCG payment scheme, and study how well they can approximate the social welfare in auction settings. We present a novel technique for setting lower bounds on the approximation ratio of this type of mechanisms. Specifically, for combinatorial auctions among submodular (and thus also subadditive) bidders we prove an $\Omega(m^{\frac{1}{6}})$ lower bound, which is close to the known upper bound of $O(m^{\frac{1}{2}})$, and qualitatively higher than the constant factor approximation possible from a purely computational point of view.

1 Introduction

1.1 Background

Algorithmic Mechanism design attempts to design protocols for distributed environments, such as the Internet, where the different participants each have their own selfish goals and are assumed to rationally attempt optimizing their own goals rather than just follow any prescribed protocol. The key target in this area is the design of *incentive-compatible* mechanisms – also called truthful or strategy-proof mechanisms – whose payment schemes motivate the participants to correctly report their private information[1]. For a general introduction to the economic field of mechanism design see [21] and for an introduction to *algorithmic* mechanism design and further motivation see [25].

Typical problems in this setting involve allocation of various resources and a paradigmatic abstraction is that of *combinatorial auctions*. In this problem m heterogenous "items" need to be allocated between n "bidders". Each bidder i holds a *valuation* function v_i that specifies for each subset of the items $S \subseteq \{1...m\}$ the bidder's value $v_i(S)$ from winning the "bundle" S. The challenge is to find a partition $S_1...S_n$ of the items that maximizes the social welfare $\Sigma_i v_i(S_i)$. This problem presents a combination of algorithmic difficulty (it is NP-complete), representational difficulty (the valuation functions are exponential size objects) and strategic difficulty (ensuring incentive compatibility).

The key positive technique for achieving incentive compatibility is that of VCG mechanisms [29, 6, 13]: if player i's value from the chosen algorithmic outcome a is $v_i(a)$, then we charge player i the quantity $h_i(v_{-i}) - \Sigma_{j \neq i} v_j(a)$, where h_i is an arbitrary fixed function that does not depend on

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[1]This paper – as most previous work in algorithmic mechanism design – considers the simple and strong notion of incentive compatibility in dominant strategies. This covers also protocols where the desired results are obtained as arbitrary dominant-strategy equilibria, as the revelation principle allows converting such mechanisms to incentive compatible ones.

v_i. A powerful observation is that if the algorithmic outcome a always maximizes the *social welfare*, $\Sigma_i v_i(a)$, then the VCG payment rule results in an incentive compatible mechanism. However, in most interesting computational scenarios, including combinatorial auctions, achieving exact optima is computationally intractable, and one must settle for heuristics or approximations. A key clash between the strategic and algorithmic considerations is that once only approximations or heuristics are chosen, the VCG payment rule no longer leads to incentive compatibility. See, e.g., [20, 24].

The heart of "algorithmic mechanism design" is trying to overcome this clash: design mechanisms that are both computationally tractable (and thus only approximate the optimum) and strategically incentive compatible. The key question is always to what extent do the strategic requirements degrade the quality of the solution beyond the degradation implied by the purely computational constraints. At this point one can summarize the state of the art as having clear characterizations and many good upper bounds (approximation mechanisms) for various classes of "single parameter" problems (e.g. [20, 22, 1]). On the other hand, for more general "multi-parameter" problems like combinatorial auctions, one can safely say that – almost – no progress has been made in obtaining upper bounds or lower bounds. In this paper we wish to address exactly this issue, attempting to prove lower bounds – showing that the strategic constraints do impose an additional burden beyond the computational ones.

1.2 VCG-based mechanisms

We are unable to prove general lower bounds, so we limit ourselves to a natural – and interesting in itself – class of mechanisms: VCG-based ones. These are the ones that use the same payment idea as in VCG: each bidder i pays $h_i(v_{-i}) - \Sigma_{j \neq i} v_j(a)$, where a is the algorithmic output). This is a rather natural choice given the complete lack of other techniques for multi-parameter settings. This class was studied in [24], who observed that the following family of allocation algorithms do yield incentive-compatible VCG-based mechanisms:

Definition: An allocation algorithm (that produces an output $a \in \mathcal{A}$ for each input $v_1 ... v_n$, where $\mathcal{A}$ is the set of possible alternatives) is called "maximal-in-range" (henceforth MIR) if it completely optimizes the social welfare over some subrange $\mathcal{R} \subseteq \mathcal{A}$. I.e., for some $\mathcal{R} \subseteq \mathcal{A}$, we have that for all $v_1 ... v_n$, $a \in \arg\max_{a \in \mathcal{R}} \Sigma_i v_i(a)$.

I.e., MIR algorithms use the following natural and simple strategy to find an approximately optimal solution: they just optimally search within a pre-specified sub-range of feasible solutions – a subrange over which optimal search is algorithmically feasible. An example for such a strategy, in a different context, is approximating the optimal Steiner tree by taking the best spanning tree [28]. Another example is Arora's approximation for the traveling salesman problem [2].

The main result of [24] states that this is essentially it – no other VCG-based mechanisms are incentive compatible.

