Reasoning About Others: Representing and Processing Infinite Belief Hierarchies

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Abstract

In this paper we focus on the problem of how infinite belief hierarchies can be represented and reasoned with in a computationally tractable way. When modeling nested beliefs one usually deals with two types of infinity: infinity of beliefs on every level of reflection and infinity of levels. In this work we assume that beliefs are finite at every level, while the number of levels may still be infinite. We propose a method for reducing the infinite regress of beliefs to a finite structure. We identify the class of infinite belief trees that allow finite representation. We propose a method for deciding on an action based on this presentation. We apply the method to the analysis of auctions. We prove that if the agents' prior beliefs are not common knowledge, the revenue equivalence theorem ceases to hold. That is, different auctions yield different expected revenue. Our method can be used to design better auction protocols, given the participants' belief structures.

1. Introduction

Reasoning about others and interactive knowledge have been the subject of continuous interest in multiagent systems [11,12,13,20], artificial intelligence [6,7,8] and game theory [1,3,15]. In multiagent interaction, where an agent's action interferes with other agents' actions, hierarchies of beliefs arise in an essential way. Usually an agent's optimal decision depends on what he believes the other agents will do, which in turn depends on what he believes the other agents believe about him, and so on. An infinite regress of this kind gives rise to several important issues. The first issue concerns the methods for representing infinitely nested beliefs. In order to maintain and update such beliefs, agents need some finite and computationally tractable way to represent them. The second issue that deserves consideration is the feasibility of decision making based on infinitely nested beliefs.

Finite hierarchies of beliefs have been studied by Gmytrasiewicz, Durfee and Vidal [11,12,13, 20]. The main advantage of their recursive modeling method is that a solution can always be derived. The recursive modeling method is based on the assumption that once an agent has run out of information his belief hierarchy can be cut at the point where there is no sufficient information. At the point of cutting, absence of information is represented by a uniform distribution over the space of all possible states of the world. The absence of information with which to model other agents implies that belief hierarchies are potentially finite. In our work we take a different approach. We suppose that every belief hierarchy is potentially infinite and we consider the problem of how such a hierarchy can be represented using a finite structure and manipulated in a computationally tractable way. We show that some infinite belief trees allow finite representation in the form of pointed accessible graphs.

The most typical and the most studied example of infinite belief hierarchies is *common knowledge* [4,7,10]. Common knowledge means that everyone knows that everyone knows that… Common knowledge, however, is a very special case of a belief hierarchy, namely, a hierarchy that consists of infinite repetition of some event. Such a hierarchy can be reduced to a finite representation if we treat all the repetitions in the hierarchy as identical. In this paper we extend this idea to infinite belief hierarchies with more complex structure.

The research presented in this paper is closely related to the work in game theory devoted to games with incomplete information [9]. In our work, the epistemic state of each agent is modeled as an infinite hierarchy of beliefs. Harsanyi suggested that each hierarchy of beliefs could be summarized by the notion of agent's type [9]. Later Mertens and Zamir proved that the space of all possible types is closed in the sense that it is large enough to include even higher-order beliefs about itself [15]. Brandenburger has shown that if agents' beliefs are coherent the space of all possible types is closed [4].

The paper is organized as follows. In Section 2 we propose the notion of balanced strategy labeling. In Section 3 we show how regular belief trees can be represented with finite trees. In section 4 we propose a graph representation for infinite belief trees. We apply our approach to the analysis of auctions in Section 5. Finally, the paper concludes by summarizing the results and providing directions for future research.

2. Infinite belief hierarchies

In every multiagent interaction, an agent faces two types of uncertainty: basic and belief uncertainty. Basic uncertainty relates to the elements of the physical environment, which are uncertain to agents. We model basic uncertainty by a finite set T, $T=\{t_1,t_2,\ldots,t_n\}$, including all uncertain elements of the physical environment. We represent agents' basic beliefs as a subjective probability distributions on T.

While basic uncertainty deals with the elements of the physical environment, belief uncertainty relates to other agents' beliefs. That is, belief uncertainty includes an agent's beliefs about other agents' beliefs, his beliefs about other agents' beliefs about other agents' beliefs, and so on.

