AVL Trees

Cost of the BST Operations

Our Goal

- Develop a data structure that has **guaranteed** O(log n) worst-case complexity for lookup, insert and find_min
  - **always!**

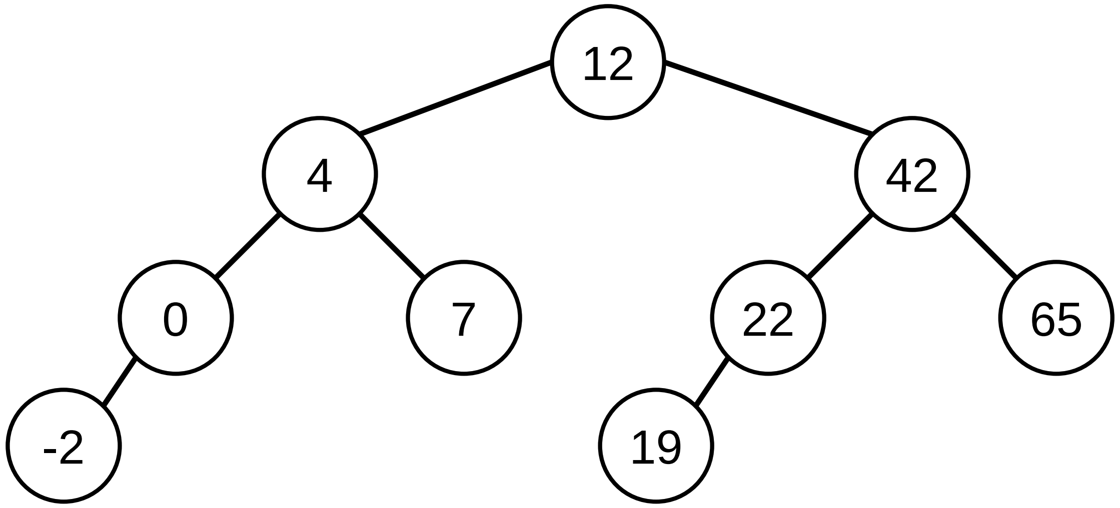
BST?

	Unsorted array	*Array sorted by key*	*Linked list*	*Hash Table*	
lookup	*O(n)*	*O(log n)*	*O(n)*	*O(1)* *average*	*O(log n)*
insert	*O(1)* *amortized*	*O(n)*	*O(1)*	*O(1)* *average and amortized*	*O(log n)*
find_min	*O(n)*	*O(1)*	*O(n)*	*O(n)*	*O(log n)*

- Do binary search trees achieve this?

Complexity

- Do lookup, insert and find_min have O(log n) complexity?
  - Yes, in this tree



  - But we are interested in the **worst-case** complexity

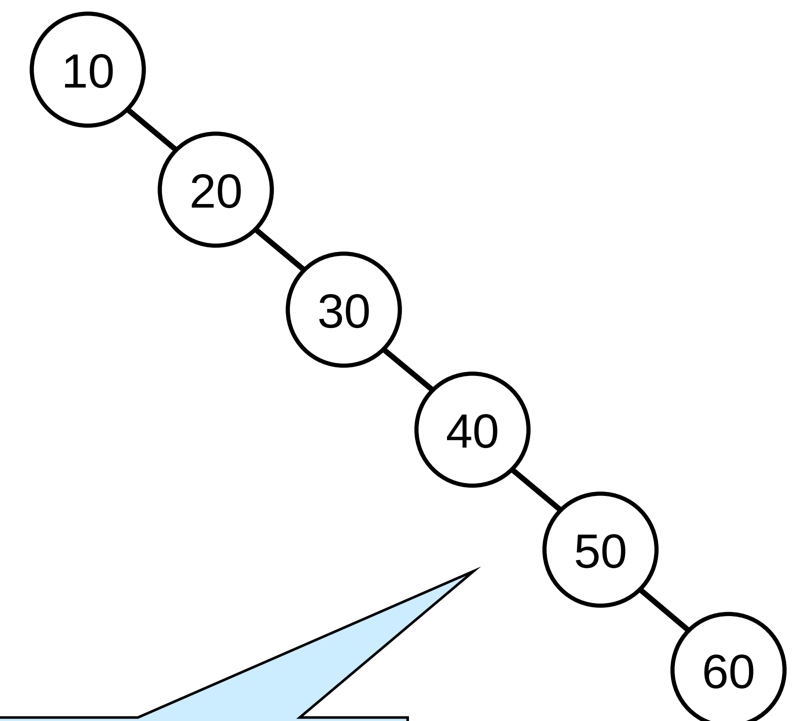
- Do lookup, insert and find_min have *O(log n)* complexity for *every* BST?

Complexity

- *Do lookup, insert and find_min have O(log n) complexity for every BST?*
  - Consider this sequence of insertions into an initially empty BST

insert 10
insert 20
insert 30
insert 40
insert 50
insert 60

  - It produces this tree:



  - Then to lookup 70, we have to go through all the nodes
    - This is O(n)

This tree has degenerated into a linked list!

- If the insertion sequence is sorted, lookup costs O(n)

Inserting 70 would also cost O(n)

Exercise: find a sequence that yields O(n) cost for find_min

Back to Square One

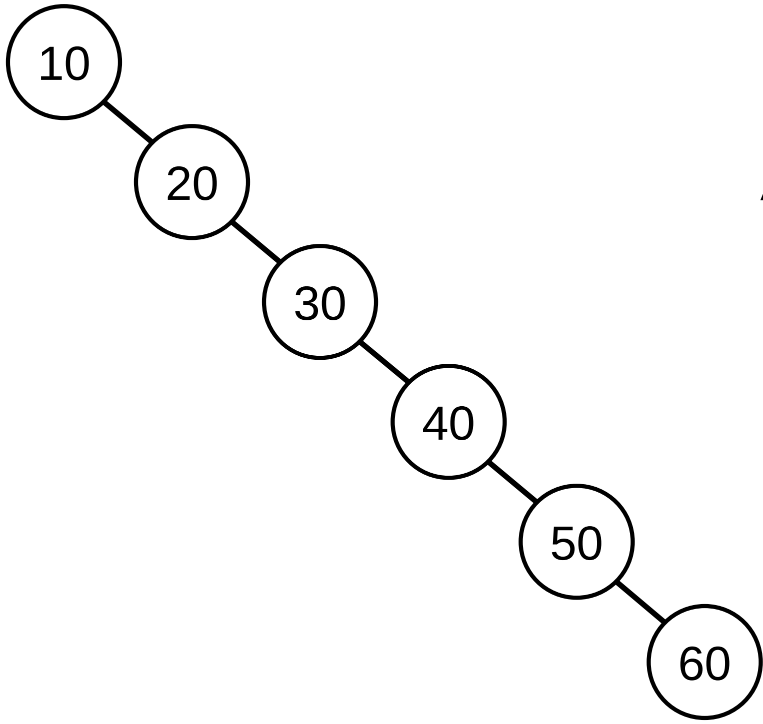
- Develop a data structure that has **guaranteed** O(log n) worst-case complexity for lookup, insert and find_min
  - **always!**

Something else …

	Unsorted array	*Array sorted by key*	*Linked list*	*Hash Table*	*BST*	
lookup	$O(n)$	$O(\log n)$	$O(n)$	$O(1)$ *average*	$O(n)$	$O(\log n)$
insert	$O(1)$ *amortized*	$O(n)$	$O(1)$	$O(1)$ *average and amortized*	$O(n)$	$O(\log n)$
find_min	$O(n)$	$O(1)$	$O(n)$	$O(n)$	$O(n)$	$O(\log n)$

- BSTs are **not** the data structure we were looking for
  - What else?

