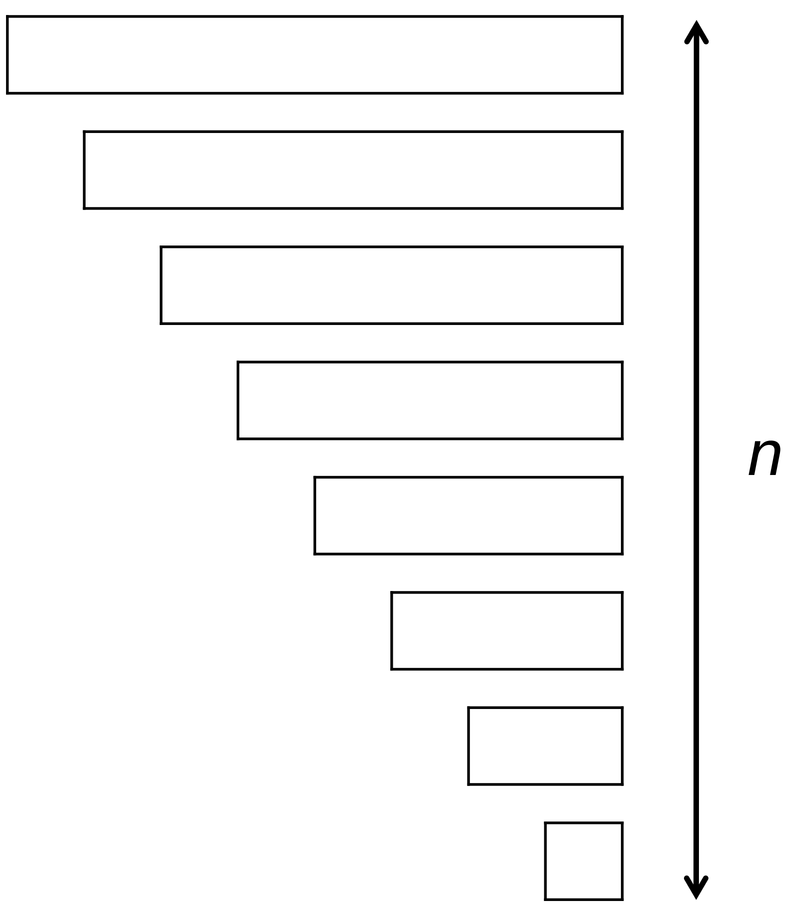
Sorting

Divide and Conquer

Searching an *n*-element Array

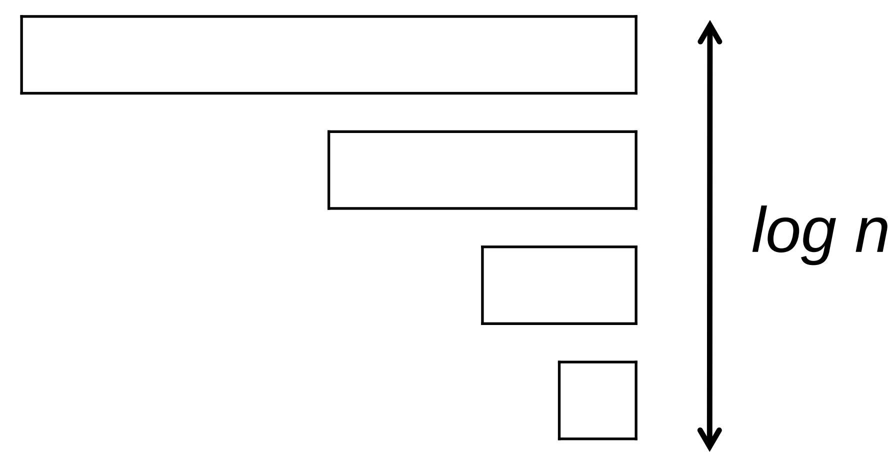
Linear Search

- Check an element
- If not found,
  search an (n-1)–element array



Binary Search

- Check an element
- If not found,
  search an (n/2)–element array

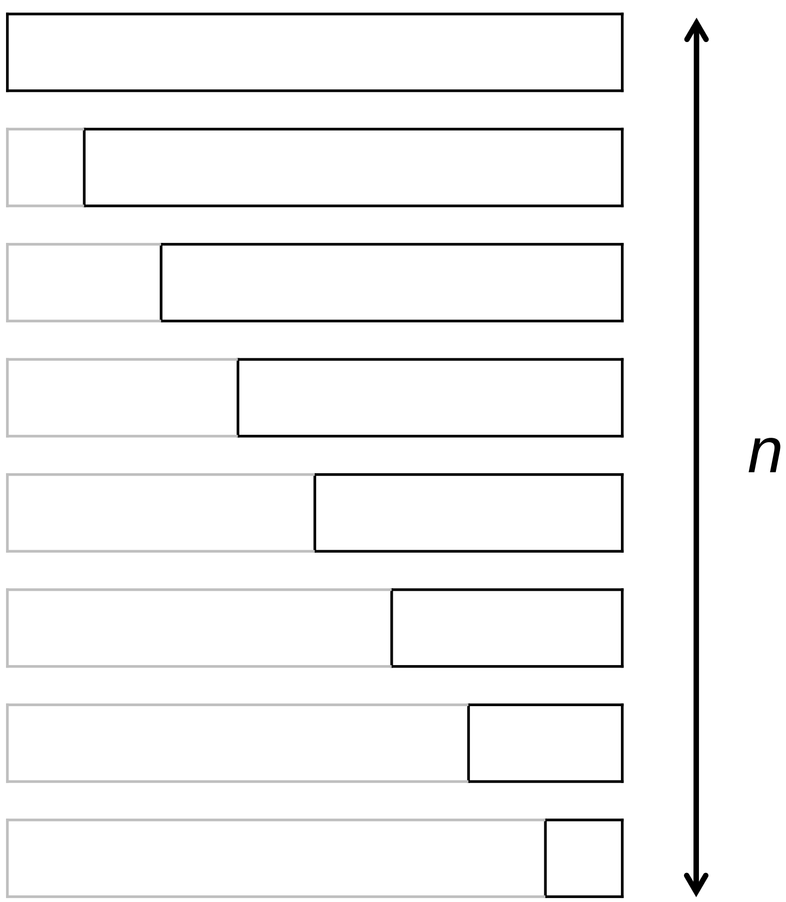


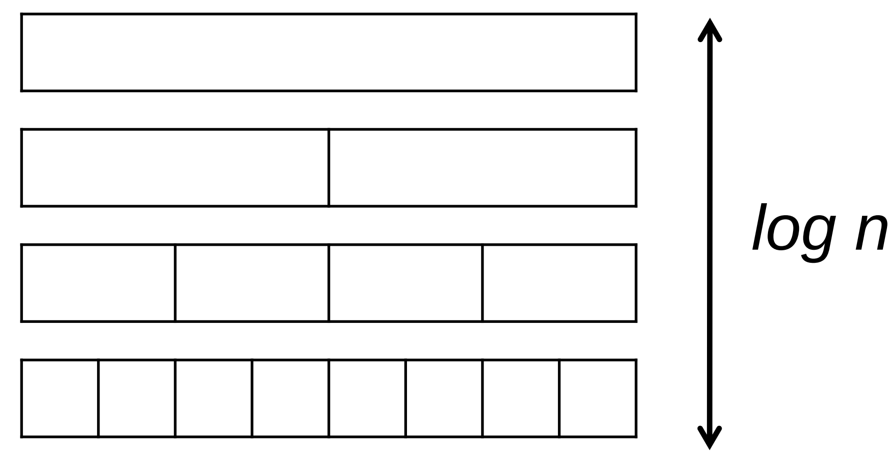
Huge benefit by **dividing** problem (in **half**)

$O(n) \Rightarrow O(\log n)$

Sorting an *n*-element Array

- Can we do the same for sorting an array?
- This time, we need to work on **two half-problems**
  - and combine their results





- This is a general technique called

  divide and conquer

Term variously attributed to Ceasar, Macchiavelli, Napoleon, Sun Tzu, *and many others*

Sorting an *n*-element Array

	Naïve algorithm	⇨	Divide and Conquer algorithm
Searching	Linear search $O(n)$	⇨	Binary search $O(\log n)$
Sorting	Selection Sort $O(n^2)$	⇨	??? sort *O(??)*

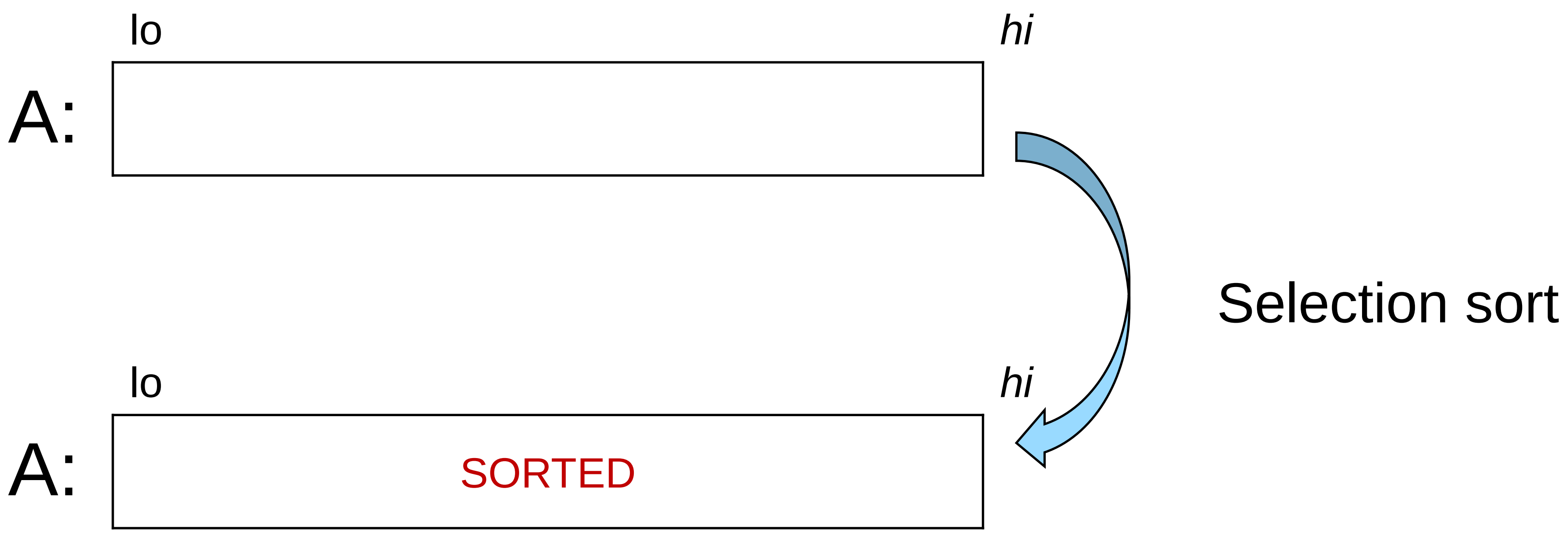
Recall Selection Sort

```
void selection_sort(int[] A, int lo, int hi)
//@requires 0 <= lo && lo <= hi  && hi <= \length(A);
//@ensures is_sorted(A, lo, hi);
{
  for (int i = lo; i < hi; i++)
  //@loop_invariant lo <= i && i <= hi;
  //@loop_invariant is_sorted(A, lo, i);
  //@loop_invariant le_segs(A, lo, i, A, i, hi);
  {
    int min = find_min(A, i, hi);
    swap(A, i, min);
  }
}
```

$O(n^2)$

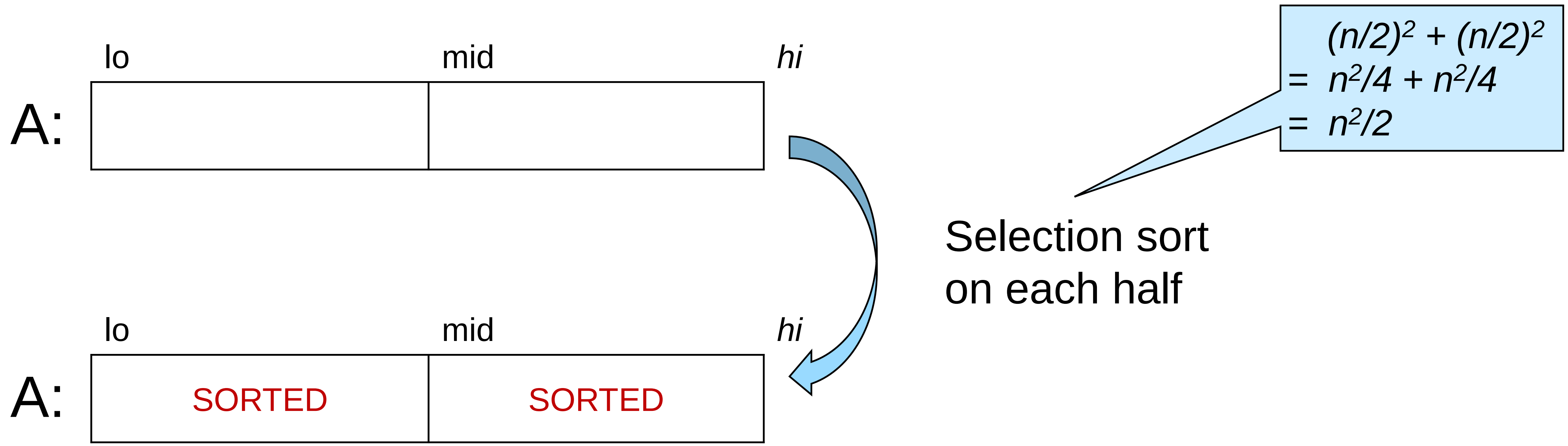
Towards Mergesort

Using Selection Sort



- If $hi - lo = n$
  - ➢ the length of array segment *A[lo, hi)*
  - ○ cost is $O(n^2)$
  - ○ let's say n^2
- But $(n/2)^2 = n^2/4$
  - ○ What if we sort the two halves of the array?