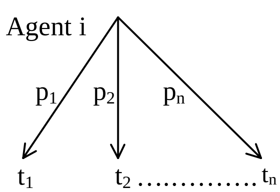

Figure 1. A first-order belief tree

Suppose that the agents under consideration are agent i and agent j. First order beliefs of agent i can be represented by a discrete probability distribution σ, $\sigma=(p_1,p_2,\ldots,p_n)$. That is, agent i believes that the true state of the environment is t_1 with probability p_1, t_2 with probability p_2, and so on. First order beliefs of agent i can be represented by a belief tree. The nodes of the tree are labeled with the elements of T and arcs are labeled with probabilities. For every node, the sum of the probabilities on outgoing arcs is 1. Figure 1 shows a belief tree that represents first-order beliefs of agent i.

Second order beliefs of agent i include his beliefs about the true state of the environment and his beliefs about agent j's beliefs about the true state of the environment. A second-order belief tree for agent i is represented in Figure 2.

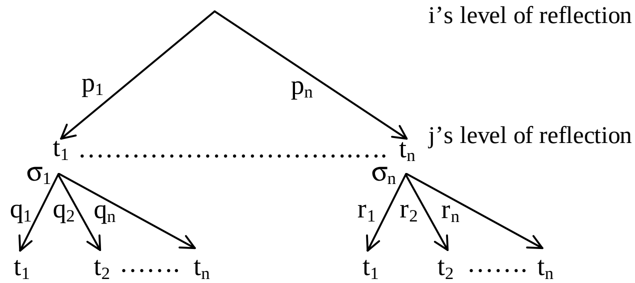

Figure 2. A second-order belief tree

Let σ_1 denote the first-order beliefs of agent j that the true state of the environment is t_1 with probability q_1 and t_2 with probability q_2, and so on. Similarly, let σ_n denote agent j's beliefs that the true state of the environment is t_1 with probability r_1 and t_2 with probability r_2, and so on. According to Figure 2, agent i assigns probability p_1 to the event that the true state is t_1 and the first order beliefs of agent j are σ_1. At the same time, agent i believes that with probability p_n the true state is t_n and agent j's beliefs are σ_n. Therefore, each level of a belief tree is associated with one of the agents. Levels alternate: the first level corresponds to agent i, the second to agent j, and so an. This alternation of levels produces alternation of beliefs: agent i believes that agent j believes that agent i believes, and so on.

In this paper we assume that all belief trees are locally finite. That is, the branching factor of every node is finite. At first sight this may seem to be a substantial constraint. Since the cardinality of the space of all first-order beliefs is a continuum, one might expect that trees are not an adequate representation of an agent's beliefs. In interactive epistemology and game theory [1,3,15] recursive beliefs are usually represented as nested probability distributions. Such representations, although theoretically elegant, are not often tractable from a computational point of view. In this paper we focus on the problem of how beliefs can be represented in a computationally tractable way. In recursive modeling, one usually deals with two types of infinity: infinity of beliefs on every level of reflection and infinity of levels. In this work we assume that beliefs are finite at every level, while the number of levels can be infinite. We propose a method for reducing the infinite regress of beliefs to a finite structure. The idea behind our approach is that some infinite belief hierarchies display repetitive patterns and regularities. This makes it possible to “merge” some parts of an infinite hierarchy and to “cut” other parts. Previous research of Gmytrasiewicz and Durfee [11,20] assumes that both the number of levels of beliefs and the number of beliefs at each level are finite.

According to the game-theoretic tradition [1], the belief structure of an agent can be represented by a hierarchy of beliefs.

Definition 1. A *belief hierarchy*, B_i^∞, of agent i is an infinite sequence of finite belief trees. That is, $B_i^\infty=(b_i^1,b_i^2,b_i^3,\ldots)$, where b_i^k represents agent i’s k-order beliefs.

A belief hierarchy is a complete enumeration of all orders of belief. Since a rational agent can hold beliefs of an arbitrary order, every belief hierarchy is inherently infinite.

Since every k-order belief tree implicitly determines k-1-order beliefs, it is natural to assume that all trees in a belief hierarchy are consistent. By this we mean that b_i^{k-1} represents the k-1-order beliefs determined by b_i^k. In other words, b_i^{k-1} is a subtree of b_i^k. Belief consistency says that the different levels of beliefs do not contradict one another.