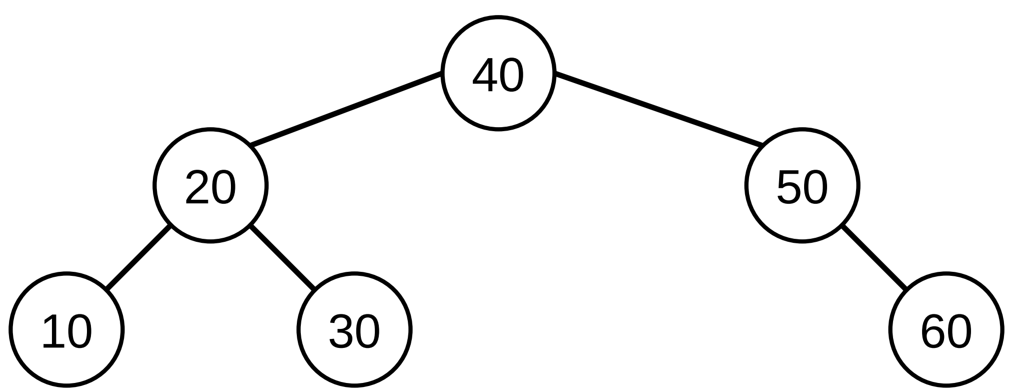
Balanced Trees



An Equivalent Tree

- Is there a BST with the same elements that yields O(log n) cost?

- How about this one?

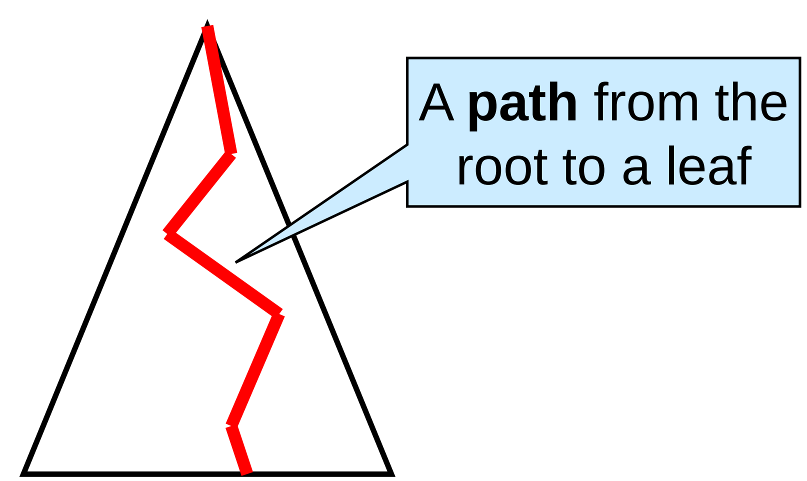


  - It contains the same elements,
  - it is sorted,
  - but the nodes are arranged differently

Reframing the Problem

- Depending on the tree, BST lookup can cost
  - O(log n) *or*
  - O(n)

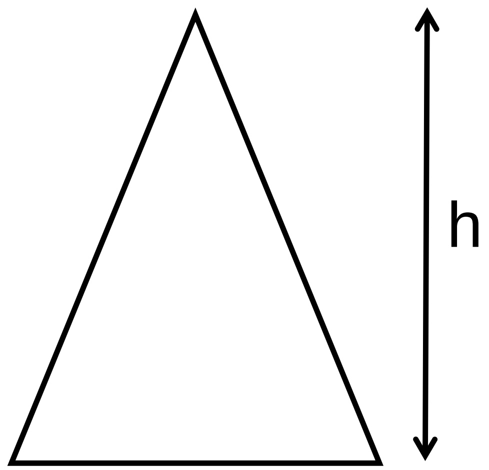
- Is there something that remains the same cost-wise?
  - Can we come up with a cost parameter that gives the same complexity in every case?
  - The cost of lookup is determined by how far down the tree we need to go
    - if the key is in the tree, the worst case is when it is in a leaf
    - if it is not in the tree, we have to reach a leaf to say so



  - The number of nodes on the longest path from the root to a leaf is called the **height** of the tree

Reframing the Problem

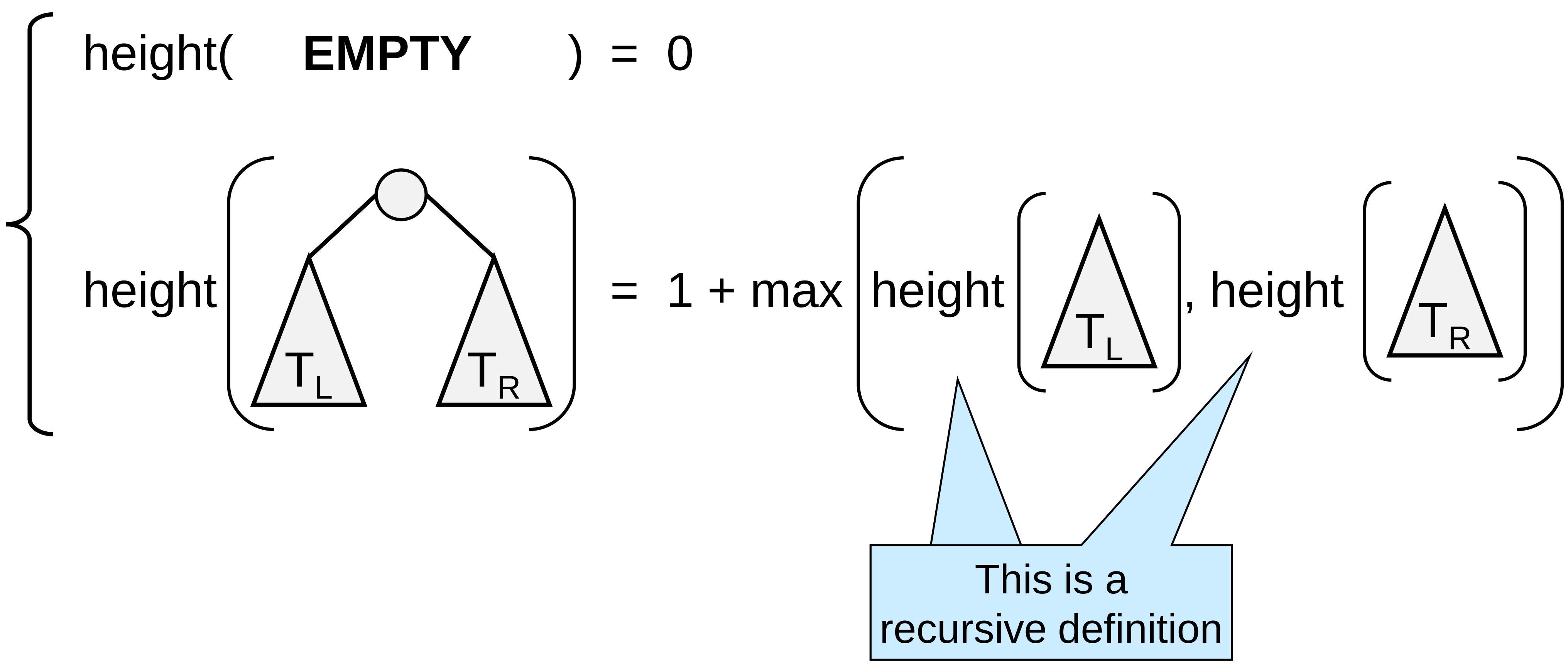
- lookup for a tree of height *h* has complexity $O(h)$
  - always!
  - same for insert and find_min