Theorem [24]: The allocation algorithm of any incentive-compatible VCG-based mechanism for combinatorial auctions is equivalent to a maximal-in-range algorithm.

"Equivalent" here means that the social utilities are identical for all inputs, i.e. if a and b are the outputs of the two allocation algorithms for input $v_1 ... v_n$ then $\Sigma_i v_i(a) = \Sigma_i v_i(b)$. In particular, the outputs must coincide generically – except perhaps in case of ties. In [24] this theorem was presented for two specific types of mechanism design problems, but the result is more general. For completeness, we spell this out in Section 3.

In this paper, we prove bounds on the power of such MIR algorithms, equivalently, for VCG-based mechanisms. Let us explicitly justify why are lower bounds for this restricted class interesting:

1. **Their power:** Many incentive compatible mechanisms put forward do fall into this category. This includes a slightly non-trivial $O(m/\sqrt{\log m})$-approximation for general combinatorial auctions [14], an $O(\sqrt{m})$-approximation for combinatorial auctions with complement-free bidders [9], a 2-approximation for multi-unit auctions, which is improved to a PTAS for certain bidding languages [7], welfare maximization in congestion games [5], and several auctions for "geometric figures" on the plane [3].

2. **Lack of alternatives:** Not only are VCG mechanisms the only general method known for constructing incentive compatible mechanisms in multi-parameter settings, there is just a single additional example of any non-VCG incentive compatible mechanism in any multi-parameter domain [4]. Moreover, Roberts' classic theorem [27] states that in completely unrestricted domains the *only* incentive compatible mechanisms are weighted versions of VCG-based mechanisms. In [17], it was suggested that Roberts' theorem could be extended to many other domains including combinatorial auctions and multi-unit auctions. In [17], they were only able to prove this for special cases or under additional assumptions, and left the general question open. If this line of attack reaches its conclusion, then our lower bounds would apply in general.

We should note that if randomization is allowed then the second point no longer holds with such force, as several *randomized* incentive compatible mechanisms are known [8, 18, 11]. It is not known, however, whether any of these can be de-randomized (even if P=BPP). The resolution of this question relies, of course, on the successful characterization of deterministic incentive compatible mechanisms.

1.3 Our Main Result

Our main result provides a lower bound for the approximation factor that can be achieved by incentive-compatible VCG-based mechanisms for combinatorial auctions. Our lower bound applies to the subclass of submodular valuations ($v_i(S \cup T) + v_i(S \cap T) \leq v_i(S) + V_i(T)$ for all S, T) and thus also to its superset class of complement-free valuations ($v_i(S \cup T) \leq v_i(S) + v_i(T)$ for all S, T) – two classes of valuations which have been extensively studied [19, 9, 11, 10, 12, 15]. The best deterministic mechanism for this case is the VCG-based $O(\sqrt{m})$-mechanism of [9][2]. We note that the technique we present is quite general, and we believe it will turn even more useful by extending our results to other domains as well.

A word about the computational model is in place here: the "inputs" to the mechanism, the v_i's, are exponential sized objects (in the number of items m), but the mechanisms should run in time polynomial in n and m. Thus it is assumed that the mechanism repeatedly queries the bidders. The upper bounds in the table always assume some specific natural type of query (usually a "demand query"), while all lower bounds apply for every type of query and are in fact communication lower bounds.

Theorem: Every VCG-based mechanism for approximating the welfare in combinatorial auctions with submodular bidders that uses a sub-exponential number of queries to the bidders achieves an approximation factor of $\Omega(\min(n, m^{1/6}))$.

The proof proceeds by combinatorially analyzing maximum in range allocation algorithms. The analysis shows that if the range is "large" then optimizing over it requires exponential communication, while if it is "small" then it can not achieve a good approximation ratio. It turns out that "large"

[2]This is also the best incentive-compatible mechanism for subadditive bidders in general, as the *randomized* $O(\log^2 m)$-approximation mechanism of [8] does not apply to the class of complement-free bidders.

and "small" in this sense cannot just be interpreted in terms of the size of the range. Instead we define two "complexity measures" of a set of partitions (which is what the range is). One of them, termed the intersection number, is shown to bound from below the communication complexity of optimization over the range. The other, termed the cover number, is shown to bound from above the approximation ratio achieved by allocations in the range. Our main combinatorial lemma, which may be of independent interest, shows that these two complexity measures are related to each other.

We stress that although communication complexity methods were already used in [23, 26] in the context of combinatorial auctions, our methods are completely different and this difference is inherent: [23, 26] did not consider incentives issues at all.

1.4 Open Problems

Open problems are left at various levels of generality. At the most specific level, the problem is to close the gap between the $m^{1/6}$ lower bound and the $m^{1/2}$ upper bound. (We can only improve the lower bound to $m^{1/5-\epsilon}$.) At the next level, the question is how well can VCG-based mechanisms approximate the social welfare in combinatorial auctions with general valuations? We only have a slightly better lower bound than what we have for the submodular case, but the upper bound of [14] is nearly linear.