The principle of consistency allows us to reduce the infinite sequence of belief trees, B_i^∞, to a single infinite tree. Since in B_i^∞ every tree carries all the information presented in the preceding trees, the sequence of belief trees is monotonically increasing and we can take the limit. Let $\Gamma(B_i^\infty)$ denote the limit tree that corresponds to the sequence B_i^∞. For the sake of simplicity we will denote $\Gamma(B_i^\infty)$ by Γ_i whenever this is not a source of confusion.

Infinity of Γ_i creates several problems. First, it is problematic how an agent can build, update and manipulate such a structure. Second, how can an agent predict the behavior of other agents if his own beliefs run to infinity?

In this paper we will show that some infinite belief trees can be represented by finite belief trees or by finite graphs. By this, we provide a uniform way to deal with uncertainty that does not depend on the depth of agents’ beliefs.

Let S_k be the strategy set of agent k, k=i,j. Γ_i denotes a belief tree (infinite or finite) of agent i. We represent Γ_i as a pair $(N(\Gamma_i),A(\Gamma_i))$, where $N(\Gamma_i)$ is the set of nodes and $A(\Gamma_i)$ is the set of arcs. We also use the following notation:

- $\Gamma_i^{ag}(v)$ returns the name of the agent whose beliefs are represented at node v, $v\in N(\Gamma_i)$. That is, Γ_i^{ag}: $N(\Gamma_i)\rightarrow\{i,j\}$. For example, $\Gamma_i^{ag}(v)$ returns i for every node that is located at an odd-numbered level of Γ_i.
- $\Gamma_i^{succ}(v)$ denotes the set of the successors of a node v, $v\in N(\Gamma_i)$.
- $\Gamma_i^{prob}(v)$ denotes the probability distribution assigned to node v, $v\in N(\Gamma_i)$.

With each node, v, of the belief tree, Γ_i, we assign a strategy that tells what agent $\Gamma_i^{ag}(v)$ will do at that node. For a node corresponding to agent i’s beliefs, the assigned strategy belongs to agent i’s strategy set S_i. Similarly, for a node corresponding to agent j’s beliefs the strategy belongs to S_j. Formally, we denote a strategy labeling by $\phi:N(\Gamma_i)\rightarrow S_i\cup S_j$. Every strategy labeling of Γ_i represents agent i’s beliefs about the strategies of both agents. It is clear that agent i’s strategy depends on his believes about agent j’s strategy. Agent j’s strategy depends on agent j’s beliefs about agent i’s strategy, and so on. That is, a strategy labeling is a solution to the problem of strategy choice for agent i given his beliefs Γ_i.

For every infinite belief tree, Γ_i, there exists an infinite number of strategy labelings. However, only a few of them (if any) meet the Bayesian rationality requirement, i.e., that a strategy at each node, v, has to be a best response to the other agents’ strategies at the successors of v. That is, if at level m of his reflection, agent i believes that at level m+1 agent j is going to use some (possibly mixed) strategy, then the strategy of agent i at level m should be a best response to agent j’s strategy given the probability distribution at level m+1. The following definition introduces the class of balanced strategy labelings. A strategy labeling, ϕ, is balanced if the strategy associated with each node is a best response to the strategies associated with the successor nodes, given the probabilities assigned to the successors. Formally,

Definition 2. A strategy labeling ϕ is *balanced* iff for every node v, $v\in N(\Gamma_i)$, $\phi(v)$ is a best response to the mixture of strategies $[\phi(\Gamma_i^{succ}(v)), \Gamma_i^{prob}(v)]$.

Another way to look at a balanced strategy labeling ϕ of Γ_i is to see it as the limit of balanced strategy labelings for finite trees of the infinite hierarchy B_i^∞, $B_i^\infty=(b_i^1,b_i^2,b_i^3,\ldots)$. That is,

$$\lim_{k\rightarrow\infty} \phi_i^k=\phi \qquad \textbf{(1)}$$

Here ϕ_i^k represents a balanced strategy labeling of b_i^k.