- But …
  - *h* can be in $O(n)$ or in $O(\log n)$
    - where *n* is the number of nodes in the tree

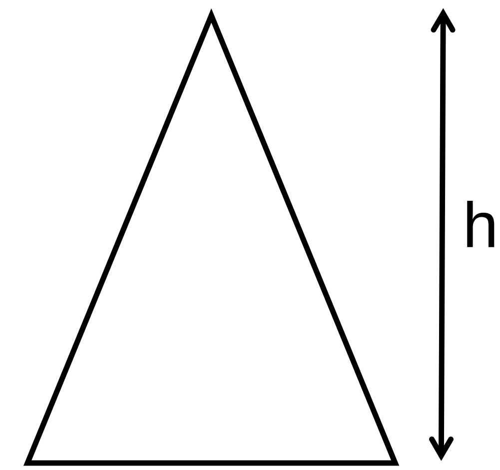
The Height of a Tree

- The length of the longest path from the root to a leaf
- Let’s define it mathematically



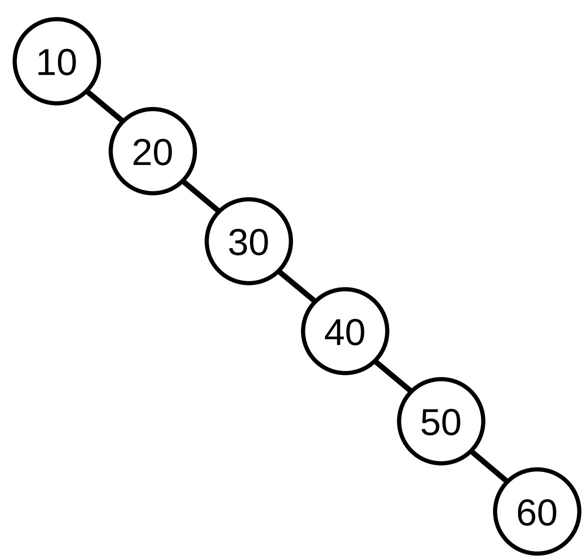
Balanced Trees

- A tree is **balanced** if $h \in O(\log n)$
  - where h is its height and n is the number of nodes

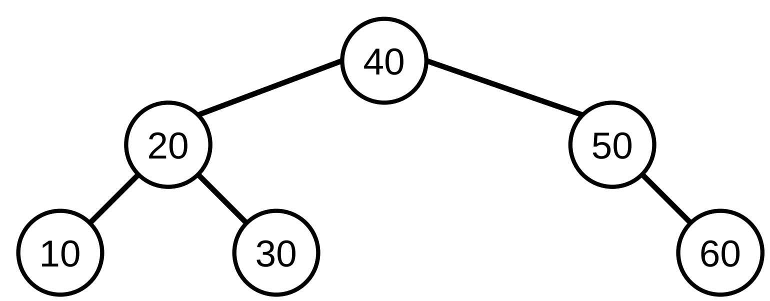


Not balanced

Well, kind of:
we can't talk about
asymptotic complexity
on a single instance



Balanced



- On a balanced tree, lookup, insert and find_min cost $O(\log n)$

Self-balancing Trees

New goal:

- make sure that a tree remains balanced as we insert new nodes

… and continues to be a valid BST

- Trees with this property are called **self-balancing**
  - There are lots of them
    - AVL trees
    - Red-black trees
    - Splay trees
    - B-trees
    - …

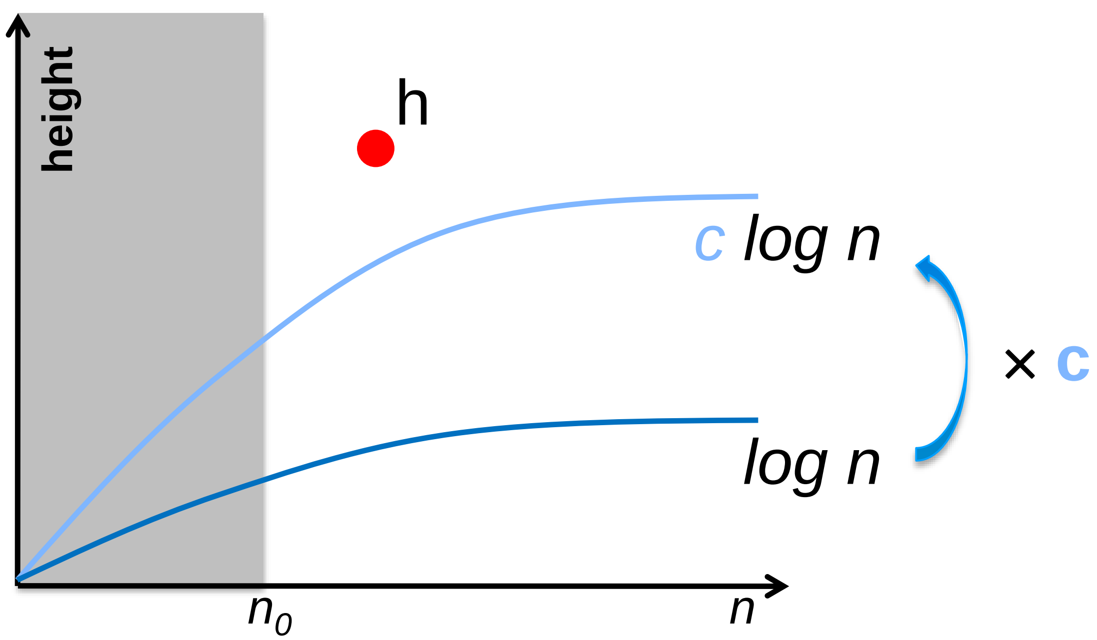
We will study this one

Why so many?

