12-759: Computational Optimization of Systems Governed by PDEs
Assignment 1: FEM review
Due 17 September 2003

Consider a square membrane of length l under tension $t(x, y)$, resting on an elastic foundation of stiffness $k(x, y)$, acted upon by pressure $p(x, y)$, and fixed on all four sides (i.e. the displacement is zero on the boundary). We wish to determine the transverse displacement $u(x, y)$ under the assumption of small displacements. The total potential energy of the membrane is given by

$$\frac{1}{2}\int_{\Omega} t\,\nabla u \cdot \nabla u\,dx\,dy + \frac{1}{2}\int_{\Omega} ku^2\;dx\,dy - \int_{\Omega} pu\;dxdy,$$

where Ω represents the interior of the membrane. The first term of the potential energy represents the "strain" energy of the membrane, the second the strain energy of the foundation, and the third the loss of potential of applied forces.

1. Using variational calculus, derive the strong form of the boundary value problem governing this problem. The following Green's identity will prove useful: for scalar functions $a(x, y)$, $u(x, y)$, and $v(x, y)$,

   $$\int_{\Omega} v\nabla \cdot (a\nabla u)\;dx\,dy = -\int_{\Omega} a\nabla u \cdot \nabla v\,dx\,dy + \int_{\Gamma} va\nabla u \cdot \boldsymbol{n}\,ds,$$

   where $\boldsymbol{n}$ is the outward unit normal to the boundary Γ, and s is the arc length.

2. Application of the Ritz method to this problem, using the finite element approximation

   $$u_h(x, y) = \sum_{i=1}^{N} u_i \phi_i(x, y),$$

   produces a linear algebraic system of the form $\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}$. Give expressions for typical elements K_{ij} and f_i of this system. Show that $\boldsymbol{K}$ is symmetric positive semidefinite, by showing that $\boldsymbol{u}^{\mathsf{T}}\boldsymbol{K}\boldsymbol{u} \geq 0$ for $\boldsymbol{u} \neq \boldsymbol{0}$.

3. Prove that the Ritz method minimizes the square of the error measured in the energy norm for this problem; i.e. if $e(x, y) \equiv u - u_h$ is the error, then prove that the Ritz method minimizes

   $$\|e(x, y)\|_E^2 = \frac{1}{2}\int_{\Omega} \left\{ t\,\nabla e \cdot \nabla e + ke^2 \right\}\;dx\,dy.$$

4. Show that the Galerkin method produces the same algebraic system $\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}$ as does the Ritz method (and therefore the two methods are the same).

5. Use Sundance to compute finite element solutions for the case $t = l = 1$, $k = \pi^2$, and $p(x, y) = 3\pi^2 \sin \pi x \sin \pi y$. Note that the exact solution is

   $$u(x, y) = \sin \pi x \sin \pi y, \quad 0 < x < 1, \;\; 0 < y < 1.$$

   Verify the asymptotic convergence rate estimates in the L^2 and H^1 norms. Use both linear and quadratic elements, and plot the log of the error in each norm against the log of the mesh size.