# 12-759: Computational Optimization of Systems Governed by PDEs
## Assignment 4: An Inverse Problem
## Estimating a Thermal Conduction Constitutive Law from Distributed Temperature Measurements
## Due 17 December 2003

Consider the following inverse problem. Suppose we know that heat conduction in a 2D square medium is governed by the following boundary value problem:

$$
\begin{aligned}
-\nabla \cdot (k(u)\nabla u) &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \Gamma
\end{aligned}
$$

where $\Omega$ is the unit square $(0,1)\times(0,1)$, $u$ is the temperature, $f$ is the heat source, and the thermal conductivity $k(u)$ is *temperature dependent* and of the form

$$k(u) = \alpha u^{\beta} + \gamma.$$

Here, $\alpha$, $\beta$, and $\gamma$ are material constants. The *forward* or *state* problem is to find the temperature distribution $u$ given the source $f$ and the material constants. The *inverse* or *parameter estimation* problem is to estimate values of the material constants, given some measurements on the temperature. For example, we would like to minimize the $L_2$ norm of the misfit between measured and predicted temperature in the interior, i.e.

$$\min_{\alpha,\beta,\gamma} \mathcal{F} = \int_{\Omega} (u - \hat{u})^2 \, dx,$$

where $\hat{u}$ is the "measured" temperature.

1. Derive (infinite-dimensional) expressions for the sensitivity of the temperature $u$ and the least squares objective $\mathcal{F}$ to each of the material constants $\alpha$, $\beta$, and $\gamma$.

2. Write a Sundance code to compute the sensitivity of the objective to the three material constants. You can use the code `membraneSens.cpp` (available from the class website) as a guide. You should first "synthesize" the temperature measurements by solving a forward problem given the specific values $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$. The resulting temperature distribution can be taken as the "measured" temperature $\hat{u}$.

3. Use your Sundance code to find the sensitivity of the least squares objective to the three material constants, when the measured temperature $\hat{u}$ corresponds to $\hat{\alpha} = 0.5$, $\hat{\beta} = 2.0$, and $\hat{\gamma} = 1.0$. Do this at two "points" in the parameter space:

   - for $\alpha = 0.5$, $\beta = 2.0$, and $\gamma = 1.0$ (in this case, you should verify that the objective and objective sensitivities are all zero)
   - for $\alpha = 0.75$, $\beta = 1.25$, and $\gamma = 2.0$

   Use quadratic elements and a sufficiently fine mesh[1] for the above computations. Show by example that a mesh that is sufficiently fine for the objective may not be sufficiently fine for the sensitivities.

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[1] That is, a uniform refinement of the mesh would result in a change of $< 5\%$ in the objective function sensitivities.