- there are many ways to guarantee that the tree remains balanced after each insertion
- some of these tree types have other properties of interest

Self-balancing Trees

- *"the tree stays balanced after each insertion"* is too vague
  - $h \in O(\log n)$ is an asymptotic behavior
    - we can't check it on any given tree
- Recall the definition



- We can fit any given h by
  - picking a bigger n_0
  - picking a bigger c

We can't say a given tree is **not** balanced

- More fundamentally, h needs to be a function in $h \in O(\log n)$

Self-balancing Trees

- *"the tree stays balanced after each insertion"* is too vague
  - $h \in O(\log n)$ is an asymptotic behavior
    - we can't check it on any given tree

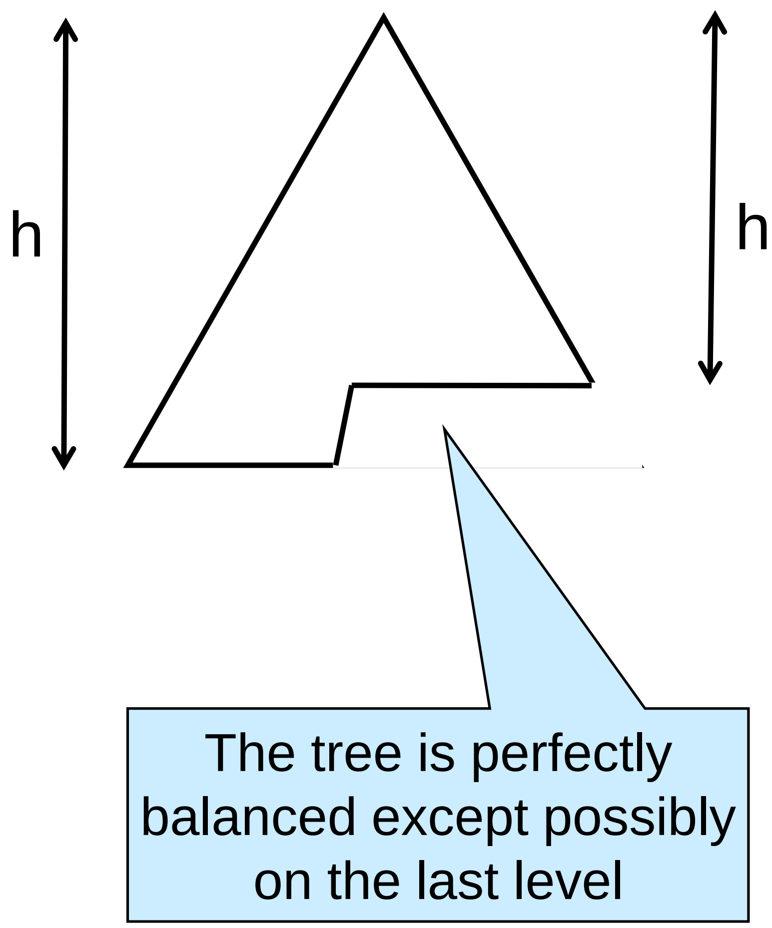
Specifically, we can't say a given tree is **not** balanced

- We want **algorithmically-checkable** constraints that
  1. guarantee that $h \in O(\log n)$
  2. are cheap to maintain
     - at most $O(\log n)$

- We do so by imposing an **additional representation invariants** on trees
    - on top of the ordering invariant
  - this *balance invariant*, when valid, ensures that $h \in O(\log n)$

A Bad Balance Invariant

- Require that
  - (the tree be a BST)
  - all the paths from the root to a leaf have height either *h* or *h-1*
  - the leaves at height *h* be on the left-hand side of the tree

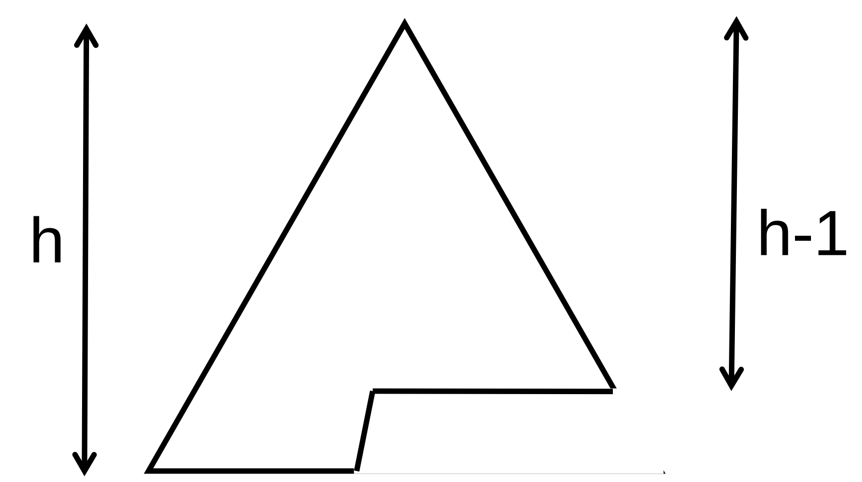


- Does it satisfy our requirements?
  1. guarantees that $h \in O(\log n)$ ✓
     - Definitely!
  2. cheap to maintain — at most $O(\log n)$
     - Let's see