Of course, the real questions are always how well can arbitrary computationally efficient incentive compatible mechanism do – not just VCG-based ones – and obtaining any such lower bound would be of great interest. This would likely require some advances in the "LMN-program" [17] of characterizing incentive compatible mechanisms in multi-parameter domains.

1.5 Paper Structure

In Section 2 we prove the main theorem. Section 3 brings the characterization of VCG-based mechanisms of [24], generalized for our setting.

2 Combinatorial Auctions with Submodular Bidders

2.1 Combinatorial Auctions: Preliminaries

In a combinatorial auction we have a set M, $|M| = m$, of heterogeneous items and a set of N bidders, $|N| = n$. Each bidder i has a valuation function $v_i : 2^M \rightarrow \mathbb{R}$. We assume that each valuation v_i is normalized (i.e., $v_i(\emptyset) = 0$) and monotone (for each $S \subseteq T$, $v_i(S) \leq v_i(T)$). An *allocation* is an n-tuple $S = (S_1, ..., S_n)$, where for each i, $S_i \subseteq M$, and for each $i \neq i'$, $S_i \cap S_{i'} = \emptyset$. Our goal is to find an allocation S that maximizes the welfare $\Sigma_i v_i(S_i)$.

A valuation v is said to be *submodular* if it exhibits decreasing marginal utilities. I.e., for each $S \subseteq T \subseteq M$ and $j \notin S$, we have that $v_i(T \cup \{j\}) - v_i(T) \leq v_i(S \cup \{j\}) - v_i(S)$. We will also use a very simple subset of submodular valuations called additive valuations. A valuation v is said to be *additive* if for each $S \subseteq M$, we have that $v(S) = \Sigma_{j \in S} v(\{j\})$.

2.2 The Main Result

In this section we analyze the power of MIR algorithms in the context of combinatorial auctions with submodular bidders. For this setting, an $O(\sqrt{m})$-approximation MIR algorithm is known [9]. We will show that this is (almost) the best approximation one can get using MIR algorithms. The theorem is stated only for MIR algorithms but we will point out how it can be extended to algorithms that are *equivalent* to MIR algorithms, and thus to all VCG-based mechanisms [3].

Theorem 2.1 *Every MIR mechanism for approximating the welfare in combinatorial auctions with submodular bidders that uses* $O(e^{m^{\frac{1}{15}}})$ *bits of communication achieves an approximation factor of* $\Omega(\min(n, m^{1/6}))$*. This result also holds for the randomized and non-deterministic settings.*

We define two complexity measures for the range $\mathcal{R}$ of an MIR algorithm A: the cover number, and the intersection number. The cover number roughly corresponds to the size of the range $\mathcal{R}$. We will show, using the probabilistic method, that if the cover number is "small" then there exists an instance such that A fails to provide a good approximation. Therefore, the range $\mathcal{R}$ must be "large". In this case we will show that the intersection number of A must be exponential. We will see that the intersection number serves as a lower bound to the communication complexity of A, and so we get that any MIR-approximation algorithm that provides a good approximation ratio must have exponential communication complexity.

The proof of the theorem starts with Subsection 2.3, where the cover number is formally defined and its relation to the approximation ratio is shown. In Subsection 2.4 we define and discuss the second measure: the intersection number. The proof concludes in Subsection 2.5 by showing the relationship between the measures.

2.3 Complexity Measure I: The Cover Number

Intuitively we wish to rely on the size of the range. Yet, naive counting will fail to provide good results, since a single allocation in the range may contain many "degenerate allocations". For example, if the range contains an allocation that assigns all items to some bidder i, it also contains all allocations such that i is assigned any subset of the items, and the rest of the bidders get nothing. These exponentially many allocations are degenerate in the sense that we can assume that they are not in the range of the algorithm without changing the guaranteed approximation ratio of the A. We therefore use an alternative measure for describing the "size" of the range.

Definition 2.2 *A set* $\mathcal{C}$ *of allocations is said to be a* cover set *of another set of allocations* $\mathcal{R}$ *if for each* $S \in \mathcal{R}$ *there exists some* $C \in \mathcal{C}$ *such that for all* i*,* $S_i \subseteq C_i$*.*

The cover number *of a set of allocations* $\mathcal{R}$ *is defined to be the size of the minimum cardinality cover set of* $\mathcal{R}$*. The cover number is denoted by* $cover(\mathcal{R})$*.*

In the next lemma we prove that if $cover(\mathcal{R})$ is small, then there exists some instance in which A provides only $\Omega(n)$-approximation.

Lemma 2.3 *Let* A *be an MIR-algorithm with range* $\mathcal{R}$*. If* $cover(\mathcal{R}) < e^{\frac{m}{300n}}$ *then there is an instance in which* A *provides no more than* $\frac{1.01}{n}$*-fraction of the welfare.*

Proof: We randomly construct an instance of a combinatorial auction with additive valuations. Since the valuations are additive, we only need to specify the value of $v_i(\{j\})$ for each bidder i and item j. This is done in the following way: for each item $j \in M$ choose exactly one bidder, where each bidder is selected with probability of exactly $\frac{1}{n}$. Let i be the selected bidder. We set the value of $v_i(\{j\})$ to be 1. For each $i' \neq i$ we set the value of $v_{i'}(\{j\})$ to be 0.