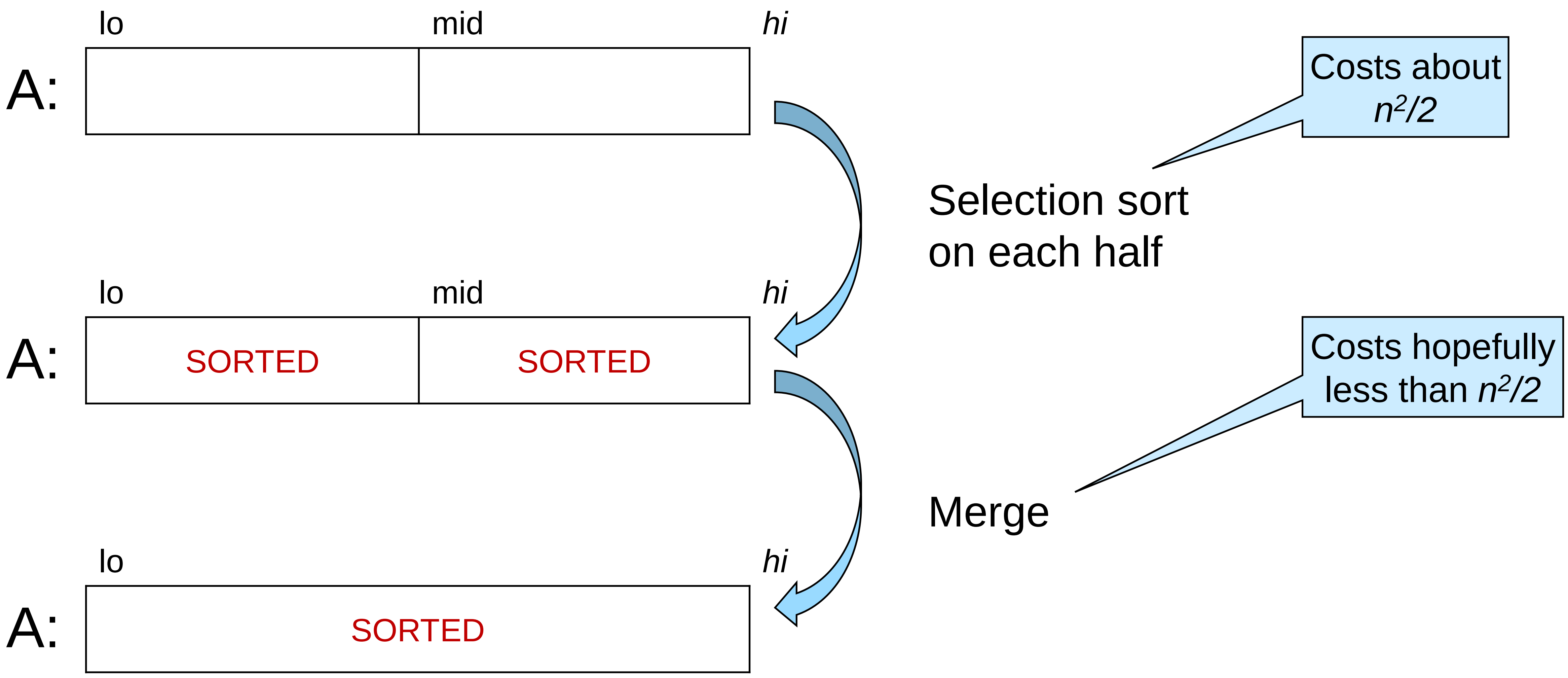
Using Selection Sort Cleverly



- Sorting each half costs $n^2/4$
- altogether that's $n^2/2$
- that's a saving of **half** over using selection sort on the whole array!

- But the overall array is not sorted
  - If we can turn two sorted halves into a sorted whole for less than $n^2/2$, we are doing better than plain selection sort

Using Selection Sort Cleverly



- merge: turns two sorted half arrays into a sorted array
  - (cheaply)

Implementation

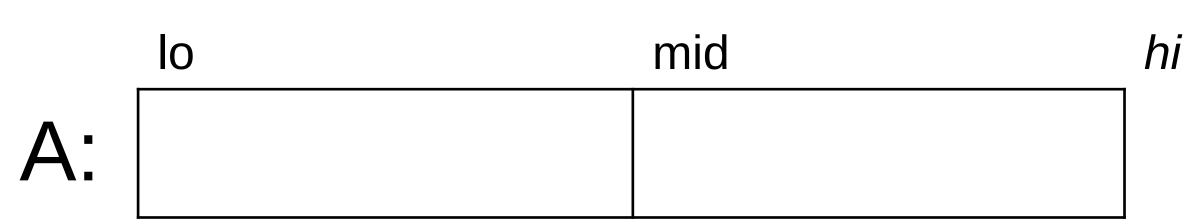
- Computing mid

```
void sort(int[] A, int lo, int hi)
//@requires 0 <= lo && lo <= hi  && hi <= \length(A);
//@ensures is_sorted(A, lo, hi);
{
  int mid = lo + (hi - lo) / 2;
  //@assert  lo <= mid && mid <= hi;
  // … call selection sort on each half …
  // … merge the two halves …
}
```

We learned this from binary search

if hi == lo,
then mid == hi

This was not possible in the code for binary search



Implementation

```
void selection_sort(int[] A, int lo, int hi)
//@requires 0 <= lo && lo <= hi  && hi <= \length(A);
//@ensures is_sorted(A, lo, hi);
```

- Calling selection_sort on each half

```
1.  void sort(int[] A, int lo, int hi)
2.  //@requires 0 <= lo && lo <= hi  && hi <= \length(A);
3.  //@ensures is_sorted(A, lo, hi);
4.  {
5.    int mid = lo + (hi - lo) / 2;
6.    //@assert  lo <= mid && mid <= hi;
7.    selection_sort(A, lo, mid);
8.    selection_sort(A, mid, hi);
9.    // … merge the two halves
10. }
```

To show: 0 ≤ lo ≤ mid ≤ \length(A)

• 0 ≤ lo	by line 2
• lo ≤ mid	by line 6
• mid ≤ hi	by line 6
hi ≤ \length(A)	by line 2
mid ≤ \length(A)	by math

To show: 0 ≤ mid ≤ hi ≤ \length(A)
Left as exercise

- Is this code safe so far? ✓
- Since selection_sort is correct, its postcondition holds
  - A[lo, mid) sorted
  - A[mid, hi) sorted

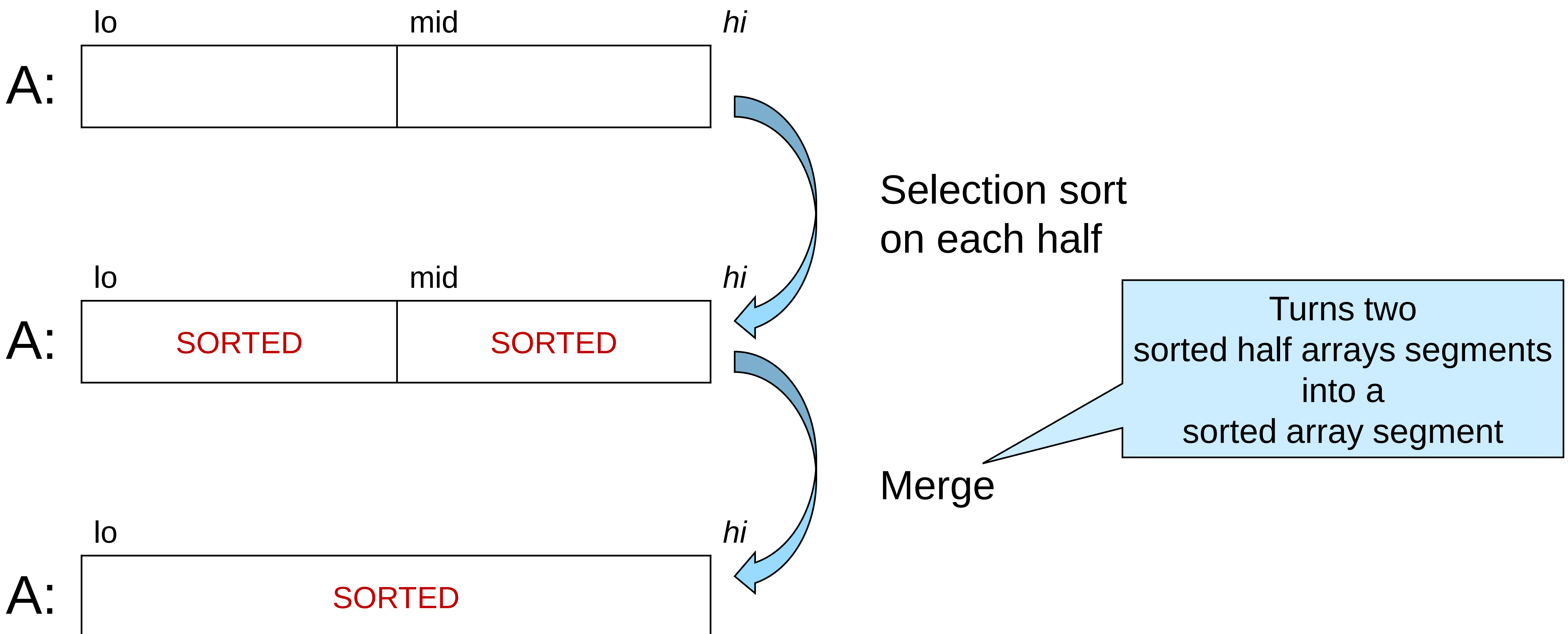
Implementation

```
void selection_sort(int[] A, int lo, int hi)
//@requires 0 <= lo && lo <= hi  && hi <= \length(A);
//@ensures is_sorted(A, lo, hi);
```

```
void sort(int[] A, int lo, int hi)
//@requires 0 <= lo && lo <= hi  && hi <= \length(A);
//@ensures is_sorted(A, lo, hi);
{
  int mid = lo + (hi - lo) / 2;
  //@assert  lo <= mid && mid <= hi;
  selection_sort(A, lo, mid);  //@assert is_sorted(A, lo, mid);
  selection_sort(A, mid, hi);  //@assert is_sorted(A, mid, hi);
  // … merge the two halves
}
```

- We are left with implementing merge

Implementation



Assume we have an implementation

```
void merge(int[] A, int lo, int mid, int hi)
//@requires 0 <= lo && lo <= mid && mid <= hi  && hi <= \length(A);
//@requires is_sorted(A, lo, mid) && is_sorted(A, mid, hi);
//@ensures is_sorted(A, lo, hi);
```

Implementation

```
void selection_sort(int[] A, int lo, int hi)
//@requires 0 <= lo && lo <= hi  && hi <= \length(A);
//@ensures is_sorted(A, lo, hi);

void merge(int[] A, int lo, int mid, int hi)
//@requires 0 <= lo && lo <= mid && mid <= hi  && hi <= \length(A);
//@requires is_sorted(A, lo, mid) && is_sorted(A, mid, hi);
//@ensures is_sorted(A, lo, hi);
```

```
void sort(int[] A, int lo, int hi)
//@requires 0 <= lo && lo <= hi  && hi <= \length(A);
//@ensures is_sorted(A, lo, hi);
{
  int mid = lo + (hi - lo) / 2;
  //@assert  lo <= mid && mid <= hi;
  selection_sort(A, lo, mid);  //@assert is_sorted(A, lo, mid);
  selection_sort(A, mid, hi);  //@assert is_sorted(A, mid, hi);
  merge(A, lo, mid, hi);
}
```

To show: 0 ≤ lo ≤ mid ≤ hi ≤ \length(A)
Left as exercise

To show: A[lo, mid) sorted **and** A[mid, hi) sorted
- by the postconditions of selection_sort

- Is this code safe? ✓
- if merge is correct, its postcondition holds
  - A[lo, hi) sorted

Implementation

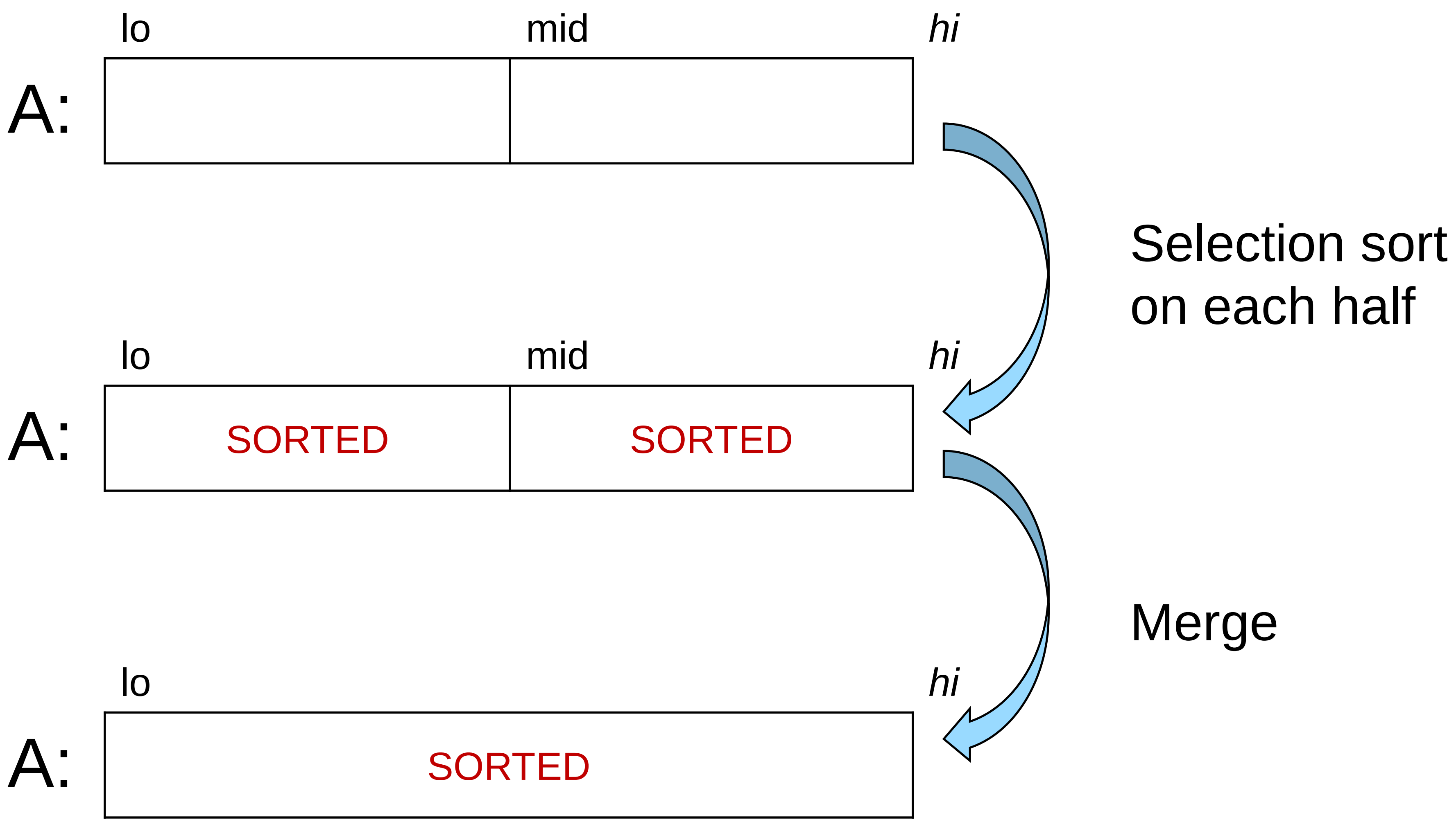
```
void selection_sort(int[] A, int lo, int hi)
//@requires 0 <= lo && lo <= hi  && hi <= \length(A);
//@ensures is_sorted(A, lo, hi);

void merge(int[] A, int lo, int mid, int hi)
//@requires 0 <= lo && lo <= mid && mid <= hi  && hi <= \length(A);
//@requires is_sorted(A, lo, mid) && is_sorted(A, mid, hi);
//@ensures is_sorted(A, lo, hi);
```

```
void sort(int[] A, int lo, int hi)
//@requires 0 <= lo && lo <= hi  && hi <= \length(A);
//@ensures is_sorted(A, lo, hi);
{
  int mid = lo + (hi - lo) / 2;
  //@assert  lo <= mid && mid <= hi;
  selection_sort(A, lo, mid);  //@assert is_sorted(A, lo, mid);
  selection_sort(A, mid, hi);  //@assert is_sorted(A, mid, hi);
  merge(A, lo, mid, hi);        //@assert is_sorted(A, lo, hi);
}
```