According to Definition 2, if the limit (1) exists, it should be a balanced strategy labeling for Γ_i. This constructive interpretation of balanced strategy labelings gives us an idea of how to compute them. First, it is necessary to compute a balanced strategy labeling for the first-order belief tree b_i^1 (in fact, computation can start from any tree b_i^k). This can be done, for example, by applying backward induction. The strategies computed at the first stage are further refined by computing a balanced strategy labeling for the second-order belief tree b_i^2. After that, the computation proceeds with the third-order belief tree b_i^3, and so on. This process can stop at any time or when some predefined solution precision is achieved. There is no guarantee that this process of iterative refinement is convergent. If it is convergent, however, the limit is a balanced strategy labeling. It is worth noting that any stage of the computation can yield several intermediate solutions. Therefore, uniqueness of the final balanced strategy labeling is not guaranteed. If, however, two balanced strategy labelings exist, they will be payoff equivalent. That is, they give agent i the same expected payoff.

3. Representation of infinite belief trees with finite *trees*

In this section we show how regular infinite belief trees can be represented in a finite way. Such representation has several important advantages. First, agents can cope with infinite belief hierarchies by reducing them to finite graphs. Second, agents can apply to infinite beliefs the same techniques they use to handle finite beliefs.

Definition 3. An infinite tree is *regular* if and only if the number of its distinct subtrees is finite [5].

It is evident that a regular belief tree is a repetition of a finite number of subtrees. This means that by extending a regular tree to infinity we do not add new strategically relevant information. The following proposition states that for every regular belief tree there exists some finite level of reflection that is sufficient for finding a balanced strategy labeling. In other words, analyzing the tree beyond that level is strategically useless.

Definition 4. A subtree α of a belief tree Γ_i is at *level of reflection* n+1, if the distance between the root of α and the root of Γ_i is n.

Proposition 1. For every regular belief tree Γ_i there exists an integer N such that for every subtree α at level of reflection K, K>N, there exists a subtree β at level of reflection M, $M\le N$, such that $\alpha=\beta$.

Proof. Let $\alpha_1,\alpha_2,\ldots,\alpha_n$ be the distinct subtrees of Γ_i. Then $N=\max \rho(\alpha_k)$, k=1,..,n, where $\rho(\alpha_k)$ is the level of reflection of α_k. ❑

If Γ_i is a regular tree, then every balanced strategy labeling ϕ of Γ_i will have at most N different strategies, where N is the number from Proposition 1. In order to find ϕ we need to consider only the first N+1 levels of reflection. Let Γ_i^* denote the tree that is obtained from Γ_i by removing all subtrees which level of reflection is greater than N+1. Since ϕ is a balanced strategy labeling, the strategy s_v, $s_v=\phi(v)$, assigned to a node v, should be a best response to the strategies assigned to the successors of v. That is,

$$s_v=f(\Gamma_i^{prob}(v),\ s_{v1},s_{v2},\ldots,s_{vk}),$$

where $s_{v1},s_{v2},\ldots,s_{vk}$ are the strategies assigned to the successors $v_1,v_2,\ldots,v_k$ and f() is a best response function.

If we write this equation for every node of Γ_i^*, we will obtain a system of N simultaneous equations:

$$\begin{aligned} s_1&=f(\Gamma_i^{prob}(v_1),\ s_{11},s_{12},\ldots,s_{1k}),\\ s_2&=f(\Gamma_i^{prob}(v_2),\ s_{21},s_{22},\ldots,s_{2k}),\\ &\ldots\\ s_N&=f(\Gamma_i^{prob}(v_N),\ s_{N1},s_{N2},\ldots,s_{Nk}), \end{aligned} \tag{2}$$

where $s_{nm}\in\{s_1,s_2,\ldots,s_N\}$, n=1,..,N, m=1,..,$n_k$. Let $S_N=\{s_1,s_2,\ldots,s_N\}$. Some strategies from S_N appear only on the left side of equations (2) and some strategies appear on both sides. Let S_L denotes the set of all strategies from S_N that appear on the right side of at least one equation. Without loss of generality we may assume that $S_L=\{s_1,s_2,..,s_L\}$. Solving (2) consists of two stages. First, we should find all strategies in S_L and after that we should find $S_N\backslash S_L$. Since every strategy in $S_N\backslash S_L$ is a function of strategies from S_L, it is straightforward to calculate $S_N\backslash S_L$ having S_L. In order to find S_L we have to solve the following system of simultaneous equations:

$$\begin{aligned} s_1&=f(\Gamma_i^{prob}(v_1),\ s_{11},s_{12},\ldots,s_{1k}),\\ s_2&=f(\Gamma_i^{prob}(v_2),\ s_{21},s_{22},\ldots,s_{2k}),\\ &\ldots\\ s_L&=f(\Gamma_i^{prob}(v_L),\ s_{L1},s_{L2},\ldots,s_{Lk}). \end{aligned} \tag{3}$$