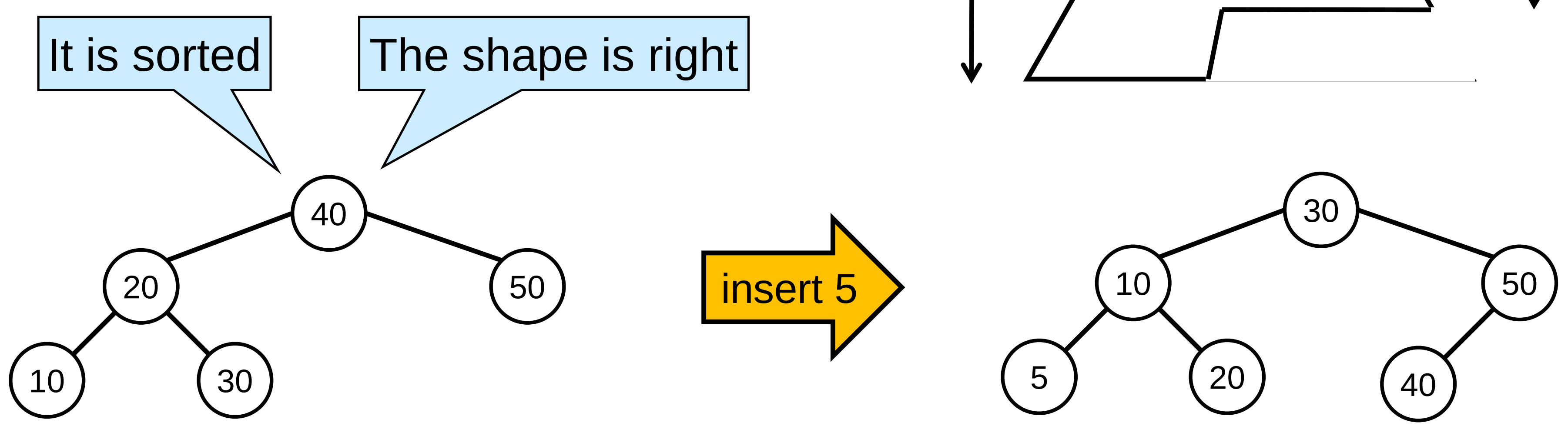
A Bad Balance Invariant

- Does it satisfy our requirements?

  1. guarantees that $h \in O(\log n)$ ✓



- Let's insert 5 in this tree



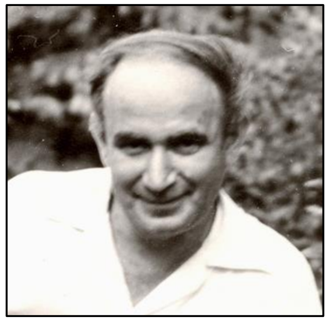
  - ○ We changed all the pointers to maintain the balance invariant!
    - ➢ O(n)

  2. cheap to maintain — at most $O(\log n)$ ✗

AVL Trees

Adelson-**V**elsky

AVL Trees



Landis

The first self-balancing trees (1962)

That’s what the balance invariant of AVL trees is called

- **Height invariant**

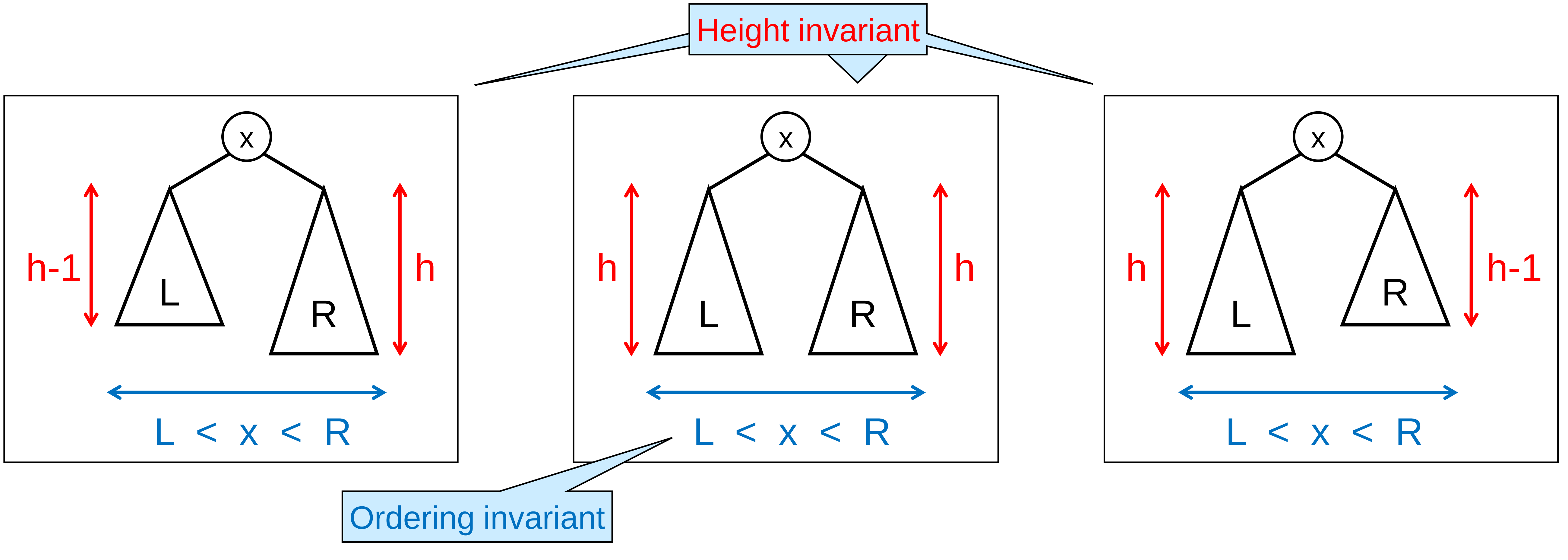
  At every node, the heights of the left and right subtrees differ by at most 1

- An AVL tree satisfies two invariants
  - the ordering invariant
  - the height invariant

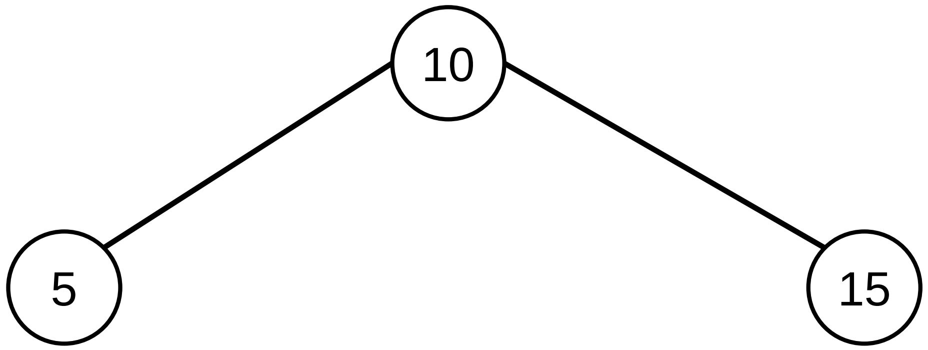
The Invariants of AVL Trees

- *The nodes are ordered*
- *At every node, the heights of the left and right subtrees differ by at most 1*