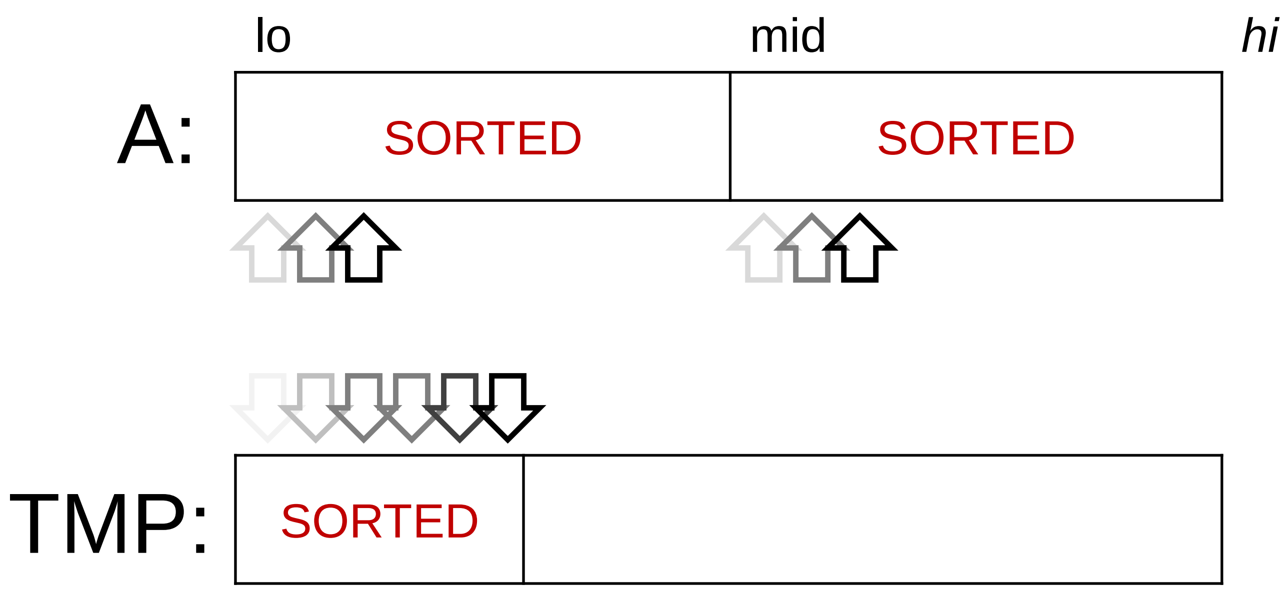
- A[lo, hi) sorted is the postcondition of sort
  - sort is **correct**

Implementation



- But how does merge work?

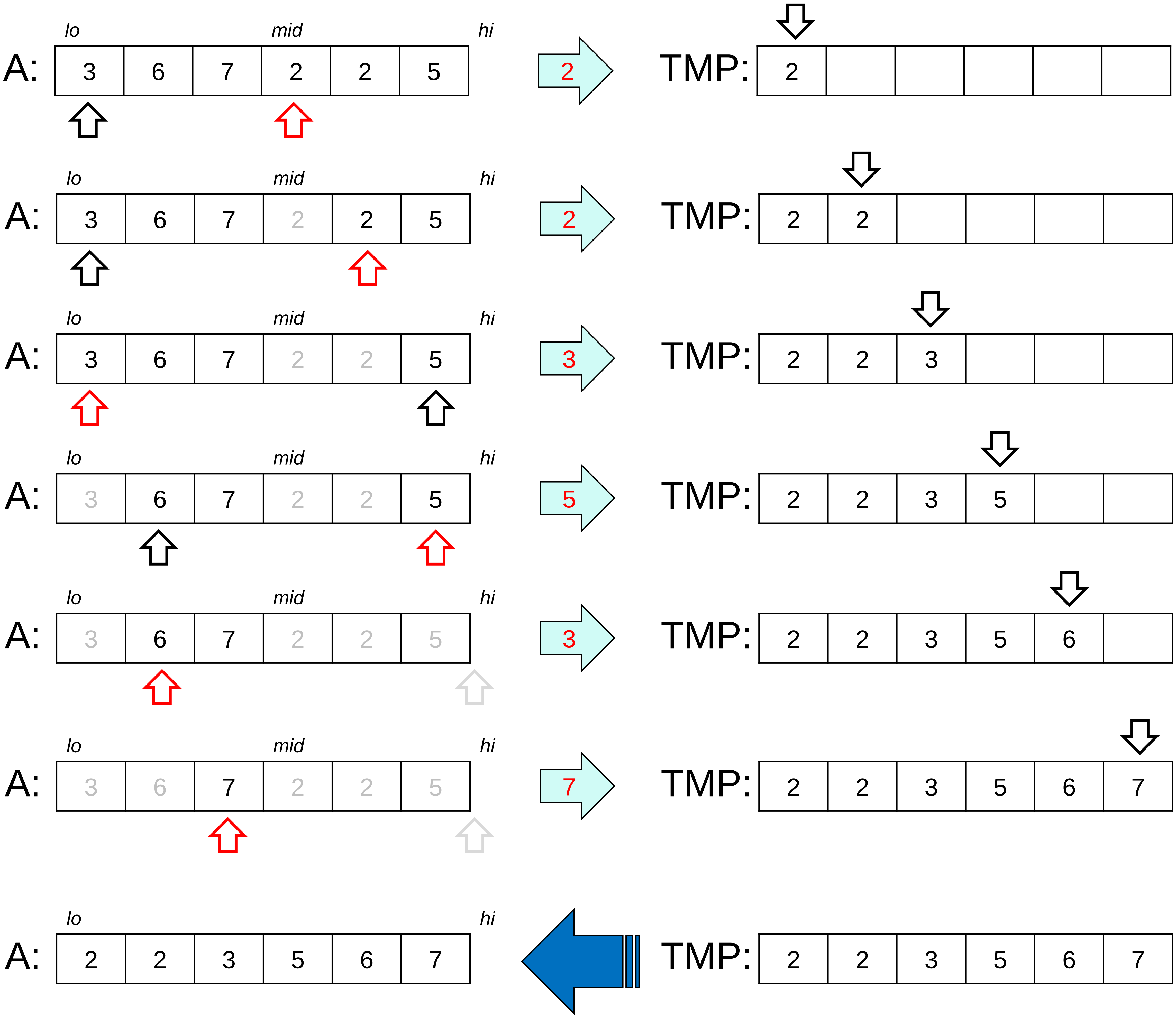
merge



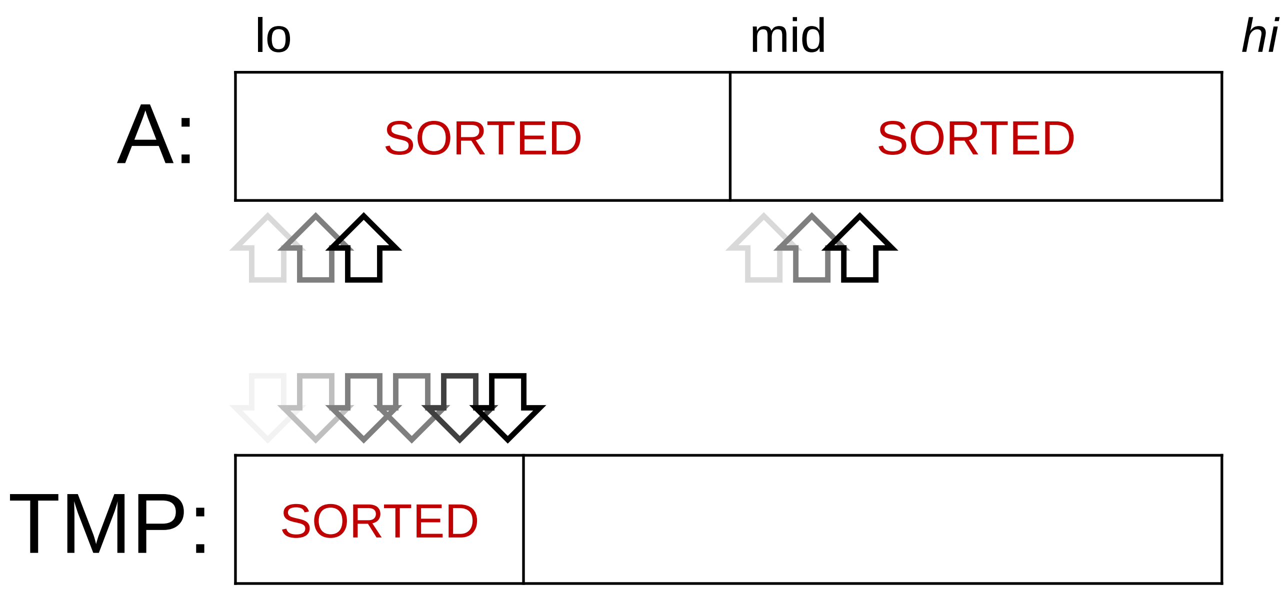
- Scan the two half array segments from left to right
- At each step, copy the smaller element in a temporary array
- Copy the temporary array back into A[lo, hi)

See code online

Example merge



merge



- Cost of merge?
  - if *A[lo, hi)* has *n* elements,
  - we copy one element to TMP at each step
    - *n* steps
  - we copy all *n* elements back to A at the end

  } ***O(n)***

- That's cheaper then $n^2/2$

In-place

- Code that uses at most a constant amount of temporary storage are called **in-place**

For example

```
void f(int[] A, int n) {
  int a = 8*n;
  bool b = false;
  char[] c = alloc_array(char, 2*n);
  string[] d = alloc_array(string, 10);
  int[] e = A;
}
```

This is a **constant amount of storage** because it takes a fixed amount of space, regardless of what n is

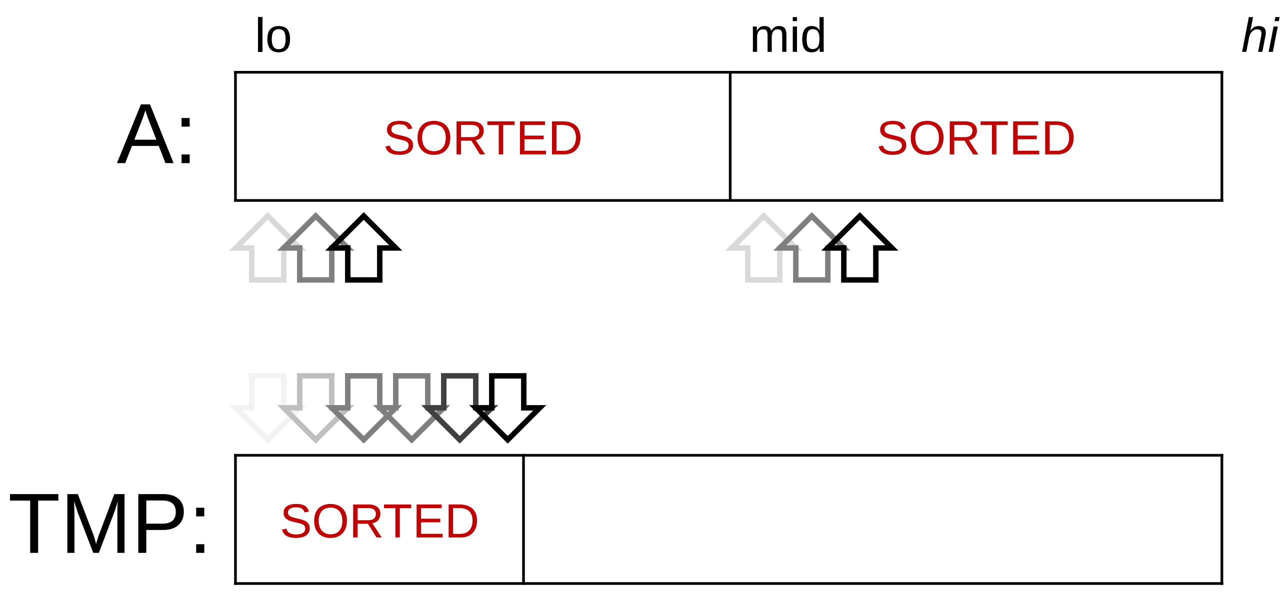
This is **not** a **constant amount of storage** because the length of c depends on the value of the parameter n

This is a **constant amount of storage** because the length of d does not depend on n