This means that the vector of strategies $\langle s_1,..,s_L\rangle$ is a fixed point of the operator F defined by

$$\begin{aligned} (&f(\Gamma_i^{prob}(v_1),\ s_{11},s_{12},\ldots,s_{1k}),\\ &f(\Gamma_i^{prob}(v_2),\ s_{21},s_{22},\ldots,s_{2k}),\\ &\ldots\\ &f(\Gamma_i^{prob}(v_L),\ s_{L1},s_{L2},\ldots,s_{Lk})). \end{aligned}$$

One way to solve (3) is by using the Banach fixed point theorem: if the set of all strategies is a complete metric space and F is a contraction, then F has a unique fixed point s, s=F(s). The Banach fixed point theorem allows us to solve (3) by iterations starting from an arbitrary point $<s_1^0,..,s_L^0>$. Other algorithms for solving systems of nonlinear equations are discussed by Rheinboldt [16].

The most typical example of finding a balanced strategy labeling includes a tree that represents common knowledge. Figure 3 shows such a belief tree for agent i. According to Figure 3, agent i believes that both agents, i and j, believe that the true state of the environment is t_1 with probability p and t_2 with probability 1-p. Agent i also believes that p and 1-p are common knowledge between agents. That is, everyone knows them, everyone knows that everyone knows them, and so on.

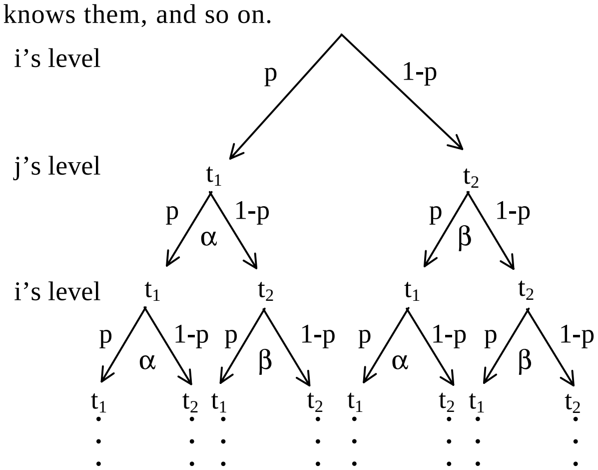

Figure 3. A tree representing common knowledge

The tree shown in Figure 3 consists of infinite repetitions of the left subtree α and the right subtree β. Therefore any extension of the belief tree one level beyond the third level does not add any strategically relevant information. If we are looking for a strategy labeling for that tree, we can ignore the infiniteness of the tree and can "cut" the tree between the third and fourth level of reflection. By this we obtain a finite tree for which we can find a solution. By "cutting" an infinite tree we do not lose any strategically relevant information, since the concept of balanced labeling guarantees that the strategies along the cutting line convey all the relevant information belonging to the infinite part of the tree.

4. Representation of infinite belief trees with finite *graphs*

In this section we present another representation that can be used to solve the same problem. Instead of cutting a regular belief tree we can represent it with a finite graph. Definition 5 and Definition 6 show how this can be done.

Definition 5. Two nodes of v_1 and v_2 of belief tree Γ_i are *identical* iff:

(i) they are labeled with the same t_k, $t_k \in T$,
(ii) $\Gamma_i^{ag}(v_1) = \Gamma_i^{ag}(v_2)$, i.e., v_1 and v_2 are both on an even or an odd-numbered level of reflection,
(iii) $\Gamma_i^{succ}(v_1) = \Gamma_i^{succ}(v_2)$, i.e., v_1 and v_2 have the same successors,
(iv) $\Gamma_i^{prob}(v_1) = \Gamma_i^{prob}(v_2)$, i.e., every two arcs starting at v_1 and v_2, that point to the same successor, are labeled with equal probabilities.

