12-759: Computational Optimization of Systems Governed by Partial Differential Equations
Fall 2003
Variatonal (weak) form of linear elasticity

In this handout I derive the weak form of the equations of linear elasticity in symbolic form. This requires some facility with tensors. First, some notation: For vector $\boldsymbol{v}$ and tensors $\boldsymbol{A}$ and $\boldsymbol{B}$,

$$(\nabla \boldsymbol{v})_{ij} = \frac{\partial v_i}{\partial x_j}, \qquad (\nabla \cdot \boldsymbol{A})_i = \sum_j \frac{\partial A_{ij}}{\partial x_j},$$
$$(\boldsymbol{A}\boldsymbol{v})_i = \sum_j A_{ij} v_j, \qquad \boldsymbol{A} \cdot \boldsymbol{B} = \sum_i \sum_j A_{ij} B_{ij}.$$

The linear elastic deformation of an isotropic solid is described in terms of the stress tensor $\boldsymbol{\sigma}$, the strain tensor $\boldsymbol{\varepsilon}$, the displacement vector $\boldsymbol{u}$, the traction vector $\boldsymbol{t}$, the body force vector $\boldsymbol{f}$, and the Lamé constants μ and λ. The governing field equations consist of the equilibrium equation

$$-\nabla \cdot \boldsymbol{\sigma} = \boldsymbol{f},$$

the strain-displacement relation

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathsf{T}}),$$

and the constitutive law

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda(\nabla \cdot \boldsymbol{u})\boldsymbol{I}.$$

The displacement form of the equilibrium equation is thus

$$-\nabla \cdot [\mu(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathsf{T}}) + \lambda(\nabla \cdot \boldsymbol{u})\boldsymbol{I}] = \boldsymbol{f}.$$

Finally, we need some boundary conditions, for example

$$\begin{aligned} \boldsymbol{u} &= \hat{\boldsymbol{u}} \quad \text{on} \quad \Gamma_D, \\ \boldsymbol{t} \equiv \boldsymbol{\sigma}\boldsymbol{n} &= \hat{\boldsymbol{t}} \quad \text{on} \quad \Gamma_N. \end{aligned}$$

Now we're ready to construct the weak form. We multiply the residual by a test (vector) function $\boldsymbol{v}$ and integrate over the domain Ω,

$$\int_\Omega \boldsymbol{v} \cdot (-\nabla \cdot \boldsymbol{\sigma} - \boldsymbol{f})\, d\Omega = 0.$$

Using a Green's formula, we obtain

$$\int_\Omega \nabla \boldsymbol{v} \cdot \boldsymbol{\sigma}\, d\Omega = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{f}\, d\Omega + \int_\Gamma \boldsymbol{v} \cdot \boldsymbol{\sigma}\boldsymbol{n}\, d\Gamma.$$

Then we substitute for the stress and traction

$$\int_\Omega \nabla \boldsymbol{v} \cdot [\mu(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathsf{T}}) + \lambda(\nabla \cdot \boldsymbol{u})\boldsymbol{I}]\, d\Omega = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{f}\, d\Omega + \int_\Gamma \boldsymbol{v} \cdot \boldsymbol{t}\, d\Gamma.$$

Finally, define $\mathcal{U}$ as the space of all vector functions whose derivatives are square integrable and that satisfy the essential boundary condition, and $\mathcal{V}$ as the space of all vector functions whose derivatives are square integrable and that vanish on Γ_D. Rearranging and making use of the boundary conditions, we obtain the weak form of the linear elasticity problem: Find $\mathbf{u} \in \mathcal{U}$ such that

$$\int_\Omega \frac{\mu}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathsf{T}}) \cdot (\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^{\mathsf{T}})\, d\Omega + \int_\Omega \lambda(\nabla \cdot \boldsymbol{u})(\nabla \cdot \boldsymbol{v})\, d\Omega = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{f}\, d\Omega + \int_{\Gamma_N} \boldsymbol{v} \cdot \hat{\boldsymbol{t}}\, d\Gamma$$

for all $\boldsymbol{v} \in \mathcal{V}$. This form is clearly symmetric.