This is a **constant amount of storage** because e is just an alias to A

So f is not in-place

merge

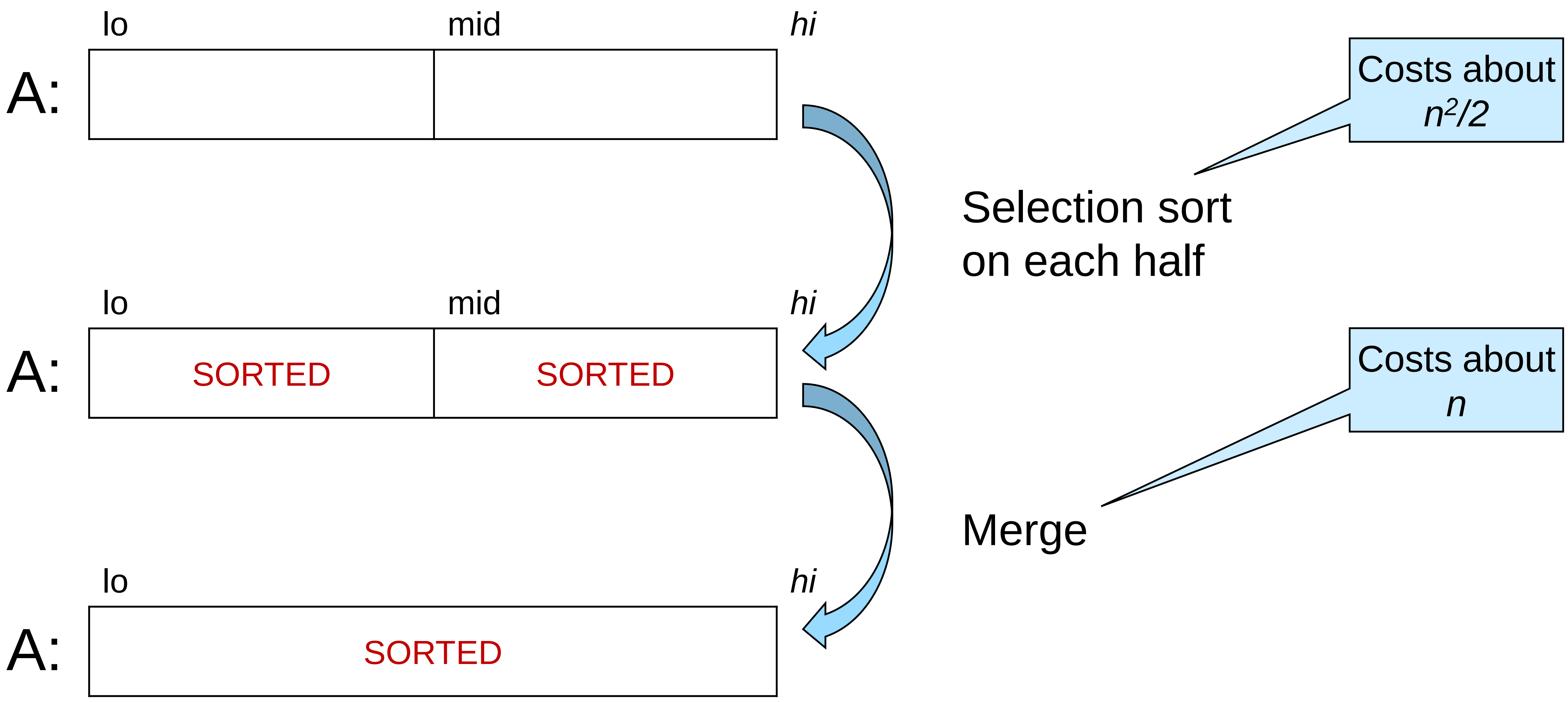


- *Algorithms that use at most a constant amount of temporary storage are called **in-place***

- merge uses lots of temporary storage
  - array TMP -- same size as A[lo, hi)
  - merge is not in-place

- In-place algorithms for merge are more expensive

Using Selection Sort Cleverly



- The overall cost is about $n^2/2 + n$
  - better than plain selection sort — n^2
  - but still $O(n^2)$

Mergesort

```
void selection_sort(int[] A, int lo, int hi)
//@requires 0 <= lo && lo <= hi  && hi <= \length(A);
//@ensures is_sorted(A, lo, hi);
```

Reflection

```
void sort(int[] A, int lo, int hi)
//@requires 0 <= lo && lo <= hi  && hi <= \length(A);
//@ensures is_sorted(A, lo, hi);
{
  int mid = lo + (hi - lo) / 2;
  //@assert  lo <= mid && mid <= hi;
  selection_sort(A, lo, mid);  //@assert is_sorted(A, lo, mid);
  selection_sort(A, mid, hi);  //@assert is_sorted(A, mid, hi);
  merge(A, lo, mid, hi);       //@assert is_sorted(A, lo, hi)
}
```

- selection_sort and sort are **interchangeable**
  - they solve the **same problem** — *sorting an array segment*
  - they have the **same contracts**
  - both are **correct**

A Recursive sort

```
void selection_sort(int[] A, int lo, int hi)
//@requires 0 <= lo && lo <= hi  && hi <= \length(A);
//@ensures is_sorted(A, lo, hi);
```

```
void sort(int[] A, int lo, int hi)
//@requires 0 <= lo && lo <= hi  && hi <= \length(A);
//@ensures is_sorted(A, lo, hi);
{
  int mid = lo + (hi - lo) / 2;
  //@assert  lo <= mid && mid <= hi;
  sort(A, lo, mid);           //@assert is_sorted(A, lo, mid);
  sort(A, mid, hi);           //@assert is_sorted(A, mid, hi);
  merge(A, lo, mid, hi);      //@assert is_sorted(A, lo, hi);
}
```

- Replace the calls to selection_sort with **recursive** calls to sort
  - same preconditions: calls to sort are safe
  - same postconditions: can only return sorted array segments
  - nothing changes for merge
    - merge returns a sorted array segment
- sort cannot compute the wrong result

A Recursive sort

```
void sort(int[] A, int lo, int hi)
//@requires 0 <= lo && lo <= hi  && hi <= \length(A);
//@ensures is_sorted(A, lo, hi);
{
  int mid = lo + (hi - lo) / 2;
  //@assert  lo <= mid && mid <= hi;
  sort(A, lo, mid);               //@assert is_sorted(A, lo, mid);
  sort(A, mid, hi);               //@assert is_sorted(A, mid, hi);
  merge(A, lo, mid, hi);          //@assert is_sorted(A, lo, hi);
}
```

- Is sort **correct**?
  - it cannot compute the wrong result
  - but will it compute the right result?

- This is a recursive function
  - but no base case!

A Recursive sort

- What if hi == lo?
  - mid == lo
  - recursive calls with identical arguments
    - infinite loop!!

- What to do?
  - A[lo,lo) is the empty array
  - always sorted!
  - simply return

```
void sort(int[] A, int lo, int hi)
//@requires 0 <= lo && lo <= hi  && hi <= \length(A);
//@ensures is_sorted(A, lo, hi);
{
  if (hi == lo) return;
  int mid = lo + (hi - lo) / 2;
  //@assert  lo <= mid && mid < hi;
  sort(A, lo, mid);              //@assert is_sorted(A, lo, mid);
  sort(A, mid, hi);              //@assert is_sorted(A, mid, hi);
  merge(A, lo, mid, hi);         //@assert is_sorted(A, lo, hi);
}
```

mid == hi now impossible

A Recursive sort

- What if hi == lo+1?
  - mid == lo, still
  - first recursive call: sort(A, lo, lo)
    - handled by the new base case
  - second recursive call: sort(A, lo, hi)
    - infinite loop!!
- What to do?
  - A[lo,lo+1) is a 1-element array
  - always sorted!
  - simply return!

```
void sort(int[] A, int lo, int hi)
//@requires 0 <= lo && lo <= hi  && hi <= \length(A);
//@ensures is_sorted(A, lo, hi);
{
  if (hi == lo) return;
  if (hi == lo+1) return;

  int mid = lo + (hi - lo) / 2;
  //@assert  lo < mid && mid < hi;
  sort(A, lo, mid);              //@assert is_sorted(A, lo, mid);
  sort(A, mid, hi);              //@assert is_sorted(A, mid, hi);
  merge(A, lo, mid, hi);         //@assert is_sorted(A, lo, hi);
}
```

mid == lo also impossible

A Recursive sort

- No more opportunities for infinite loops
- The preconditions still imply the postconditions
  - base case return: arrays of lengths 0 and 1 are always sorted
  - final return: our original proof applies

- sort is **correct!**

- This function is called

  mergesort

```
void sort(int[] A, int lo, int hi)
//@requires 0 <= lo && lo <= hi  && hi <= \length(A);
//@ensures is_sorted(A, lo, hi);
{
  if (hi - lo <= 1) return;
  int mid = lo + (hi - lo) / 2;
  //@assert  lo <= mid && mid <= hi;
  sort(A, lo, mid);             //@assert is_sorted(A, lo, mid);
  sort(A, mid, hi);             //@assert is_sorted(A, mid, hi);
  merge(A, lo, mid, hi);        //@assert is_sorted(A, lo, hi);
}
```

minor clean-up

A Recursive sort

- Recursive functions don't have loop invariants
- How does our correctness methodology transfer?
  - **INIT**: Safety of the initial call to the function
  - **PRES**: From the preconditions to the safety of the recursive calls
  - **EXIT**: From the postconditions of the recursive calls to the postcondition of the function
  - **TERM**:
    - the base case handles input smaller than some bound
    - the input of each recursive call is strictly smaller than the input of the function

Mergesort

```
void merge(int[] A, int lo, int mid, int hi)
//@requires 0 <= lo && lo <= mid && mid <= hi  && hi <= \length(A);
//@requires is_sorted(A, lo, mid) && is_sorted(A, mid, hi);
//@ensures is_sorted(A, lo, hi);
```

```
void mergesort(int[] A, int lo, int hi)
//@requires 0 <= lo && lo <= hi  && hi <= \length(A);
//@ensures is_sorted(A, lo, hi);
{
  if (hi - lo <= 1) return;

  int mid = lo + (hi - lo) / 2;
  //@assert  lo < mid && mid < hi;
  mergesort(A, lo, mid);
  mergesort(A, mid, hi);
  merge(A, lo, mid, hi);
}
```

Complexity of Mergesort

```
void mergesort(int[] A, int lo, int hi) {
  if (hi - lo <= 1) return;      // O(1)
  int mid = lo + (hi - lo) / 2;  // O(1)
  mergesort(A, lo, mid);
  mergesort(A, mid, hi);
  merge(A, lo, mid, hi);         // O(n)
}
```

n = hi - lo

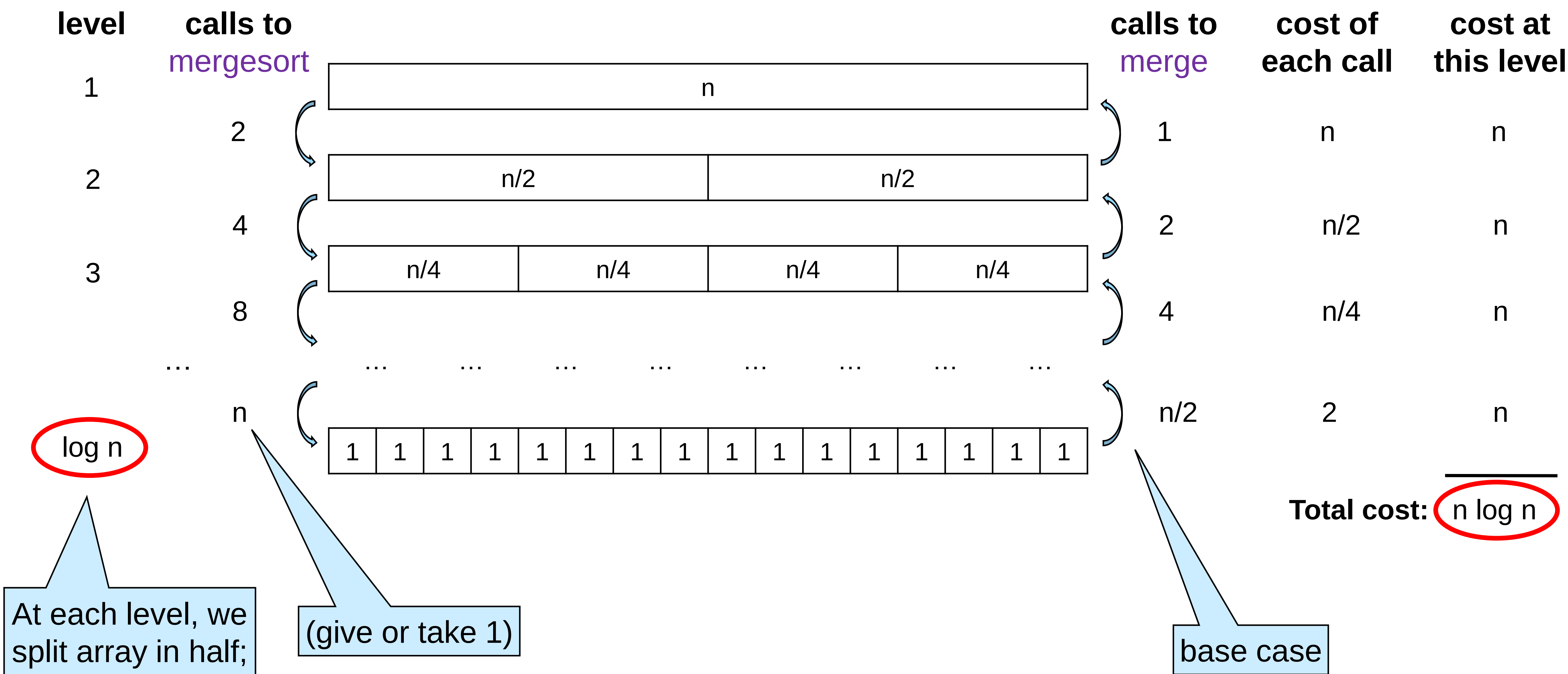
- Work done by each call to mergesort

  (ignoring recursive calls)
  - Base case: constant cost -- O(1)
  - Recursive case:
    - compute mid: constant cost -- O(1)
    - recursive calls: (ignored)
    - merge: linear cost -- O(n)

- We need to add this for all recursive calls
  - It is convenient to organize them by *level*

Complexity of Mergesort

```
void mergesort(int[] A, int lo, int hi) {
  if (hi - lo <= 1) return;        // O(1)
  int mid = lo + (hi - lo) / 2;   // O(1)
  mergesort(A, lo, mid);
  mergesort(A, mid, hi);
  merge(A, lo, mid, hi);          // O(n)
}
```



O(n log n)

Comparing Sorting Algorithms

- Selection sort and mergesort solve the **same problem**
  - mergesort is **asymptotically faster**: *$O(n \log n)$* vs. *$O(n^2)$*
    - mergesort is preferable if speed for large inputs is all that matters
  - selection sort is **in-place** but mergesort is not
    - selection sort may be preferable if space is very tight

- Choosing an algorithm involves several parameters
  - **It depends on the application**