Definition 6. An *elementary contraction* of a graph G is obtained by identifying two identical nodes v_1 and v_2 by removing v_1 and v_2, and by adding a new node v adjacent to those nodes to which v_1 and v_2 were adjacent.

After an elementary contraction the arcs starting at node v are labeled with the same probabilities as the deleted arcs starting at v_1 and v_2. If, as a result of an elementary contraction, two arcs pointing to v_1 and v_2 are merged, then the new arc is labeled with the sum of probabilities on the deleted arcs.

Definition 7. A graph G is *contractible* to a graph G' if G' can be obtained from G by applying elementary contractions.

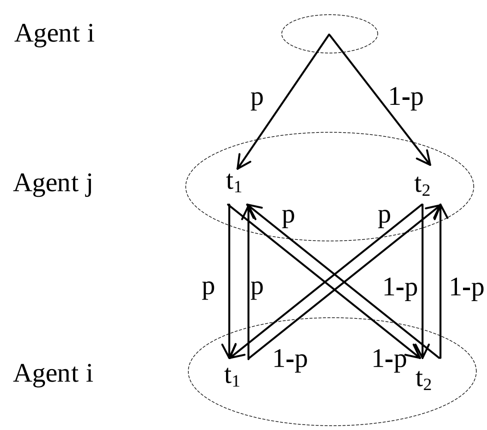

Figure 4. An accessible pointed graph

If we look at the tree represented in Figure 3, we will notice that this tree contains many identical nodes. For example, all nodes that are labeled with t_1 and that correspond to agent i are identical. Similarly, all nodes that are labeled with t_1 and that correspond to agent j are identical. If we contract all identical nodes we will obtain the graph shown in Figure 4. In order to compute a balanced strategy labeling for this graph we have to consider only 5 nodes, instead of analyzing infinitely many nodes of the original belief tree.

In general, it is not true that for any graph there exists a belief tree (infinite or finite) that is contractible to that graph. Therefore, some graphs cannot represent belief hierarchies. As we will soon show an accessible pointed graph [2] always represents some belief hierarchy.

Definition 8. A *pointed* graph is a graph together with a distinguished node called its point. A pointed graph is *accessible* if for every node, which is different from the point, there is a path from the point to that node.

If this path is always unique, then the pointed graph is a tree and the point is the root of the tree. The nodes of every pointed accessible graph are labeled with the elements of the space of basic uncertainty, T. The arcs are labeled with probabilities. For every node, the sum of the probabilities on outgoing arcs is 1.

Proposition 2. If a belief tree is *contractible* to a graph, then the graph is pointed and accessible.

Proof. Since every contraction preserves accessibility, the proposition follows from the definition of tree as an acyclic and connected graph and Definition 8. ❑

According to Proposition 2, the notion of an accessible pointed graph is a generalization of both finite and infinite belief trees. This, however, does not imply that every accessible pointed graph represents some beliefs. Nevertheless, this is the case:

Proposition 3. For every accessible pointed graph G there exists a belief tree that is contractible to G.

Proof. Every pointed graph, G, can be unfolded into a belief tree, Γ, whose root is the point of the graph. The nodes of Γ are the finite paths that start from the point of G. ❑

5. An example: auction analysis

With the growing impact of electronic commerce, auctions are going to play an increasingly important role in multiagent systems and distributed artificial intelligence [17,18,14]. In this section we demonstrate how the belief graph reasoning, presented here, can be applied to the analysis of auctions.

Most theoretical results in auction theory draw crucially on the revenue equivalence theorem [19]. According to the theorem, the first-price sealed bid, second-price sealed bid, English and Dutch auctions are all optimal selling mechanisms, provided that they are supplemented by an optimally set reserve price. The revenue equivalence theorem is based on the following assumptions: the bidders are risk neutral, payment is a function of bids alone, the auction is regarded in isolation of other auctions, the bidders' private valuations are independently and identically distributed random variables, every bidder knows his own valuation, and there is common knowledge about the distribution from which the valuations are drawn.

In this example the common knowledge assumption about prior beliefs is dropped, but all other classic assumptions are kept intact. In particular, the assumption that the agents' valuations are drawn from the same prior is kept. We will show that without common knowledge, the revenue equivalence theorem ceases to hold. This is particularly noteworthy since common knowledge is unobtainable with any amount of communication [8].