[3] We note that extending our results to *weighted* VCG-based mechanisms is straightforward.

First, observe that the value of the optimal solution in the random instance is exactly m. Nevertheless we will see that with non-negative probability the welfare provided by the MIR-algorithm A is only $\frac{1.01}{n}m$. Hence, the approximation ratio provided by A is no better than $\frac{n}{1.01}$. The following version of the Chernoff bounds will be useful.

Claim 2.4 (Chernoff bound) *Let $X_1, ...X_m$ be independent random variables that take values in $\{0,1\}$, such that for all i, $\Pr[X_i = 1] = p$ for some p. Then for every $0 \leq \delta \leq 2e - 1$ it holds that:*

$$\Pr[\Sigma_i X_i > (1+\delta)pm] \leq e^{-\frac{pm\delta^2}{3}}$$

Let $\mathcal{C}$ be the minimum cardinality cover set of $\mathcal{R}$ with $|\mathcal{C}| = cover(\mathcal{R})$. Fix some $C \in \mathcal{C}$. The probability that $v_i(\{j\}) = 1$, and that $j \in C_i$ is exactly $\frac{1}{n}$, for any bidder i and item j. By the Chernoff bound, $\Pr[\Sigma_i v_i(C_i) > \frac{1+\delta}{n}m] \leq e^{-\frac{\delta^2 m}{3n}}$. We now claim, by using the union bound, that if $cover(\mathcal{R}) < e^{\frac{\delta^2 m}{3n}}$ then there exists some instance such that no allocation in $\mathcal{C}$ provides a welfare of more than $\frac{1+\delta}{n}m$. Therefore it is obvious that no allocation in $\mathcal{R}$ can provide a welfare of more than $(\frac{1+\delta}{n})m$ for this instance. The lemma follows by choosing $\delta = .01$. □

2.4 Complexity Measure II: The Intersection Number

The second complexity measure we consider is the intersection number. We will show that the intersection number of the range of an MIR algorithm is a lower bound to its communication complexity. Before defining the intersection number, we need a structural definition of a set of allocations.

Definition 2.5 *We say that a set of allocations $\mathcal{R}$ is* regular *if there exist constants $s_1, ..., s_n$ such that for all $S \in \mathcal{R}$ and for all $1 \leq i \leq n$ it holds that $|S_i| = s_i$.*

We are now ready to define the complexity measure itself.

Definition 2.6 *A set of allocations $\mathcal{D}$ is called an (i,j)-*intersection set *if for each $D, D' \in \mathcal{D}$, $D \neq D'$, it holds that $D_i \cap D'_j \neq \emptyset$.*

Define the intersection number *of a set of allocations $\mathcal{R}$, denoted by $intersect(\mathcal{R})$, to be the maximum cardinality regular (i,j)-intersecting set which is a subset of $\mathcal{R}$, where the maximum is taken over all pairs of bidders i and j.*

Notice that in the definition of the intersection number we require that the intersection set will be regular.

The next lemma shows that we can use the intersection number as a lower bound to the communication complexity of the algorithm.

Lemma 2.7 *Let A be an MIR-algorithm for combinatorial auctions with submodular bidders with range $\mathcal{R}$. Let $intersect(\mathcal{R}) = d$. Then, the communication complexity of A is $\Omega(d)$. This result holds even for randomized protocols and for non-deterministic protocols.*

Proof: We reduce from the disjointness problem (see [16]). In this problem Alice holds a d-bit string $a_1, ..., a_d$, and Bob holds a d-bit string $b_1, ..., b_d$. The goal is to decide whether there exists some index k such that $a_k = b_k = 1$. It is known that solving the disjointness problem requires $\Omega(d)$ communication bits, even for nondeterministic and randomized protocols.

Let $\mathcal{D} = \{D^1, ..., D^d\}$ be an (i, j)-intersection set of $\mathcal{R}$. $\mathcal{D}$ is regular, so for each bidder t there exists a constant s_t such that $|D_t| = s_t$, for all $D \in \mathcal{D}$. Construct a combinatorial auction with m items in the following way: Alice will play the role of bidder i, and Bob will play the role of the rest of the bidders, in particular bidder j. We now define the valuations of the bidders. Let the valuation of bidder i played by Alice be:

$$v_i(S) = \begin{cases} |S|, & |S| \leq s_i - 1; \\ s_i, & \exists k \text{ s.t. } D_i^k \subseteq S \text{ and } a_k = 1; \\ s_i - 2^{-(|S|-s_i+1)}, & \text{otherwise.} \end{cases}$$

The valuation v_j is defined in an analogous way. Let the valuations of the rest of the bidders be zero on any bundle. The reader is encouraged to verify that all valuations are indeed submodular.