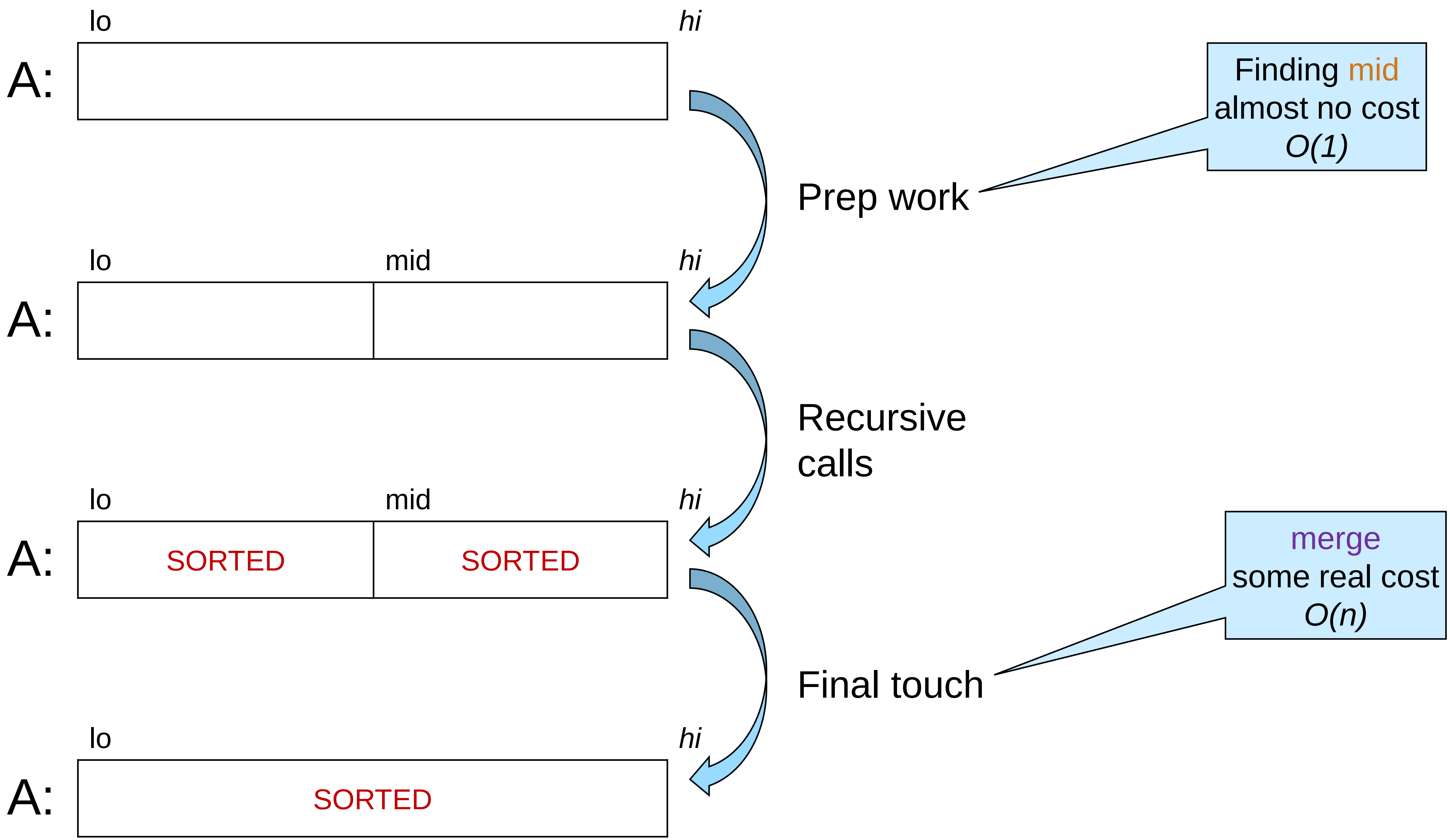
- Summary

	Selection sort	**Mergesort**
Worst-case complexity	$O(n^2)$	$O(n \log n)$
In-place?	Yes	No

Quicksort

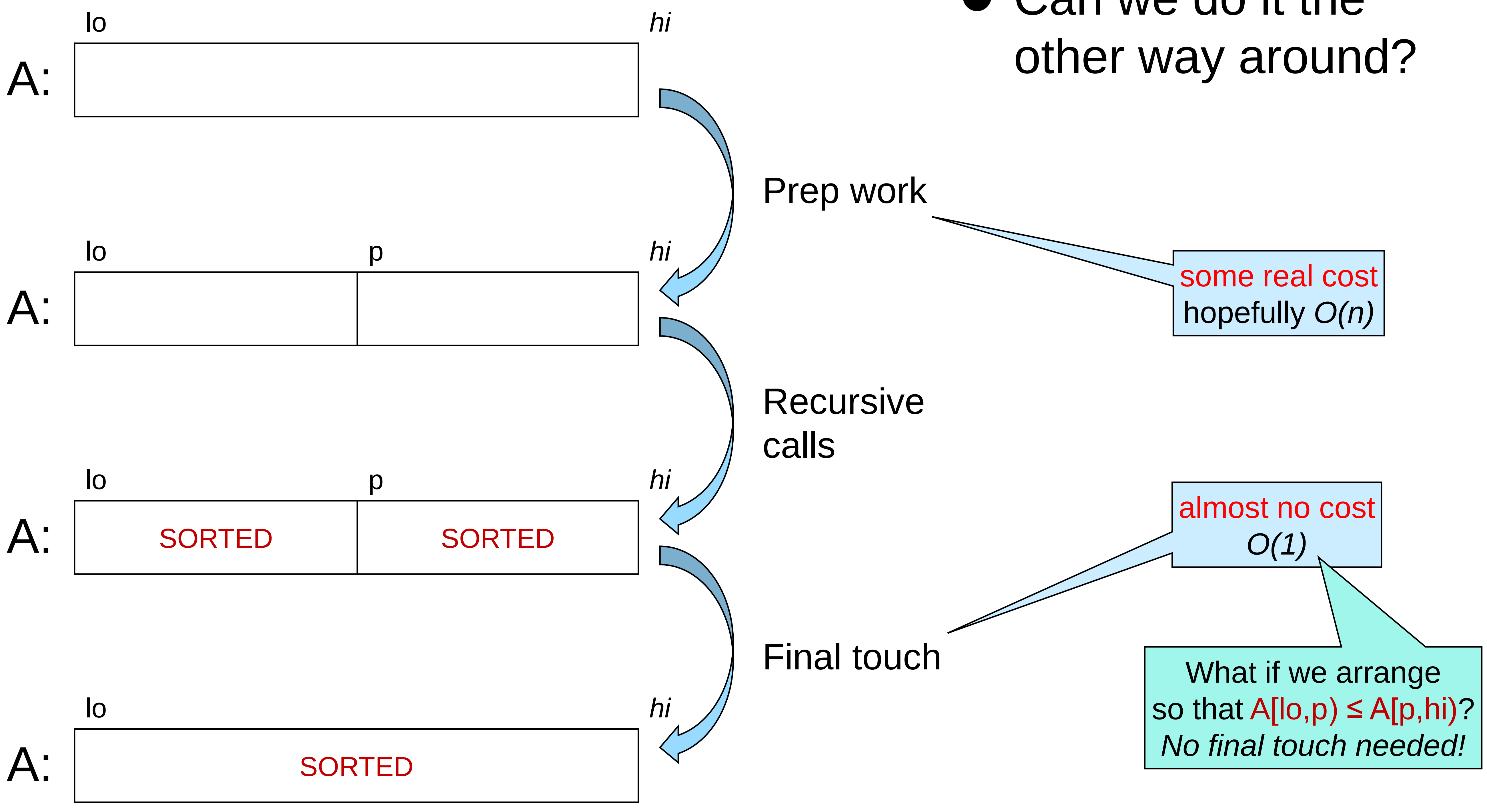
Reflections

```
void mergesort(int[] A, int lo, int hi) {
  if (hi - lo <= 1) return;
  int mid = lo + (hi - lo) / 2;
  mergesort(A, lo, mid);
  mergesort(A, mid, hi);
  merge(A, lo, mid, hi);
}
```



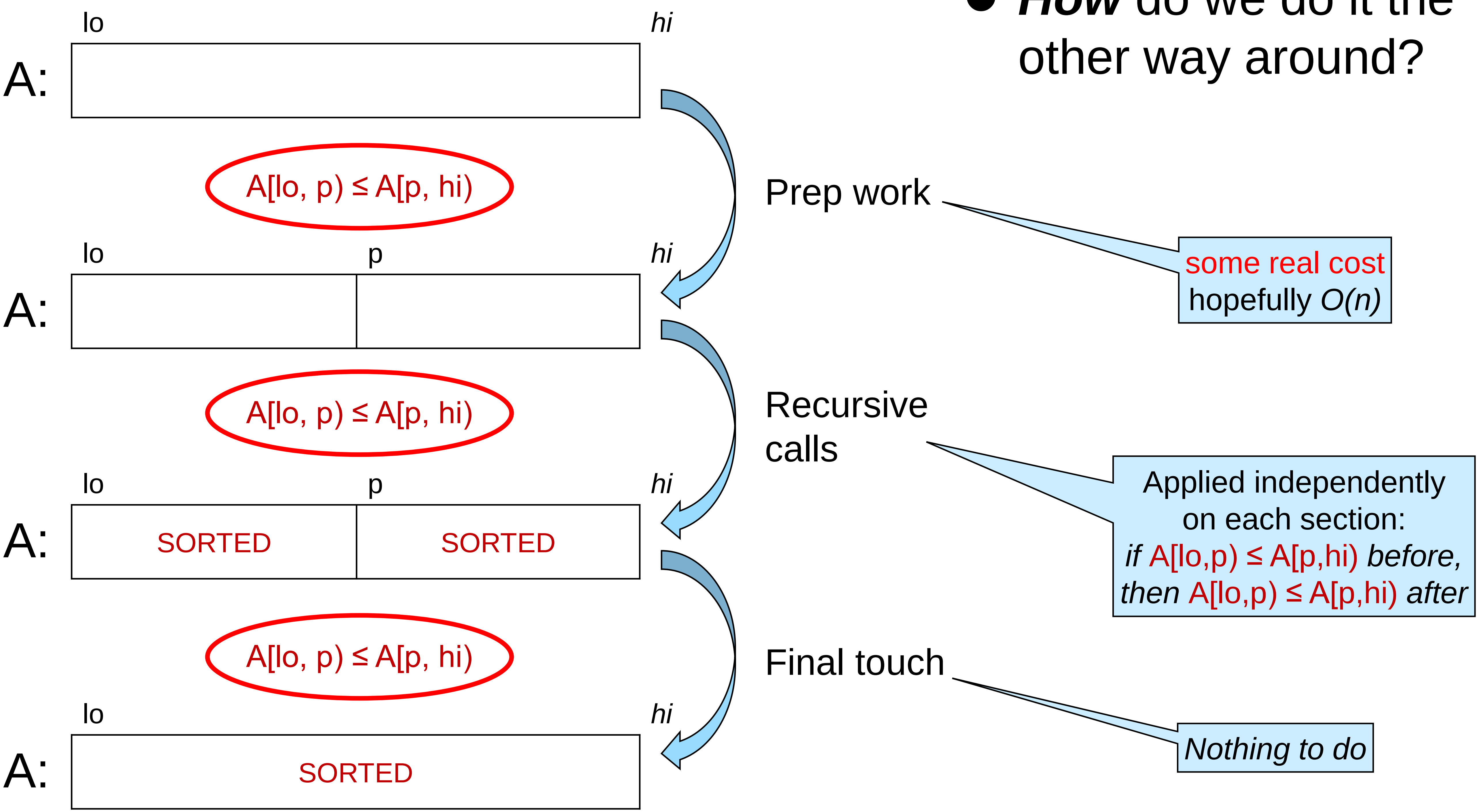
Reflections

- Can we do it the other way around?



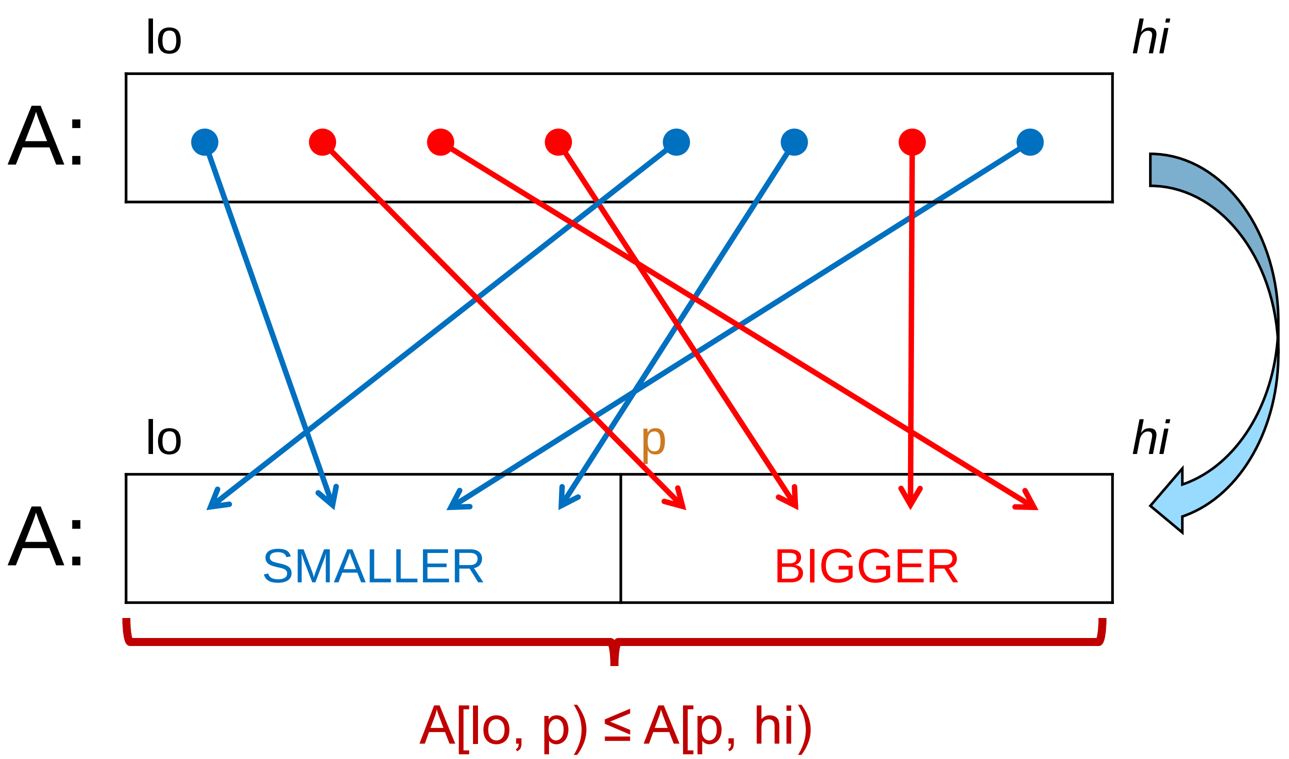
Reflections

- ***How*** do we do it the other way around?