Consider the following simple auction setting. There are two risk-neutral buyers in an auction for a single indivisible object. Suppose that each buyer has one of two possible valuations of the object: t_1 or t_2 (with $t_1<t_2$). Each bidder knows his own valuation, but is uncertain about his rival's valuation. Therefore, in our example, the space of basic uncertainty is $T=\{t_1,t_2\}$. Assume that valuations are independent and that there exists some objective distribution $\pi=(p,1-p)$ from which valuations are drawn. That is, with probability p each bidder's valuation is t_1, and with probability 1-p it is t_2. Since π is not common knowledge, each bidder can hold some private beliefs about π.

Suppose now that at some moment τ_0, before the beginning of the auction, the actual distribution π has been (1/2,1/2) and π has been common knowledge. Just before the action, at time τ_1, $\tau_0<\tau_1$, some event occurs that changes distribution π from (1/2,1/2) to (p,1-p), where $p\neq 1/2$. This event is not mutually observable and the fact that π has changed is not common knowledge. In this situation each agent believes that the actual distribution is (p,1-p) and that the other agent still believes that there is common knowledge about (1/2,1/2). This is a realistic assumption, since in many electronic commerce applications bidders do not have sufficient information about their rivals.

The belief structure of each bidder can be represented by the infinite belief tree shown in Figure 5. In order to find a balanced strategy labeling for that tree we need to analyze the first four levels of the tree (N=3). That is, we should look for 31 strategies corresponding to the first 31 nodes of the tree. Instead of doing so, we look for the accessible pointed graph

H that corresponds to the tree. The graph H is represented in Figure 6. It shows that there are only 5 non-identical nodes: the root, t_1- and t_2- nodes for the first bidder and t_1- and t_2- nodes for the second bidder. Now, instead of analyzing 31 nodes we need to consider only 5.

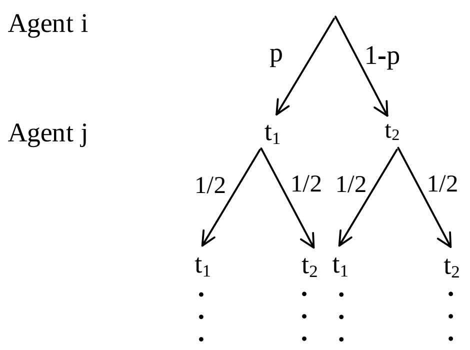

Figure 5. The belief tree of bidder i.

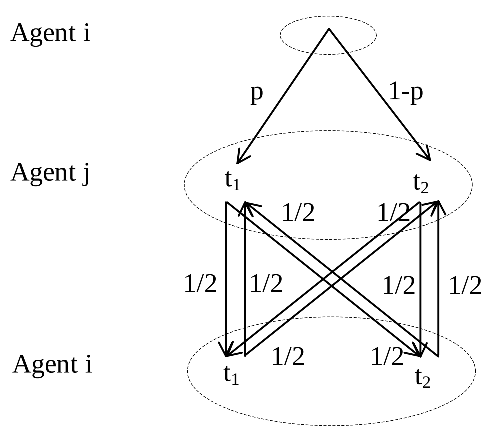

Figure 6. The belief graph of bidder i

The solution for the first-price sealed bid auction without common knowledge about prior beliefs is provided by the following proposition.

Proposition 4. When the prior beliefs are not common knowledge for the given auction setting, the first-price sealed bid auction yields expected utility 0 to the bidder with valuation t_1 and $½(t_2-t_1)$ to the bidder with valuation t_2. The optimal bid and the expected utility do not depend on the bidders' first-order prior beliefs.