- At any node, there are 3 possibilities



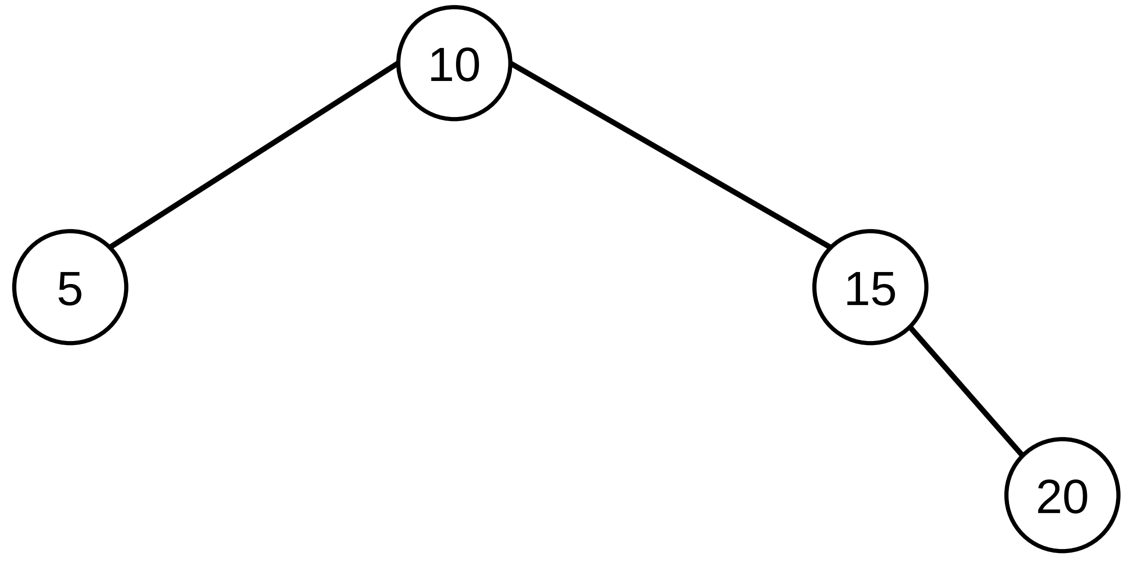
Is this an AVL Tree?



- Is it sorted? ✓
- Do the heights of the two subtrees of every node differ by at most 1? ✓

YES

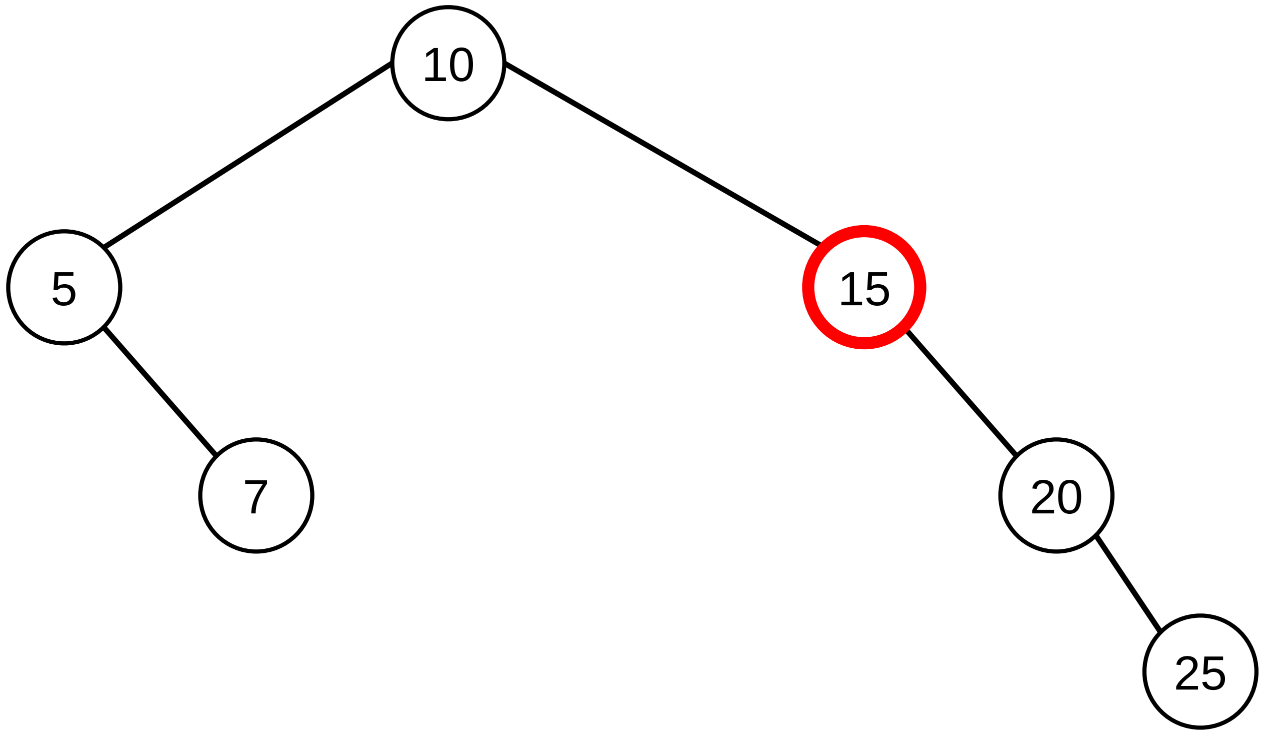
Is this an AVL Tree?



- Is it sorted? ✓
- Do the heights of the two subtrees of every node differ by at most 1? ✓

YES

Is this an AVL Tree?