Observe that if there exists some index k such that the k'th input bit of both players is 1, then the optimal welfare is $s_i + s_j$. Otherwise, the optimal welfare is strictly less than $s_i + s_j$. To see this notice that if bidder i gains a value of s_i from the bundle S_1 he was assigned by A, then there must be an index k such that $D_i^k \subseteq S_1$ and $a_k = 1$. In order of bidder j to gain a value of s_j he must have an index k' such that $D_j^{k'} \subseteq S_2$. However, $\mathcal{D}$ is an (i, j)-intersection set and so it must hold that $D_i^k \cap D_j^{k'} \neq \emptyset$, and thus $S_1 \cap S_2 \neq \emptyset$. Clearly, the optimal welfare in this case is less than $s_i + s_j$.

By construction, if the optimal welfare is $s_i + s_j$ then it can be achieved by an allocation in $\mathcal{R}$. A is a maximal-in-range algorithm, and so the value of the allocation returned by A in this case must be $s_i + s_j$. Thus, we will be able to decide if there is a some index k such that $a_k = b_k = 1$. Hence, the communication complexity of A is at least as that of the disjointness problem: $\Omega(d)$. □

Notice that our lower bound applies even for computing the *value* of the optimal allocation in $\mathcal{R}$, and thus applies not only to MIR algorithms but also to algorithms that are equivalent to MIR.

2.5 The Relationship between the Measures

It is easy to see that $cover(\mathcal{R}) \geq intersect(\mathcal{R})$. This subsection shows that the gap between the two is not too large. Specifically, if $intersect(\mathcal{R})$ is small, then $cover(\mathcal{R})$ is small too.

Lemma 2.8 *Let $\mathcal{R}$ be a set of allocations with $intersect(\mathcal{R}) \leq d$. Then*

$$cover(\mathcal{R}) < (8d)^{m^{\frac{3}{5}}n} \cdot m^{4m^{\frac{2}{5}}n^2}$$

As a corollary[4], let $n = m^{\frac{1}{6}}$. If $cover(\mathcal{R}) > e^{\frac{m}{300n}}$ then $intersect(\mathcal{R}) \geq e^{m^{\frac{1}{15}}}$. Thus, proving the lemma, together with Lemmas 2.3 and 2.7, derives Theorem 2.1.

Proof: (of Lemma 2.8) The lemma will follow from the following claim.

Claim 2.9 *Fix some w, $1 \leq w \leq m$. Let $\widetilde{\mathcal{R}}$ be a regular set of allocations. If $intersect(\widetilde{\mathcal{R}}) \leq d$ then there is a subset $\mathcal{E}$ of $\widetilde{\mathcal{R}}$ where $\frac{|\mathcal{E}|}{|\widetilde{\mathcal{R}}|} \geq (8d)^{-mn/w} 4^{-n^2}$, and $cover(\mathcal{E}) \leq w^n m^{wn^2}$.*

The lemma is proved by partitioning $\mathcal{R}$ to up to m^n classes of regular allocations, $\mathcal{R}_1, ..., \mathcal{R}_{m^n}$, one for each possible choice of constants $s_1, ..., s_n$ from Definition 2.5. Each s_i is between 1 and m, so there are at most m^n classes. For each class $\mathcal{R}_s$ we will set an upper bound on $cover(\mathcal{R}_s)$ separately:

Let $\mathcal{E}_1^s$ be the set obtained from the claim. Look at $\mathcal{R}_s \setminus \mathcal{E}_1^s$, and obtain from the claim another set $\mathcal{E}_2^s \subseteq \mathcal{R}_s \setminus \mathcal{E}_1^s$ with small cover, and so on. After $(8d)^{\frac{mn}{w}} \cdot 4^{n^2} \cdot \log |\mathcal{R}_s|$ steps $\mathcal{R}_s$ is completely

[4]The result is actually stronger: fix a constant $\epsilon > 0$, and let $n < m^{\frac{1}{5}-\epsilon}$. If $cover(\mathcal{R}) > e^{\frac{m}{300n}}$ then $intersect(\mathcal{R}) \geq e^{m^{\epsilon}}$. The statement of the theorem improves accordingly.

covered. Now $cover(\mathcal{R}_s)$ can be bounded from above by $\Sigma_k cover(\mathcal{E}^s_k)$. By bounding from above the size of each class $|\mathcal{R}_s|$ by $|\mathcal{R}| \leq n^m$, we have that (by choosing $w = m^{\frac{2}{5}}$):

$$cover(\mathcal{R}) \leq \Sigma_{a=1}^{m^n} cover(\mathcal{R}_s) \leq \Sigma_{a=1}^{m^n} \Sigma_k |\mathcal{E}^s_k| \leq m^n \cdot (8d)^{\frac{mn}{w}} \cdot 4^{n^2} \cdot m \log n \cdot w^n m^{wn^2} \leq (8d)^{m^{\frac{3}{5}n}} \cdot m^{4m^{\frac{2}{5}}n^2}$$

Before proving Claim 2.9 itself, and thus Lemma 2.8, we will need some notation.