Partition

- A function that
  - moves small values to the left of A
  - moves big values to the right of A
  - returns the index p that separates them



- This is partition

```
int partition (int[] A, int lo, int hi)
//@requires 0 <= lo && lo <= hi  && hi <= \length(A);
//@ensures lo <= \result && \result <= hi;
//@ensures le_segs(A, lo, \result, A, \result, hi);
```

Partition

- Using partition in sort

```
int partition (int[] A, int lo, int hi)
//@requires 0 <= lo && lo <= hi  && hi <= \length(A);
//@ensures lo <= \result && \result <= hi;
//@ensures le_segs(A, lo, \result, A, \result, hi);
```

```
void sort(int[] A, int lo, int hi)
//@requires 0 <= lo && lo <= hi  && hi <= \length(A);
//@ensures is_sorted(A, lo, hi);
{
  if (hi - lo <= 1) return;

  int p = partition(A, lo, hi);
  //@assert  lo <= p && p <= hi;
  sort(A, lo, p);
  sort(A, p, hi);
}
```

just like mergesort

- What if p == hi where hi > lo+1 ?
  - Infinite loop!
- We want p < hi
- Thus \result < hi in partition

Partition

- To use partition in sort, we need \result < hi

```
int partition (int[] A, int lo, int hi)
//@requires 0 <= lo && lo <= hi  && hi <= \length(A);
//@ensures lo <= \result && \result < hi;
//@ensures le_segs(A, lo, \result, A, \result, hi);
```

DANGER!
this function is **unimplementable**!
if hi==lo,
then \result can't exist

- We want lo < hi

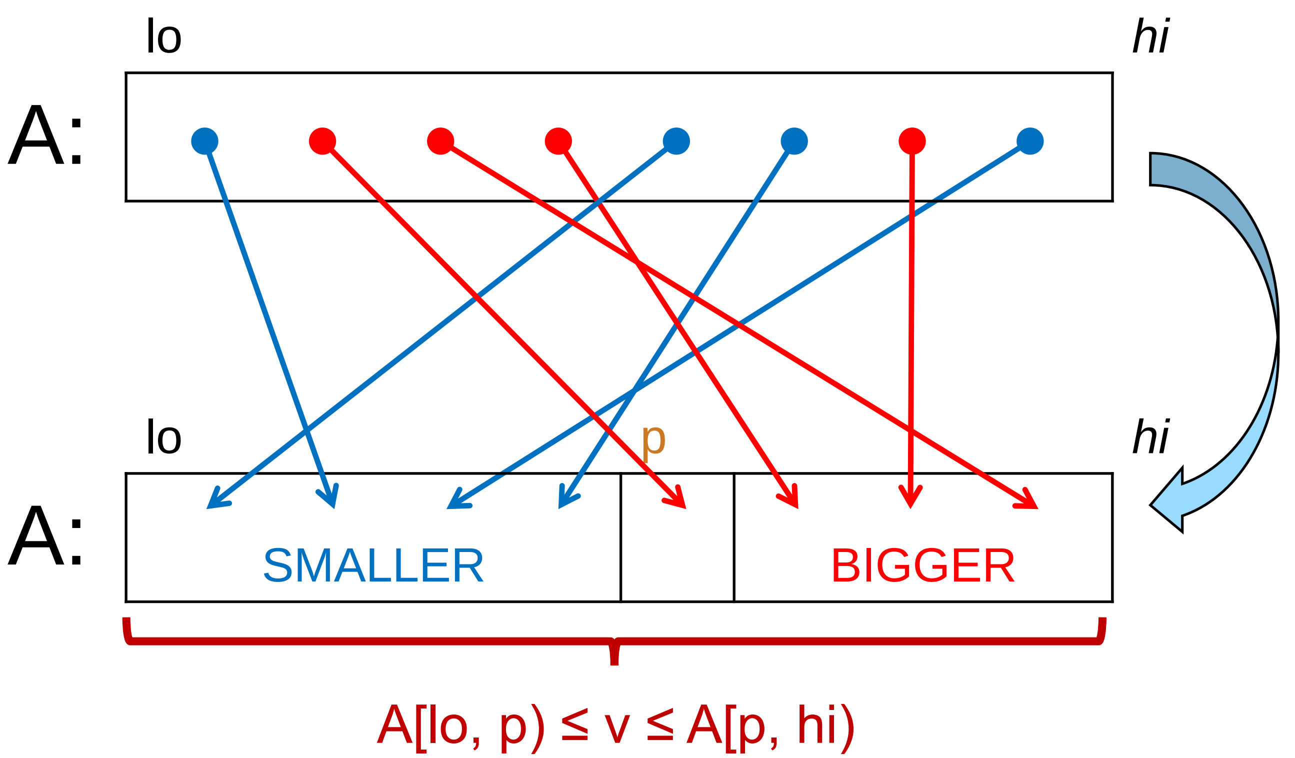
```
int partition (int[] A, int lo, int hi)
//@requires 0 <= lo && lo < hi  && hi <= \length(A);
//@ensures lo <= \result && \result < hi;
//@ensures le_segs(A, lo, \result, A, \result, hi);
```

  - There is an element at A[\result]

Partition

- If A[lo, p) ≤ A[p, hi), then there is a value v such that

A[lo, p) ≤ v ≤ A[p, hi)

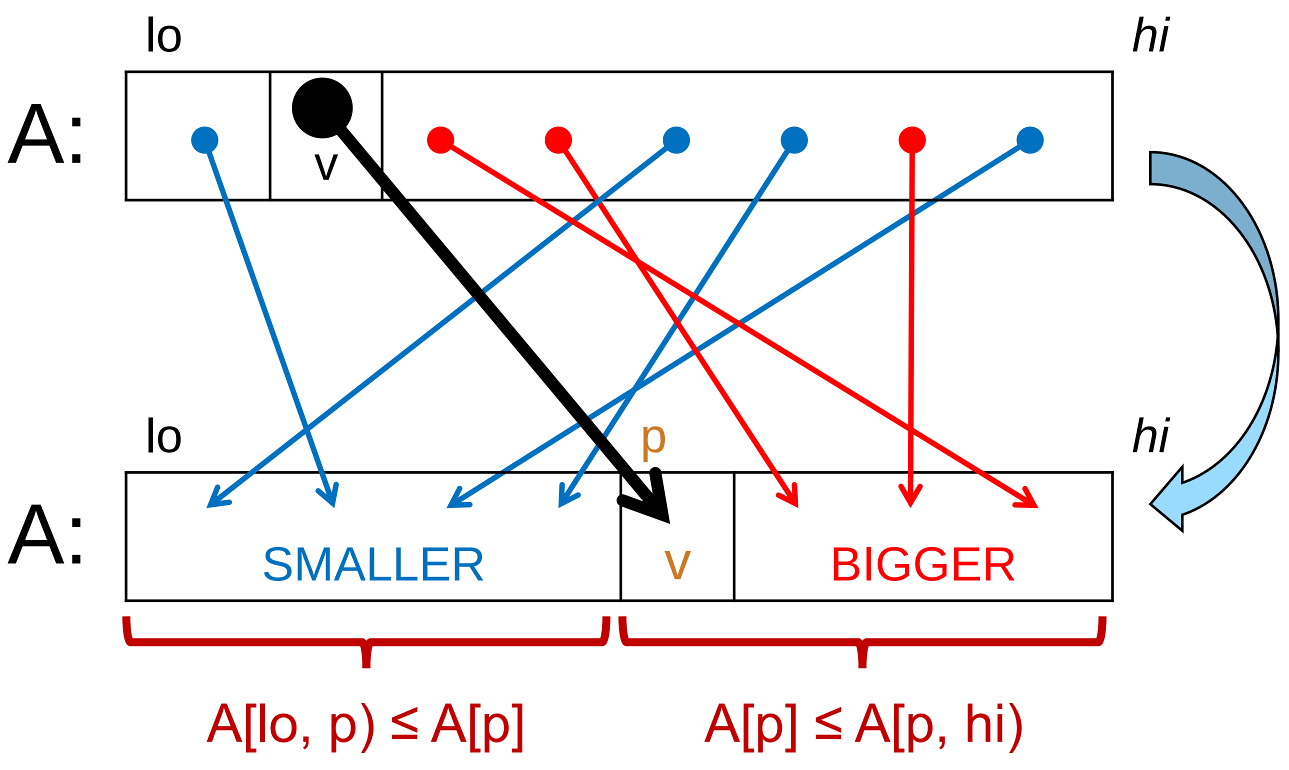


Partition

- There is a value v s.t. A[lo, p) ≤ v ≤ A[p, hi)

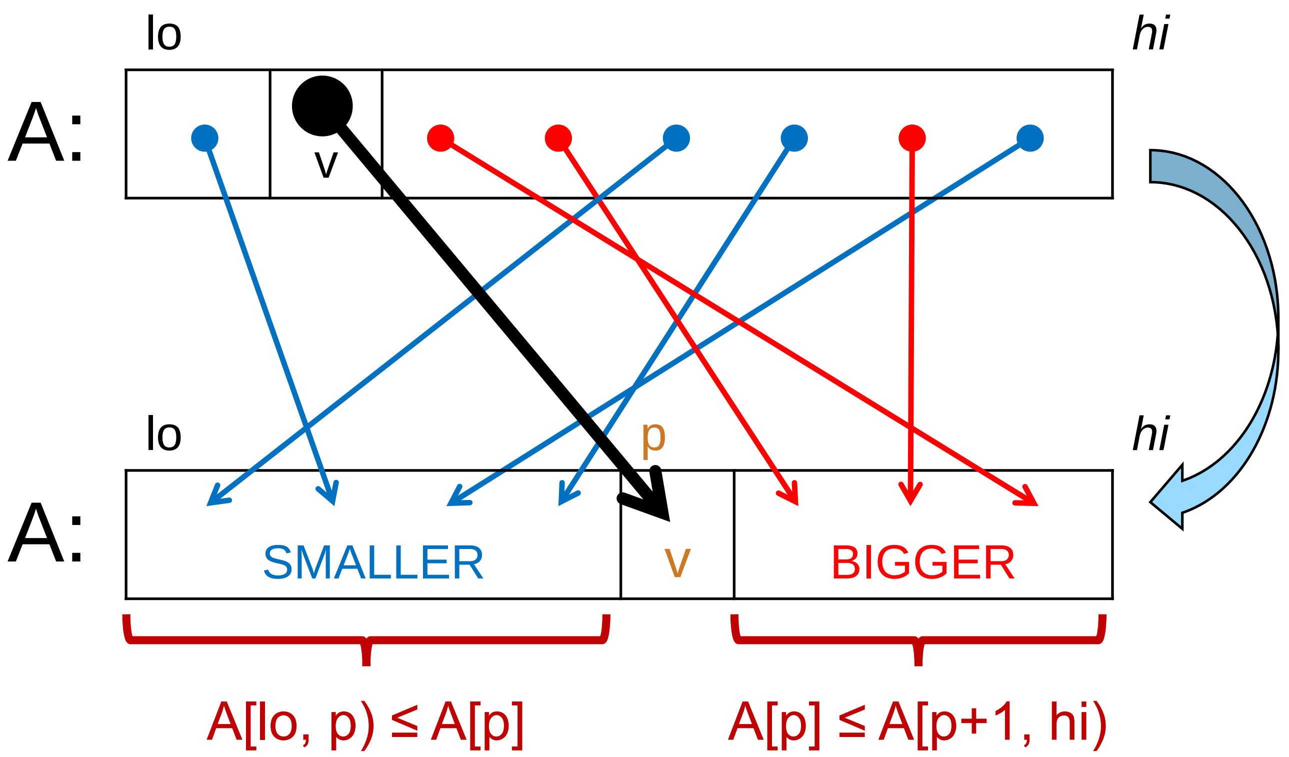
- Since lo < hi, take v to be an element of A[lo, hi)
  - It will end up in A[p]
  - A[lo, p) ≤ A[p] ≤ A[p, hi)

- v is called the **pivot**
  - p is the **pivot index**



Partition

- A[lo, p) ≤ A[p] ≤ A[p, hi)
  is equivalent to
  - A[lo, p) ≤ A[p]
  - A[p] ≤ A[p+1, hi)




```
int partition (int[] A, int lo, int hi)
//@requires 0 <= lo && lo < hi  && hi <= \length(A);
//@ensures lo <= \result && \result < hi;
//@ensures ge_seg(A[\result], A, lo, \result);
//@ensures le_seg(A[\result], A, \result+1, hi);
```

- the pivot separates the smaller and the bigger elements
- it ends up in the right place in the sorted array

Quicksort

```
int partition (int[] A, int lo, int hi)
//@requires 0 <= lo && lo < hi  && hi <= \length(A);
//@ensures lo <= \result && \result < hi;
//@ensures ge_seg(A[\result], A, lo, \result);
//@ensures le_seg(A[\result], A, \result+1, hi);
```

- This algorithm is called **quicksort**

```
void quicksort(int[] A, int lo, int hi)
//@requires 0 <= lo && lo <= hi  && hi <= \length(A);
//@ensures is_sorted(A, lo, hi);
{
  if (hi - lo <= 1) return;

  int p = partition(A, lo, hi);
  //@assert  lo <= p && p < hi;
  quicksort(A, lo, p);
  quicksort(A, p+1, hi);
}
```

pivot A[p] is already in the right place

Quicksort

```
int partition (int[] A, int lo, int hi)
//@requires 0 <= lo && lo < hi  && hi <= \length(A);
//@ensures lo <= \result && \result < hi;
//@ensures ge_seg(A[\result], A, lo, \result);
//@ensures le_seg(A[\result], A, \result+1, hi);
```

- Is it safe?

```
1.  void quicksort(int[] A, int lo, int hi)
2.  //@requires 0 <= lo && lo <= hi  && hi <= \length(A);
3.  //@ensures is_sorted(A, lo, hi);
4.  {
5.    if (hi - lo <= 1) return;
6.    int p = partition(A, lo, hi);
7.    //@assert  lo <= p && p < hi;
8.    quicksort(A, lo, p);
9.    quicksort(A, p+1, hi);
10. }
```

To show: 0 ≤ lo < hi ≤ \length(A)

- 0 ≤ lo — by line 2
- lo ≤ hi+1 — by line 5
  lo < hi — by math
- hi ≤ \length(A) — by line 2

To show: 0 ≤ lo ≤ p ≤ \length(A)
Like mergesort

To show: 0 ≤ p+1 ≤ hi ≤ \length(A)
Left as exercise

✓

Quicksort

```
int partition (int[] A, int lo, int hi)
//@requires 0 <= lo && lo < hi  && hi <= \length(A);
//@ensures lo <= \result && \result < hi;
//@ensures ge_seg(A[\result], A, lo, \result);
//@ensures le_seg(A[\result], A, \result+1, hi);
```

- Is it correct?

```
1.  void quicksort(int[] A, int lo, int hi)
2.  //@requires 0 <= lo && lo <= hi  && hi <= \length(A);
3.  //@ensures is_sorted(A, lo, hi);
4.  {
5.    if (hi - lo <= 1) return;
6.    int p = partition(A, lo, hi);
7.    //@assert  lo <= p && p < hi;
8.    //@assert ge_seg(A[p], A, lo, p);
9.    //@assert le_seg(A[p], A, p+1, hi);
10.   quicksort(A, lo, p);     //@assert is_sorted(A, lo, p);
11.   quicksort(A, p+1, hi); //@assert is_sorted(A, p+1, hi);
12. }
```