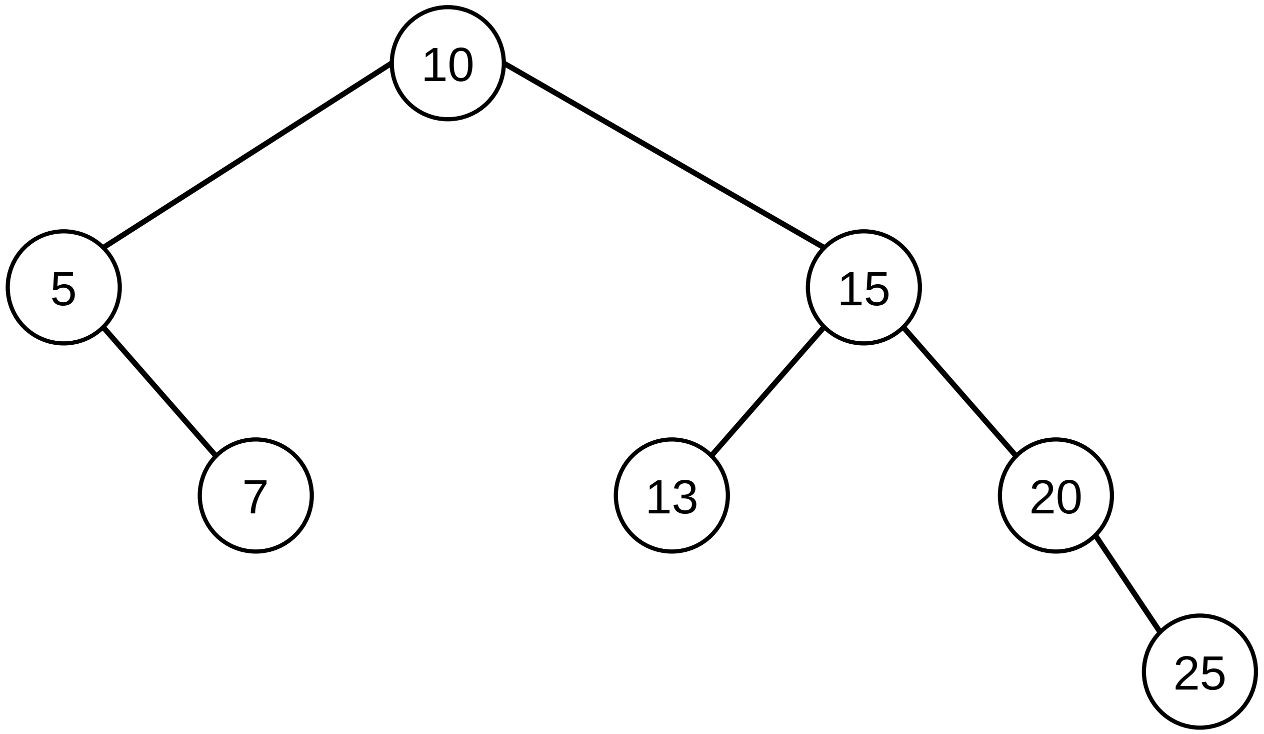


- Is it sorted? ✓
- Do the heights of the two subtrees of every node differ by at most 1? ✗
  - It doesn't hold at node 15

NO

- We say there is a **violation** at node 15

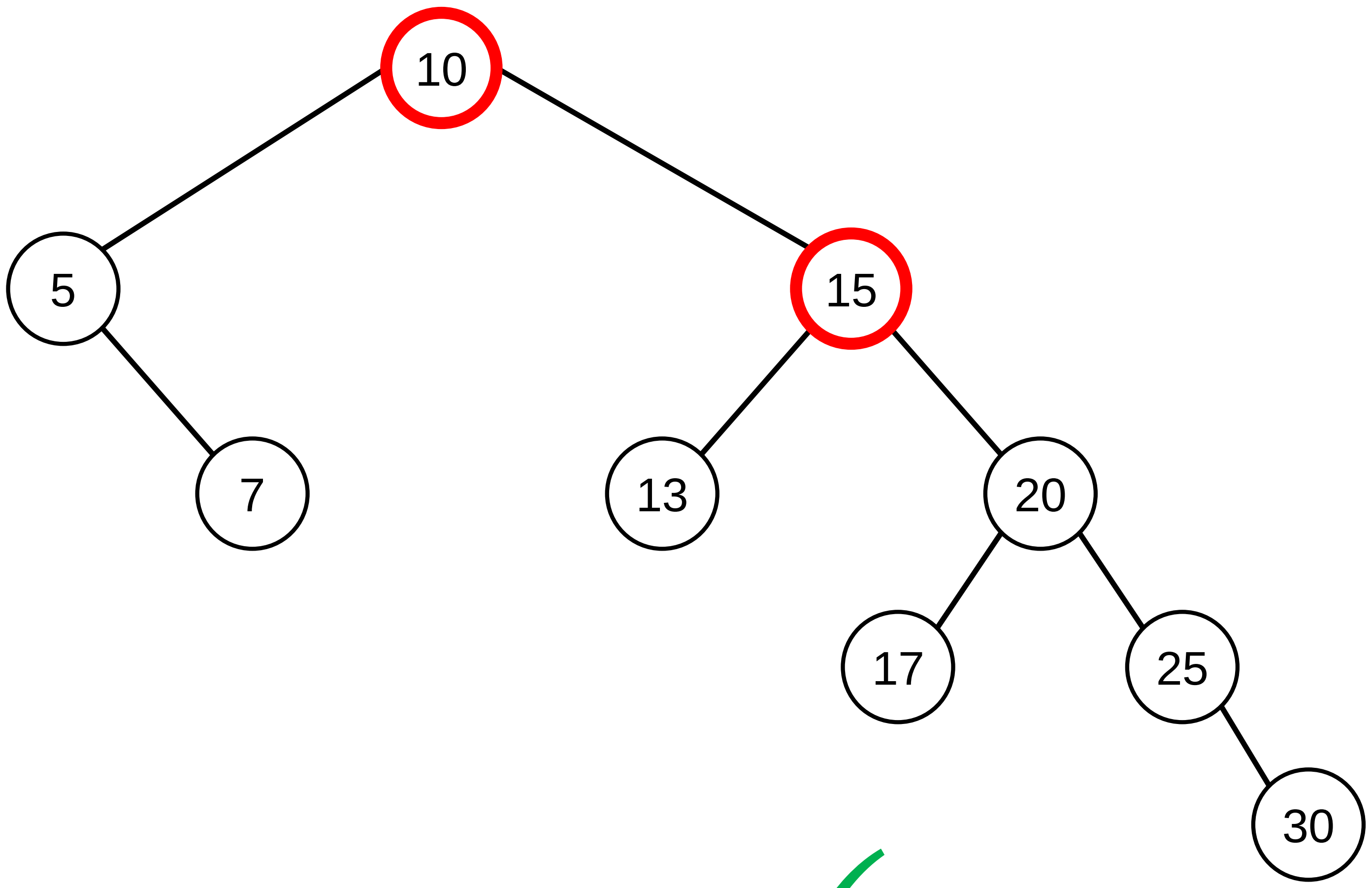
Is this an AVL Tree?



- Is it sorted? ✓
- Do the heights of the two subtrees of every node differ by at most 1? ✓