Proof. Since the situation is symmetric, we are looking for a symmetric solution. Therefore every balanced strategy labeling consist of 3 strategies: one for the point of the graph H, one for t_1-nodes and one for t_2-nodes. The strategies assigned to t_1 and t_2-nodes should be in equilibrium. That is, each of them should be a best response to the other. Since there does not exist an equilibrium in pure strategies, we look for an equilibrium where each bidder with valuation t_1 bids t_1 ($t_1<t_2$), and each bidder with valuation t_2 randomizes according to a continuous cumulative distribution function F(x) with continuous support on $[a_1,a_2]$, where $t_1 \le a_1 \le a_2 \le t_2$. It can be shown that this equilibrium is unique. Clearly, $a_1=t_1$. If $a_1>t_1$, then a bidder with valuation t_2 would be better off bidding $t_1+\varepsilon$ rather than bidding a_1. In order for a bidder with valuation t_2 to play a mixed strategy in the interval $[a_1,a_2]$ he must be indifferent *ex ante* between all bids in this interval. Hence, for every bid $x \in [a_1,a_2]$ it holds that

$$(t_2-x)(\tfrac{1}{2} + \tfrac{1}{2}F(x))=c,$$

where c is constant. Here t_2-x is the bidder's utility if he wins and $½ + ½F(x)$ is the probability of winning. Because $F(t_1)=0$, it follows that $c= ½ (t_2-t_1)$. Thus, the continuous distribution function F(x) is implicitly defined by

$$(t_2-x)(\tfrac{1}{2}+ \tfrac{1}{2})F(x)= \tfrac{1}{2} (t_2-t_1) \qquad \textbf{(4)}$$

Substituting a_2 for x in Equation (4) and taking into account that $F(a_2)=1$, we obtain

$$a_2= \tfrac{1}{2}t_1+ \tfrac{1}{2}t_2.$$

What remains to be done is to find a bidding strategy b* corresponding to the root of the tree. It is clear that b* must be a best response to the strategy mixture $[p,t_1;1\text{-}p,b^{**}]$, where b** is the strategy defined by Equation (4). We can solve Equation (4) for F(x), thereby obtaining

$$F(x)=(x-t_1)/(t_2-x).$$

The expected utility of submitting bid x, given that the rival adheres to the strategy mixture $[p,t_1;1\text{-}p,b^{**}]$ is:

$$(t_2-x)(p+(1-p)(x-t_1)/(t_2-x)) \text{ if } t_1 \le x \le \tfrac{1}{2}(t_1+t_2) \text{ and}$$
$$t_2-x \qquad \text{if} \qquad \tfrac{1}{2}(t_1+t_2)<x.$$

In order to obtain an optimal bid we have to maximize the expected utility function. There are three possible cases:

(i) $p<½$: the optimal bid is $½(t_1+t_2)$. The expected utility is $½(t_2-t_1)$;

(ii) $p=½$: every bid in the interval $[t_1,½(t_1+t_2)]$ is optimal. The expected utility is $½(t_2-t_1)$;

(iii) $p>½$: the optimal bid is $½(t_1+t_2)$. The expected utility is $½(t_2-t_1)$. ❒

Proposition 5. When there does not exist common knowledge about private beliefs, the revenue equivalence theorem ceases to hold. The bidder's expected utility is different in the first price sealed bid auction and Vickrey auction.

Proof. Consider the first-price sealed bid auction and the second-price sealed bid auction. It follows from Proposition 4 that the expected utility for the bidder with valuation t_2 is ½(t_2-t_1) in the first-price sealed bid auction without common knowledge about prior beliefs. On the other hand, for the second-price auction the dominant strategy for every bidder is to bid his own valuation. Therefore, in the second-price auction the expected utility for the bidder with valuation t_2 is (t_2-t_1)p, where p is the subjective probability that the other bidder's valuation is t_1. Thus, the two auctions yield different expected utility. ❒

6. Conclusions

In this paper we identified a class of infinite belief trees that allow finite representation. Such representation has several important advantages. First, agents can cope with infinite belief hierarchies by reducing them to finite trees or finite graphs. Second, agents can apply to infinite beliefs the same techniques they use to handle finite beliefs.

As an example of our approach we showed that without common knowledge about prior beliefs, the revenue equivalence theorem of auctions ceases to hold. Since different auctions yield different revenues, auction designers should be careful when choosing auction rules. Our solution concept for infinite belief trees provides an analytic tool for comparing different auction forms and more general interaction mechanisms

Most mechanism design today is based on Nash equilibrium which assumes common knowledge of priors. At the same time it is known that common knowledge is not obtainable with any amount of communication. Therefore, our method that does not rely on common knowledge holds promise as a future analysis and design tool.

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