Definition 2.10 *Let $T_1, ..., T_n \subseteq M$. We say that a set of allocations $\mathcal{R}$ is $(T_1, ..., T_n)$-*structured *if for all $S \in \mathcal{R}$ it holds that $S_i \subseteq T_i$.*

Definition 2.11 *We will say that an allocation S is w-(i, j)-*aligned *in structure $(T_1, ..., T_n)$, if $|S_i \cap T_j| \leq w$. We will omit w when it will be clear from the context.*

The idea in proving Claim 2.9 will be to find a large subset $\mathcal{E} \subseteq \widetilde{\mathcal{R}}$, which is "sufficiently" aligned. Next we show that such subset has a small cover number.

Claim 2.12 *Let $\mathcal{E}$ be a $T = (T_1, ..., T_n)$-structured set of allocations. If for each pair of bidders i and j either all allocations in $\mathcal{E}$ are w-(i, j)-aligned in T or all allocations in $\mathcal{E}$ are w-(j, i)-aligned in T, then $cover(\mathcal{E}) \leq w^n m^{n^2 w}$.*

Proof: (of Claim 2.12) For each bidder i define a set of bidders I_i, where bidder j is in I_i if all allocations in $\mathcal{E}$ are (i, j)-aligned in T. Clearly, for each i and j, either $j \in I_i$ or $i \in I_j$. Let $B_i = T_i \setminus (\cup_{j \in I_i} T_j)$. The construction guarantees that $(B_1, ..., B_n)$ "almost" covers $\mathcal{E}$ in the sense that for bidder i and $S \in \mathcal{E}$, $|S_i \setminus B_i| \leq nw$. Also notice that by construction for each two different bidders i and j, $B_i \cap B_j = \emptyset$. Define the cover $\mathcal{C}$ as follows:

$$\mathcal{C} = \{P | P \text{ is an allocation in the form } ((B_1 \cup Q_1) \setminus \cup_{j \neq 1} Q_j, ..., (B_n \cup Q_n) \setminus \cup_{j \neq n} Q_j), \text{ and for each } i,\ |Q_i| \leq nw\}$$

Observe that since each $|Q_i| \leq nw$ we have that $|\mathcal{C}| \leq (\Sigma_{r=1}^{w} r\binom{m}{nr})^n \leq (w(m)^{nw})^n = w^n m^{n^2 w}$. Also notice that $\mathcal{C}$ is a cover set of $\mathcal{E}$. To see this, fix an allocation $S \in \mathcal{E}$. For each i, let $Q_i = S_i \setminus B_i$. Observe that each $|Q_i| \leq nw$, and that the Q_i's define an allocation that is in $\mathcal{C}$ and covers S. □

Now we are ready to prove the main claim, and thus finish the proof of Lemma 2.8.

Proof: (of Claim 2.9) The construction of $\mathcal{E}$ will be divided into several steps. During the construction we maintain a sequence of subsets of $\widetilde{\mathcal{R}} : \mathcal{R}_0, \mathcal{R}_1, ...$ and structures T^0, T^1, ..., such that each $\mathcal{R}_t$ is T^t-structured. We start by setting $\mathcal{R}_0 = \widetilde{\mathcal{R}}$ and $T^0 = (M, ..., M)$.

In each step we look at a pair of bidders i and j such that either all allocations in $\mathcal{R}_t$ are (i, j)-aligned in T^t or all allocations in $\mathcal{E}$ are (j, i)-aligned in T^t. If there is no such pair then let $\mathcal{E} = \mathcal{R}_t$ and the construction is over. Otherwise, look at all allocations in $\mathcal{R}_t$ that are either (i, j)-aligned or (j, i)-aligned in T^t. If there are at least $|\mathcal{R}_t|/2$ such allocations then we set $\mathcal{R}_{t+1}$ to be the largest set of the two: all allocations in $\mathcal{R}_t$ that are (i, j)-aligned, or all allocations in $\mathcal{R}_t$ that are (j, i)-aligned. Set the structure T^{t+1} to be T^t. Notice that $|\mathcal{R}_{t+1}| \geq |\mathcal{R}_t|/4$, and that $\mathcal{R}_{t+1}$ is T^{t+1}-structured. We call this step an *alignment* step, and proceed to the next step.

Otherwise, let $\mathcal{R}'_t$ be the set of allocations in $\mathcal{R}_t$ that are neither (i, j)-aligned nor (j, i)-aligned. Notice that $|\mathcal{R}'_t \geq \frac{\mathcal{R}_t}{2}|$. Take a maximal (i, j)-intersection set $\mathcal{D} \subseteq \mathcal{R}'_t$ – of size at most d. Now for every allocation $S \in \mathcal{R}'_t \setminus \mathcal{D}$ there exists some $D \in \mathcal{D}$ such that $D_i \cap S_j = \emptyset$ or $D_j \cap S_i = \emptyset$. Otherwise we have that $S \in \mathcal{D}$, contradicting the fact that $\mathcal{D}$ is a maximal intersection set. Thus, for some $D \in \mathcal{D}$ we have that for at least $(|\mathcal{R}'_t| - d)/(2d)$ allocations in $\mathcal{R}'_t$ either $D_i \cap S_j = \emptyset$ or