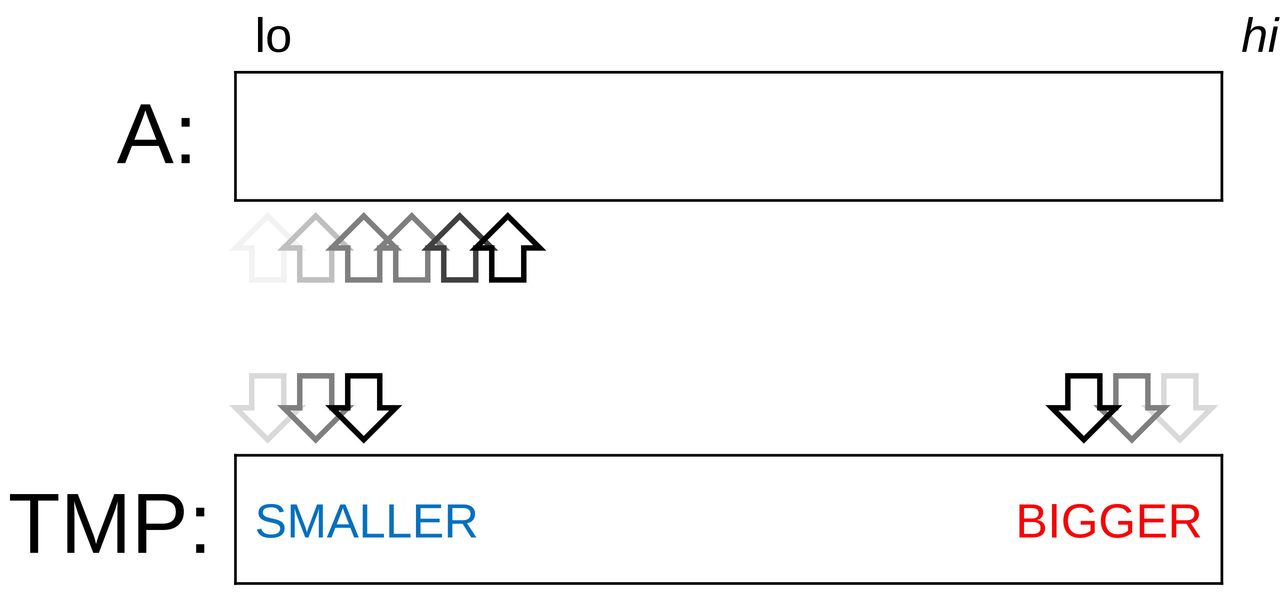
To show: A[lo, hi) sorted
All arrays of length 0 or 1 are sorted

To show: A[lo, hi) sorted

A. A[lo, p) ≤ A[p] by line 8
B. A[p] ≤ A[p+1, hi) by line 9
C. A[lo, p) sorted by line10
D. A[p+1, hi) sorted by line11
E. A[lo, hi) sorted by A-D

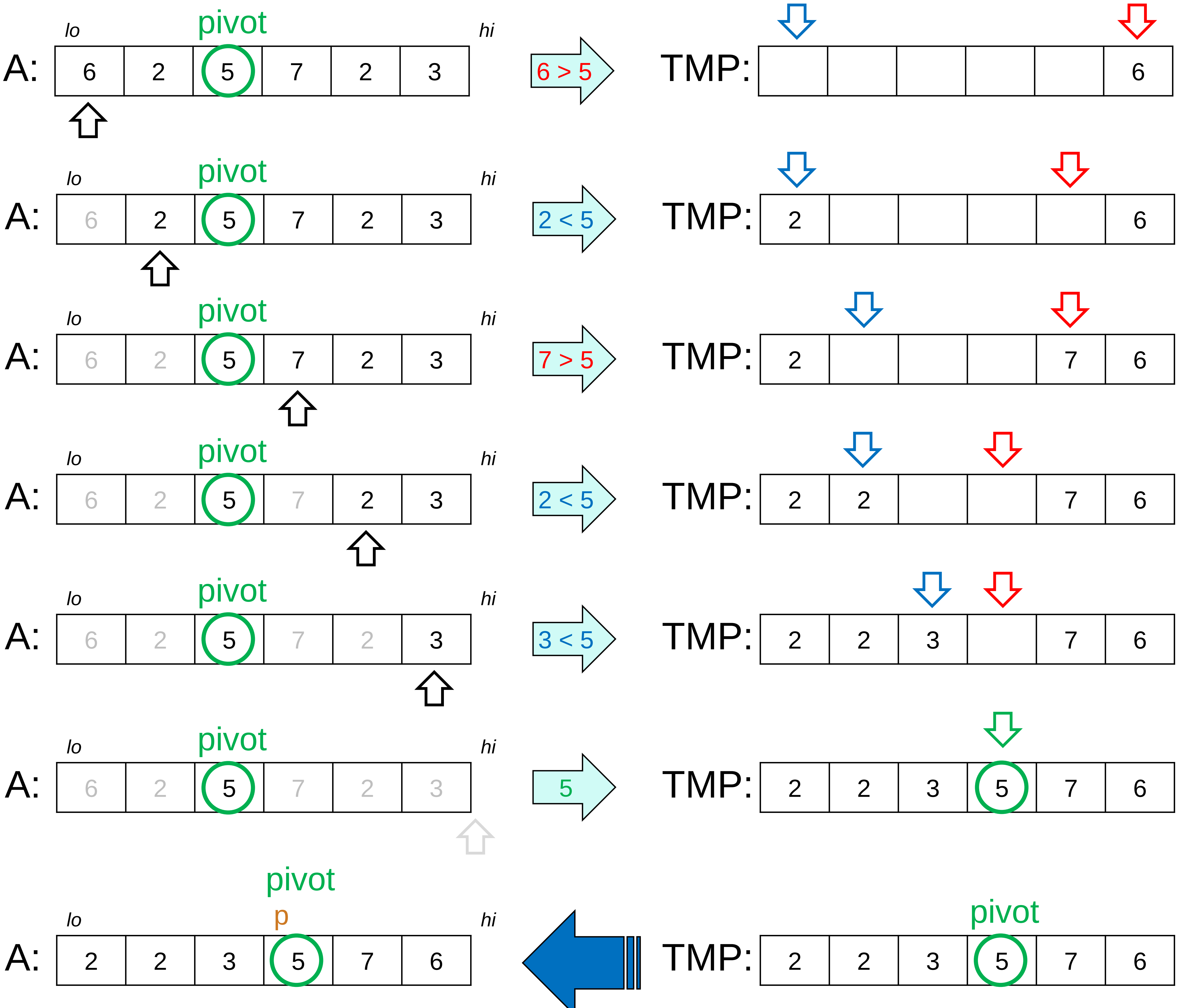
✓

How to partition

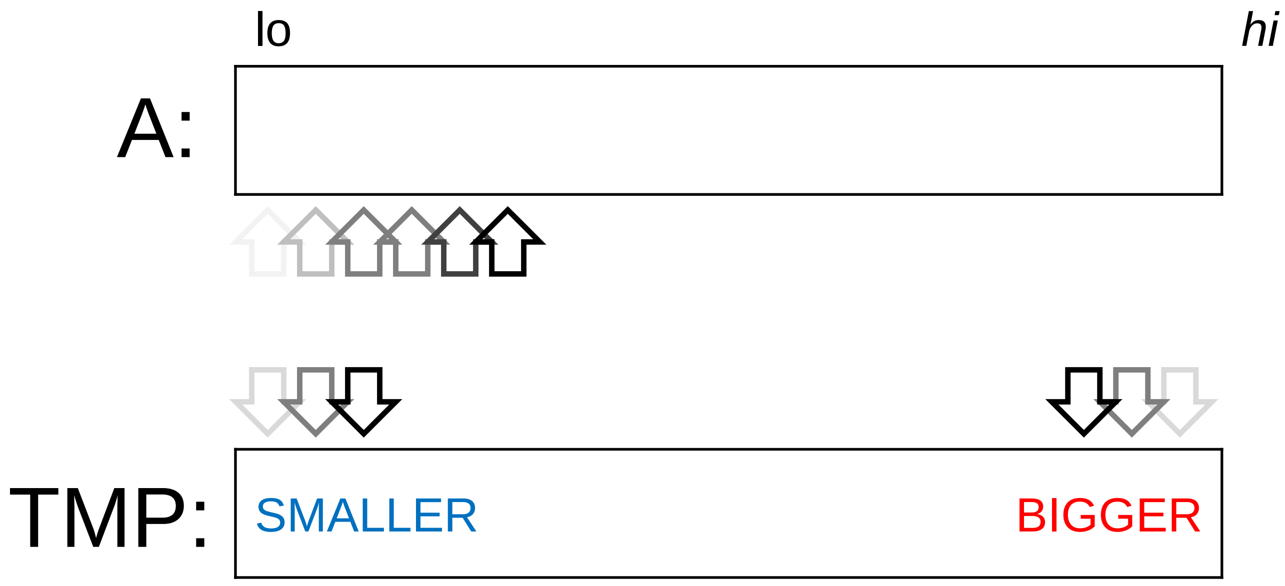


- Create a temporary array, TMP, the same size as A[lo, hi)
- Pick the pivot in the array
- Put all other elements at either end of TMP
  - smaller on the left, larger on the right
- Put pivot in the one spot left
- Copy TMP back into A[lo, hi)
- Return the index where the pivot ends up

Example partition



How to partition

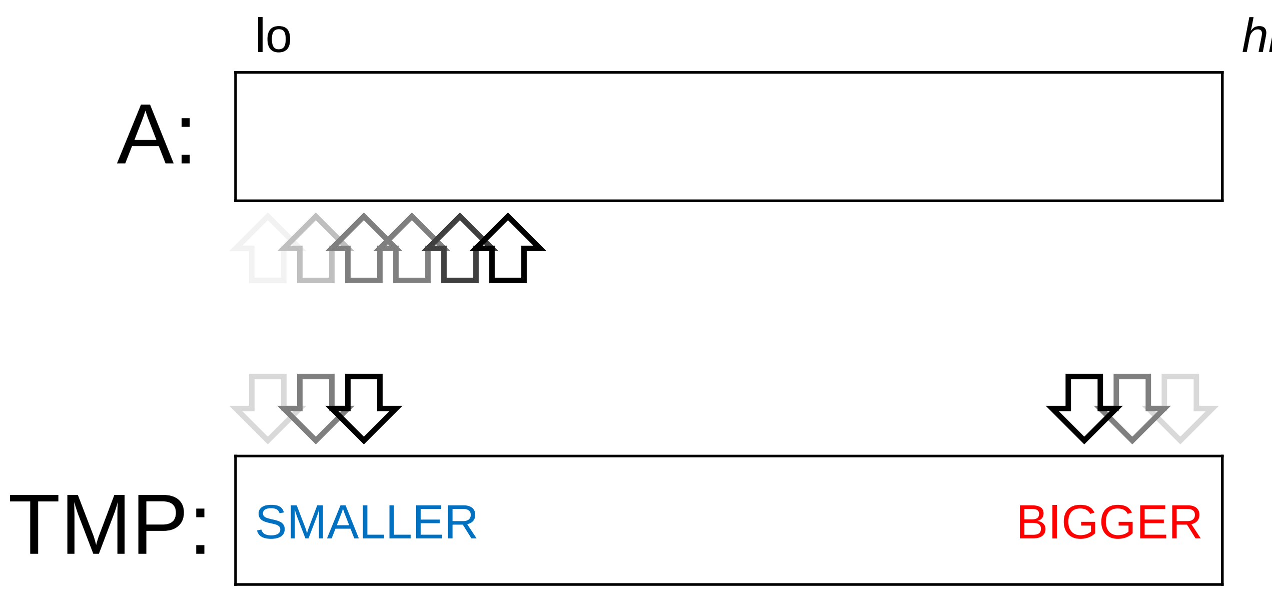


- Cost of partition?
  - if *A[lo, hi)* has *n* elements,
  - we copy one element to TMP at each step
    - *n* steps
  - we copy all *n* elements back to A at the end

O(n)

- Just like merge

How to partition



- Done this way, partition is **not** in-place
- With a little cleverness, this can be modified to be **in-place**
  - Still ***O(n)***

See code online

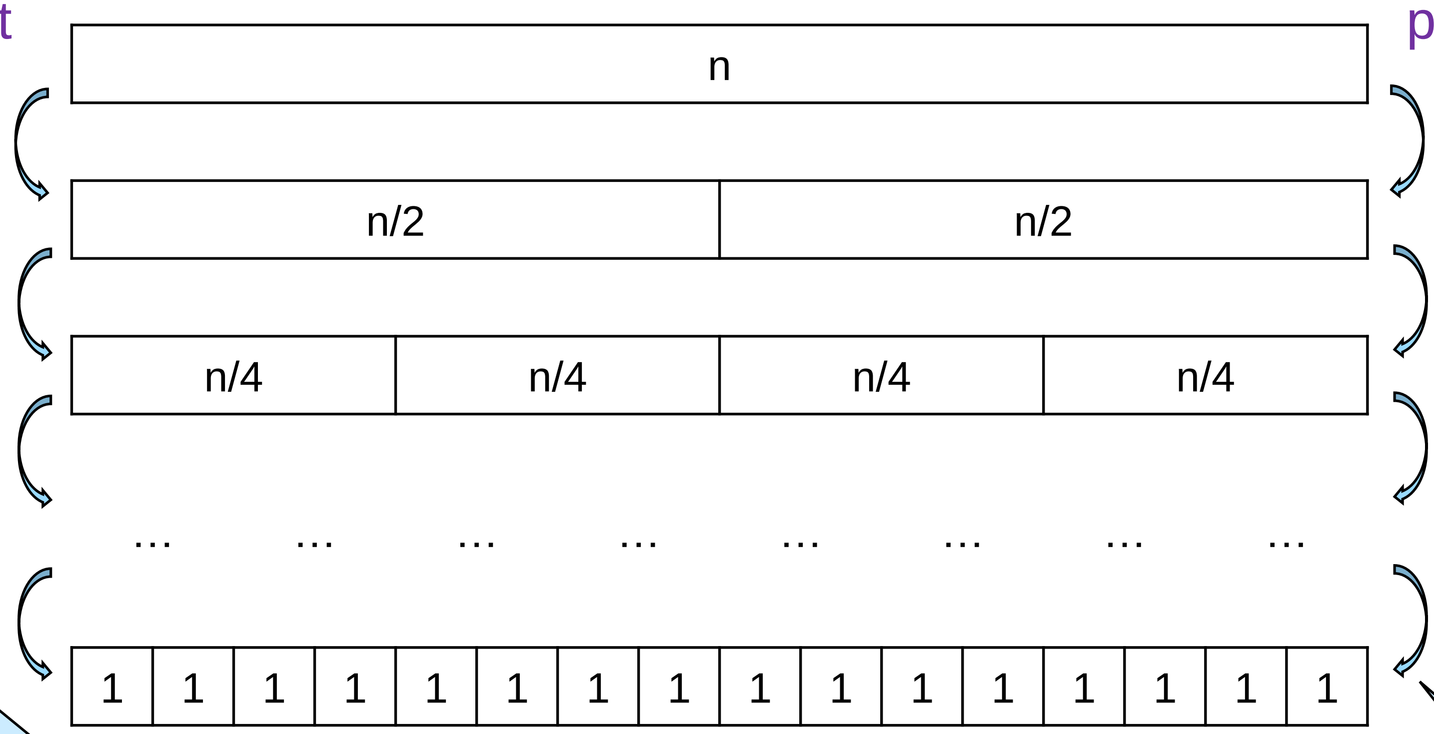
Complexity of Quicksort

```
void quicksort(int[] A, int lo, int hi) {
  if (hi - lo <= 1) return;          // O(1)
  int p = partition(A, lo, hi);    // O(n)
  quicksort(A, lo, p);
  quicksort(A, p+1, hi);
}
```