$D_j \cap S_i = \emptyset$. Let us assume that for at least $(|\mathcal{R}'_t| - d)/(2d)$ allocations in $\mathcal{R}'_t$ the first option occurs. Define $\mathcal{R}_{t+1}$ to be this set of $(|\mathcal{R}'_t| - d)/(2d) \geq |\mathcal{R}_t|/(8d)$ allocations. Let $T_j^{t+1} = T_j^t \setminus D_i$. Also let $T_k^{t+1} = T_k^t$, for each $k \neq i$. Now notice that since $\mathcal{D}$ is a set of allocations that are not (i,j)-aligned in T_t, we have that $D_i \cap T_j^t > w$. We therefore have that $|T_j^{t+1}| < |T_j^t| - w$. (The other case is handled similarly, but this time by shrinking T_i^{t+1} rather than T_j^{t+1}.) By construction we have that $\mathcal{R}_{t+1}$ is T^{t+1}-structured. Term this step a *shrinkage* step, and continue to the next step.

Denote by l the number of steps the process went on. At most $\frac{nm}{w}$ steps are shrinkage steps, since in each shrinkage step $\Sigma_i |T_i^t|$ loses an additive of at least w. In addition, there are at most $\binom{n}{2}$ alignment steps, one for each pair of bidders. Therefore $|\mathcal{E}| = |\mathcal{R}_l| \geq \frac{|\widetilde{\mathcal{R}}|}{(8d)^{mn/w} 4^{\binom{n}{2}}}$. Also note that in the end of the process for each pair of bidders i and j either all allocations in $\mathcal{E}$ are (i,j)-aligned in T^l or all allocations in $\mathcal{E}$ are (j,i)-aligned in T^l (observe that an allocation that became properly aligned after an alignment step will remain so during the rest of the process.) By Claim 2.12 we have that $cover(E) \leq w^n m^{n^2 w}$, and thus Claim 2.9 is proved. □

This concludes the proof of Lemma 2.8. □

3 A Characterization of VCG-Based Algorithms

In [24] it was proved that any VCG-based mechanism for general combinatorial auctions is equivalent to MIR algorithm. We slightly generalize this proof to hold for more settings, including the one considered in this paper.

Let $\mathcal{A}$ be a set of alternatives (in our application, $\mathcal{A}$ will be the set of allocations). For all i let $V_i \subseteq R^{\mathcal{A}}$ be a set of valuations on $\mathcal{A}$ and denote $V = V_1 \times ... \times V_n$. A mechanism is composed of an allocation rule $f : V \to \mathcal{A}$ and payment rules $p = (p_1, ..., p_n)$, where $p_i : V \to \mathbb{R}$.

Definition 3.1 *A mechanism (f,p) is called VCG-based if for every i and some $h_i : V_{-i} \to \mathbb{R}$ we have that for all v, $p_i(v) = h_i(v_{-i}) - \Sigma_{j \neq i} v_j(f(v))$.*

Definition 3.2 *A mechanism (f,p) is called incentive compatible if for every v_i, v'_i, v_{-i} we have that $v_i(f(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq v_i(f(v'_i, v_{-i}) - p_i(v'_i, v_{-i})$.*

Definition 3.3 *An allocation rule f is called maximal in its range (MIR) if for every v, $f(v) \in \arg\max_{r \in \mathcal{R}} \Sigma v_i(r)$, where $\mathcal{R} = \{f(v)|v \in V\}$ is the range of f.*

Definition 3.4 *An allocation rule f is equivalent to an allocation rule g if for all v, $\Sigma_i v_i(f(v)) = \Sigma_i v_i(g(v))$.*

Theorem 3.5 (slight extension of [24]) *: Assume that V satisfies Condition 1 and Condition 2 below. If a mechanism (f,p) is VCG-based and incentive compatible then f is equivalent to a MIR allocation rule.*

For Condition 1 and Condition 2 we will need notations:

Definition 3.6 $V' = \{v \in V | \forall a \neq b \in A,\ \Sigma_i v_i(a) \neq \Sigma_i v_i(b)\}$.

Condition 1 *V' is dense in V (in the usual metric in $R^{\mathcal{A}}$).*

Definition 3.7 *For $a \in \mathcal{A}$ and $v_i, z_i \in V_i$, We say that z_i is a-above v_i if for every $b \in \mathcal{A}$, $z_i(a) - v_i(a) \geq z_i(b) - v_i(b)$.*

Condition 2 *For every $v_i, w_i \in V_i$ there exists $z_i \in V_i$ that is a-above v_i and a-above w_i.*

Before we prove the theorem, let us just look at the two applications, one considered in this paper, and one in [7]:

1. **Multi-unit auctions**: (see [7] for an exact definition) $\mathcal{A} = \{(a_1 ... a_n) | \Sigma_i a_i \leq m\}$, V_i is all weakly monotone functions from $1...m$ to $\mathbb{R}$. Condition 1 is met since V' has measure 0. Condition 2 is met by giving a sufficiently high value q to getting at least a_i items.

2. **Combinatorial auctions with submodular bidders:** $\mathcal{A}$ is the set of all allocations $(S_1, ..., S_n)$, and v_i is the set of submodular valuations. Condition 1 is met since again V' has zero measure while V is fully dimensional. Condition 2 is met by defining an additive valuation (which in particular is submodular) that gives a sufficiently high value for each element in S_i.