- If we pick the **median** of A[lo, hi) as the pivot,
  - the median is the value such that half elements are larger and half smaller
  - the pivot index then becomes the **midpoint**, *(lo + hi)/2*

  then it's like mergesort

level	**calls to** quicksort	**calls to** partition	**cost of each call**	**cost at this level**
1	2	1	n	n
2	4	2	n/2	n
3	8	4	n/4	n
…	…	…	…	…
log n	n	n/2	2	n



At each level, we split array in half; can be done only ***log n*** times

(give or take 1)

base case

Total cost: n log n

O(n log n)

Complexity of Quicksort

- How do we find the median?
  - sort the array and pick the element at the midpoint …
  - This defeats the purpose!
  - And it costs *O(n log n)* -- using mergesort

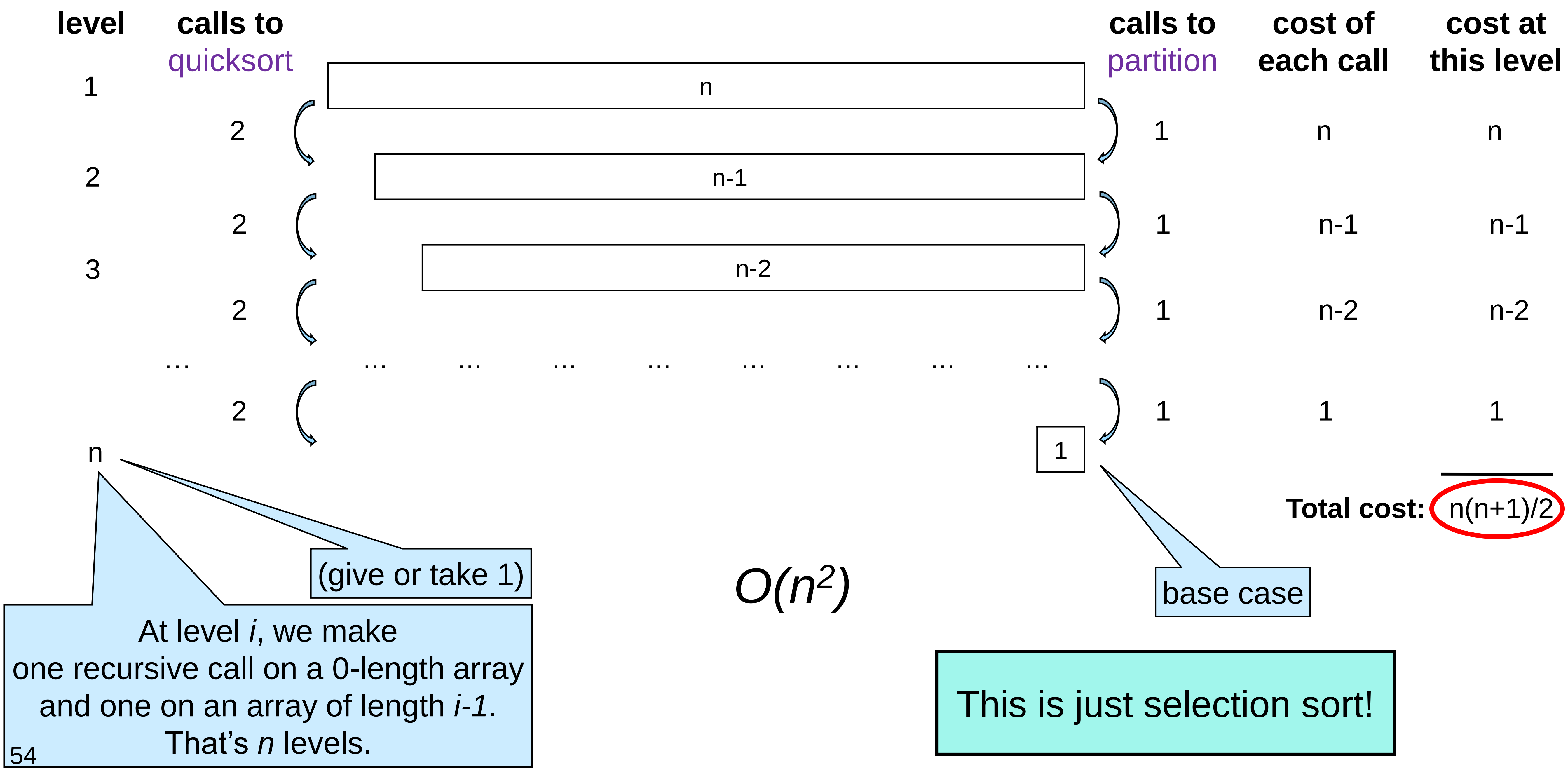
- We want to spent at most *O(n)*
- No such algorithm for finding the median!
  - Either *O(n log n)*
  - Or *O(n)* for an approximate solution
    - which may be an Ok compromise

- So, ***if we are lucky***, quicksort has cost O(n log n)

Complexity of Quicksort

```
void quicksort(int[] A, int lo, int hi) {
  if (hi - lo <= 1) return;        // O(1)
  int p = partition(A, lo, hi);   // O(n)
  quicksort(A, lo, p);
  quicksort(A, p+1, hi);
}
```

- What if we are **unlucky**?
  - Pick the **smallest** element each time *(or the largest)*



$O(n^2)$

This is just selection sort!

Complexity of Quicksort

QUICKsort ?!

A blatant case of false advertising?

- Worst-case complexity is $O(n^2)$
  - if array is (largely) already sorted
- Best case complexity is $O(n \log n)$
  - if we are so lucky to pick the median each time as the pivot
- What happens on average?
  - if we add up the cost for *each possible input* and divide by the number of possible inputs

$$O(n \log n)$$

This is what we expect if the array contains values selected at random

- but we may be unlucky and get $O(n^2)$!

- This is called **average-case complexity**

Complexity of Quicksort

- Worst-case complexity is $O(n^2)$
  - if array is (largely) already sorted
- Best case complexity is $O(n \log n)$
  - if we are so lucky to pick the median each time as the pivot
- Average-case complexity is $O(n \log n)$
  - **if we are not too unlucky**

- In practice, quicksort is pretty fast,
  - it often outperforms mergesort
  - and it is in-place!

> quicksort ?!
>
> Maybe there is something to it …

Selecting the Pivot

- How is the pivot chosen in practice?

- Common ways:
  - Pick A[lo]
    - or the element at any fixed index
  - Choose an index i at **random** and pick A[i]
  - Choose 3 indices i1, i2 and i3,
    and pick the median of A[i1], A[i2] and A[i3]

Comparing Sorting Algorithms

- Three algorithms to solve the **same problem**
  - and there are many more!
  - mergesort is asymptotically faster: *O(n log n)* vs. $O(n^2)$
  - selection sort and quicksort are in-place but merge sort is not
  - quicksort is ***on average*** as fast as mergesort

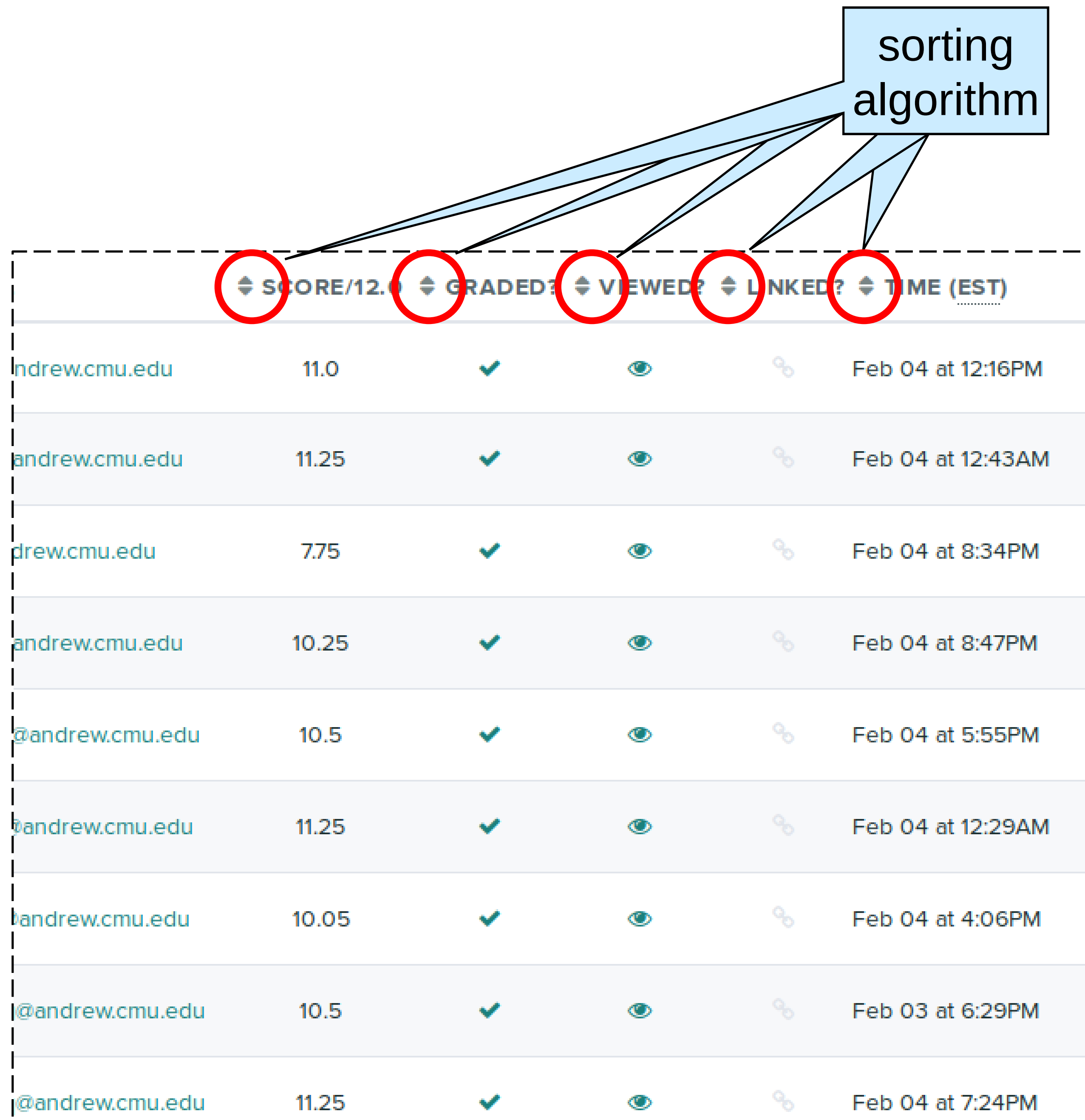
	Selection sort	Mergesort	Quicksort
Worst-case complexity	$O(n^2)$	*O(n log n)*	$O(n^2)$
In-place?	Yes	No	Yes
Average-case complexity	$O(n^2)$	*O(n log n)*	*O(n log n)*

- *Exercises:*
  - *Check that selection sort and mergesort have the given average-case complexity*
    - *Hint: there is no luck involved*

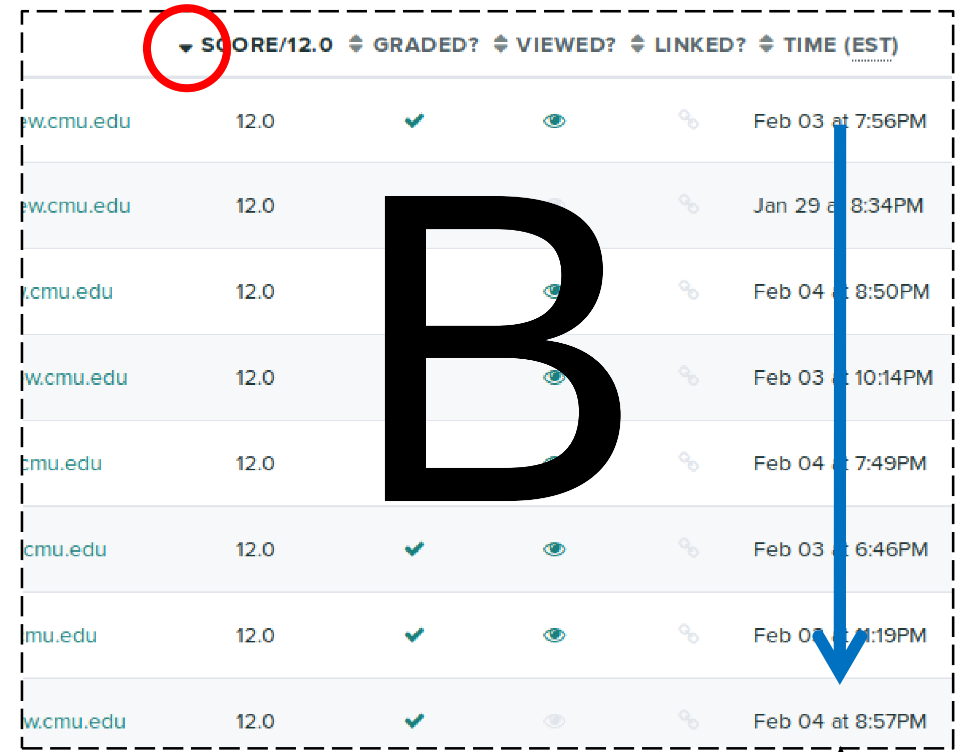
Stable Sorting

Sorting in Practice

- We are not interested in sorting just numbers
  - also strings, characters, etc
- and **records**
  - e.g., student records in tabular form



Stability



time ordering is **not preserved** for any given score

- Say the table is already sorted by time and we sort it by score

- Two possible outcomes:
  A. relative time order within each score is preserved
  B. relative time order within each score is lost

- A sorting algorithm that always does A is called **stable**
  - stable sorting is desirable for spreadsheets and other consumer-facing applications
  - it is irrelevant for some other applications

- New parameter to consider when choosing sorting algorithms

Stability

- In general,

  a sorting algorithm is **stable** if the relative order of duplicate elements doesn't change after sorting

  - the 1st occurrence of x in the input array is the 1st occurrence of x in the sorted array
  - the 2nd occurrence of x is till the 2nd occurrence
  - etc

Comparing Sorting Algorithms

- Three algorithms to solve the **same problem**
  - mergesort is asymptotically faster: *$O(n \log n)$* vs. *$O(n^2)$*
  - selection sort and quicksort are in-place but merge sort is not
  - quicksort is on average as fast as mergesort
  - mergesort is stable

	Selection sort	**Mergesort**	**Quicksort**
Worst-case complexity	$O(n^2)$	$O(n \log n)$	$O(n^2)$
In-place?	Yes	No	Yes
Average-case complexity	$O(n^2)$	$O(n \log n)$	$O(n \log n)$
Stable?	No	Yes	No

- *Exercises:*
  - *check that mergesort is stable*
  - *check that selection sort and quicksort are not*