Proof: Let us denote $\mathcal{R}' = \{f(v) | v \in V'\}$. Notice that by definition $\Sigma_i v_i(a) \neq \Sigma_i v_i(b)$ for every $v \in V'$ and $a \neq b \in \mathcal{R}'$ and in particular the argmax is unique. We will follow [24] and first show that over V', f is exactly maximal in the range $\mathcal{R}'$. I.e. that for all $v \in V'$, $f(v) = \arg\max_{r \in \mathcal{R}'} \Sigma v_i(r)$. Let us assume wlog that all $h_i = 0$. Before proceeding with the proof let us note two simple claims:

Claim 3.8 *If $f(w) = a$ and z_i is a-above w_i then $f(z_i, w_{-i}) = a$.*

Proof: Assume to the contrary $f(z_i, w_{-i}) = b \neq a$. Since the VCG mechanism based on f is incentive compatible, we get by looking at a player with valuation w_i that $w_i(a) + \Sigma_{j \neq i} w_j(a) \geq w_i(b) + \Sigma_{j \neq i} w_j(b)$ while by looking at a player with valuation z_i we get $z_i(a) + \Sigma_{j \neq i} w_j(a) \leq z_i(b) + \Sigma_{j \neq i} w_j(b)$. Subtracting the two inequalities we get $w_i(a) - z_i(a) \geq w_i(b) - z_i(b)$ but notice that the fact that z_i is a-above w_i gives the opposite inequality which means that in fact $w_i(a) - z_i(a) = w_i(b) - z_i(b)$. Adding this equality to the second inequality above gives $w_i(a) + \Sigma_{j \neq i} w_j(a) \leq w_i(b) + \Sigma_{j \neq i} w_j(b)$, and thus equality holds in contradiction to w being in V'. □

Claim 3.9 *If $f(v_i, v_{-i}) \neq a = \arg\max_{c \in \mathcal{R}'} \Sigma_i v_i(c)$ and z_i is a-above v_i then $f(z_i, v_{-i}) \neq a = \arg\max_{c \in \mathcal{R}'} z_i(c) + \Sigma_{j \neq i} v_j(c)$.*

Proof: The fact that $a = \arg\max_{c \in \mathcal{R}'} z_i(c) + \Sigma_{j \neq i} v_j(c)$ is simply since in moving from v_i to z_i, the value of the argument to the argmax increased more for a than for any other $c \in A$. The fact that $f(z_i, v_{-i}) \neq a$ is since otherwise a bidder with valuation v_i will benifit from reporting z_i. □

We are now ready to prove that f is exactly maximal in the range $\mathcal{R}'$. Assume towards contradiction that for $v, w \in V'$, $f(v) = b \neq a = \arg\max_{c \in \mathcal{R}'} \Sigma_i v_i(c)$, and $f(w) = a$. For every i fix z_i that is a-above both v_i and w_i (using Condition 2). Using Claim 3.8 repeatedly, for all i, we get that $f(z) = a$ (at every stage i we look at $z_1 ... z_{i-1}, w_i, w_{i+1} ... w_n$ vs $z_1 ... z_{i-1}, z_i, w_{i+1} ... w_n$.) On the other hand, using Claim 3.9 repeatedly we get that $f(y) \neq a$ (at every stage i we look at $z_1 ... z_{i-1}, v_i, v_{i+1} ... v_n$ vs $z_1 ... z_{i-1}, z_i, v_{i+1} ... v_n$). Contradiction.

We now need to handle $V - V'$. Due to Condition 2, for every $v \in V - V'$ we can find an infinite sequence of points $v^j \in V'$ such that $v^j \to v$ and for all j, $f(v^j) = a$ for some fixed $a \in \mathcal{R}'$. Our equivalent MIR allocation rule f' will define $f'(v) = a$ (using e.g. the lexicographic first possible a). It remains to see that $\Sigma_i v_i(a) = \Sigma_i v_i(f(v))$. This follows since (1) $\Sigma_i v_i(a) = \lim_{j \to \inf} \Sigma_i v_i^j(a)$ (simply since $v^j \to v$) and (2) $\Sigma_i v_i(f(v)) = \lim_{j \to \inf} \Sigma_i v_i^j(f(v^j))$. This last equality just means the continuity of the function $\Sigma_i v_i(f(v))$ in v and can be established by looking at $|\Sigma_i v_i(f(v)) - \Sigma_i w_i(f(w))|$ which can be bounded by a telescopic sum of n elements of a similar form but with only a single index i with $v_i \neq w_i$, i.e. $|(v_i(a) + \Sigma_{j \neq i} v_j(a)) - (w_i(b) + \Sigma_{j \neq i} v_j(b))|$, where $a = f(v_i, v_{-i})$ and $b = f(w_i, v_{-i})$.

This last difference can be bounded by $\max(v_i(a) - w_i(a), v_i(b) - w_i(b))$ since otherwise a player with valuation v_i would rather declare w_i (in case the LHS of the difference is smaller), or vice versa (otherwise). $\square$

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