YES

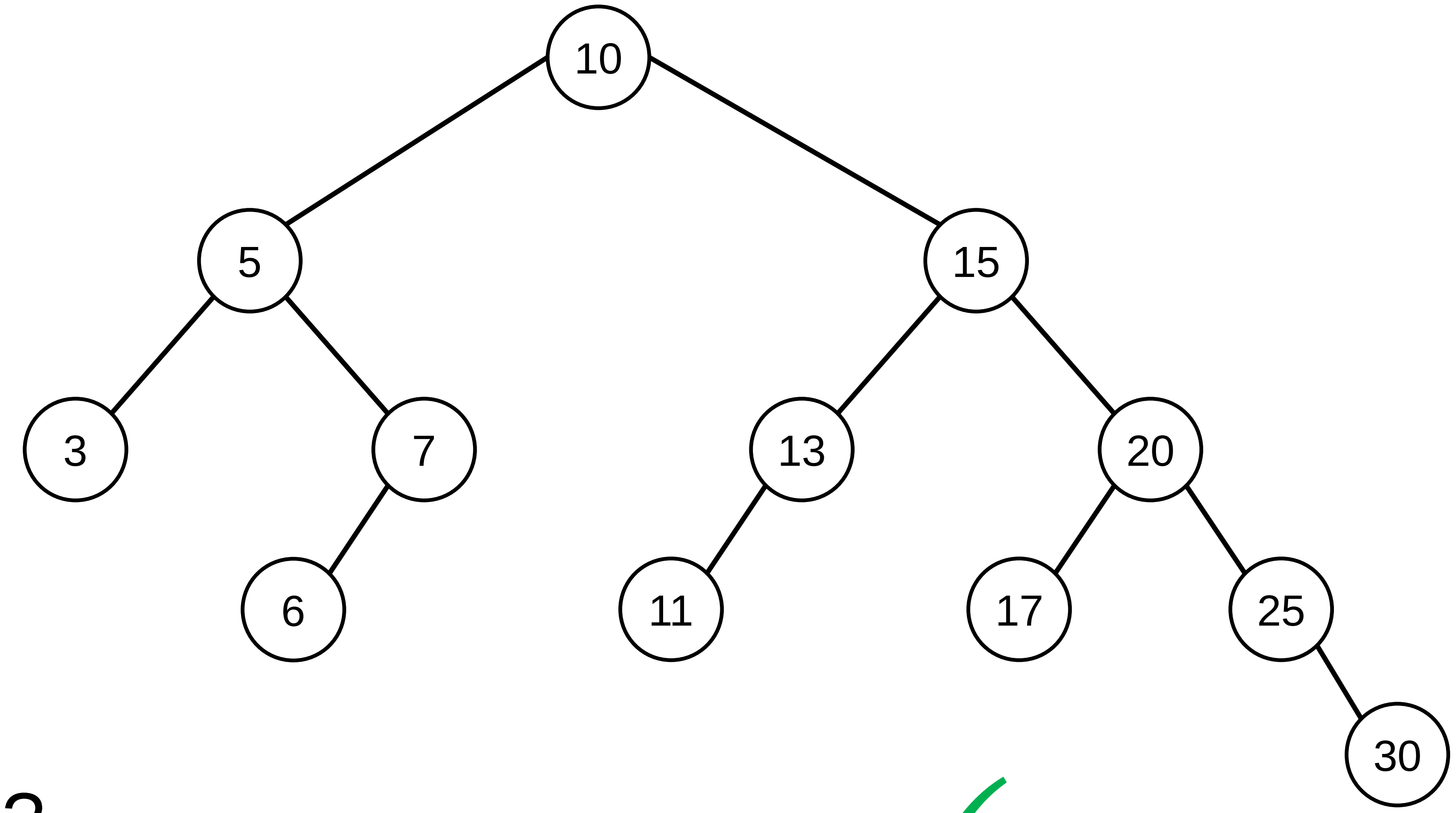
Is this an AVL Tree?



- Is it sorted? ✓
- Do the heights of the two subtrees of every node differ by at most 1? ✗
  - There is a violation at node 15 and another violation at node 10

NO

Is this an AVL Tree?



- Is it sorted? ✓
- Do the heights of the two subtrees of every node differ by at most 1? ✓

The height invariant does **not** imply that the length of every path from the root to a leaf differ by at most 1

YES

Rotations

Insertion Strategy

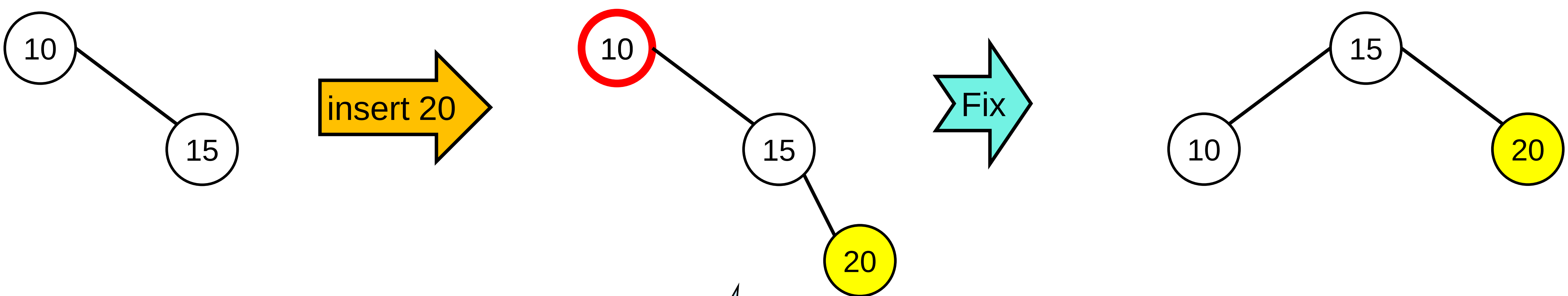
1. Insert the new node as in a BST
   - this preserves the ordering invariant
   - but it **may break the height invariant**

2. Fix any height invariant violation
   - fix the **lowest** violation
     - this will take care of all other violations

We will see why later

- This is a common approach
  - of two invariants, preserve one and temporarily break the other
  - then, patch the broken invariant
    - cheaply

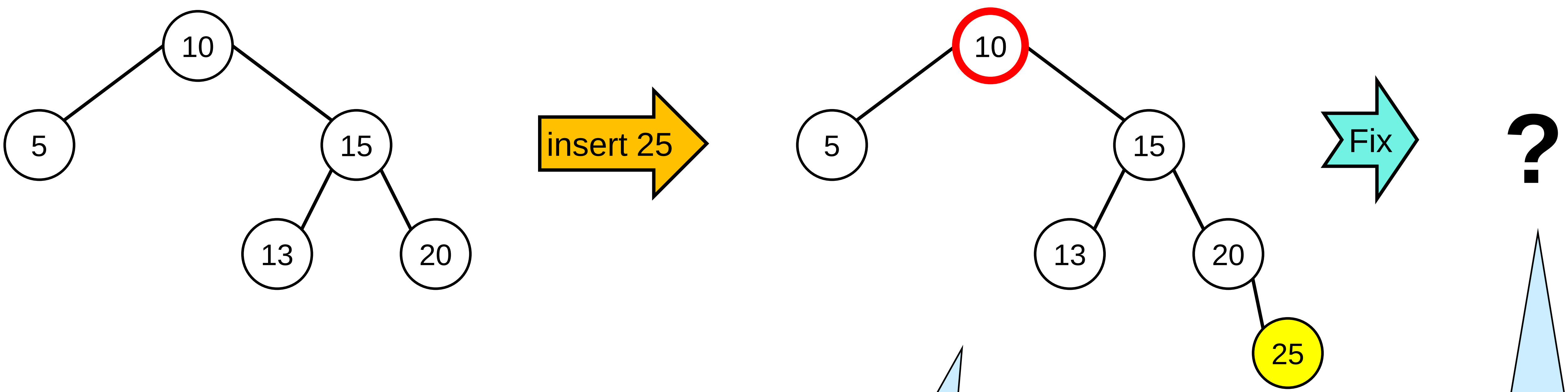
Example 1



Inserting 20 **as in a BST** causes a violation at node 10

This is the only tree with these elements that satisfies both the ordering and the height invariants

Example 2

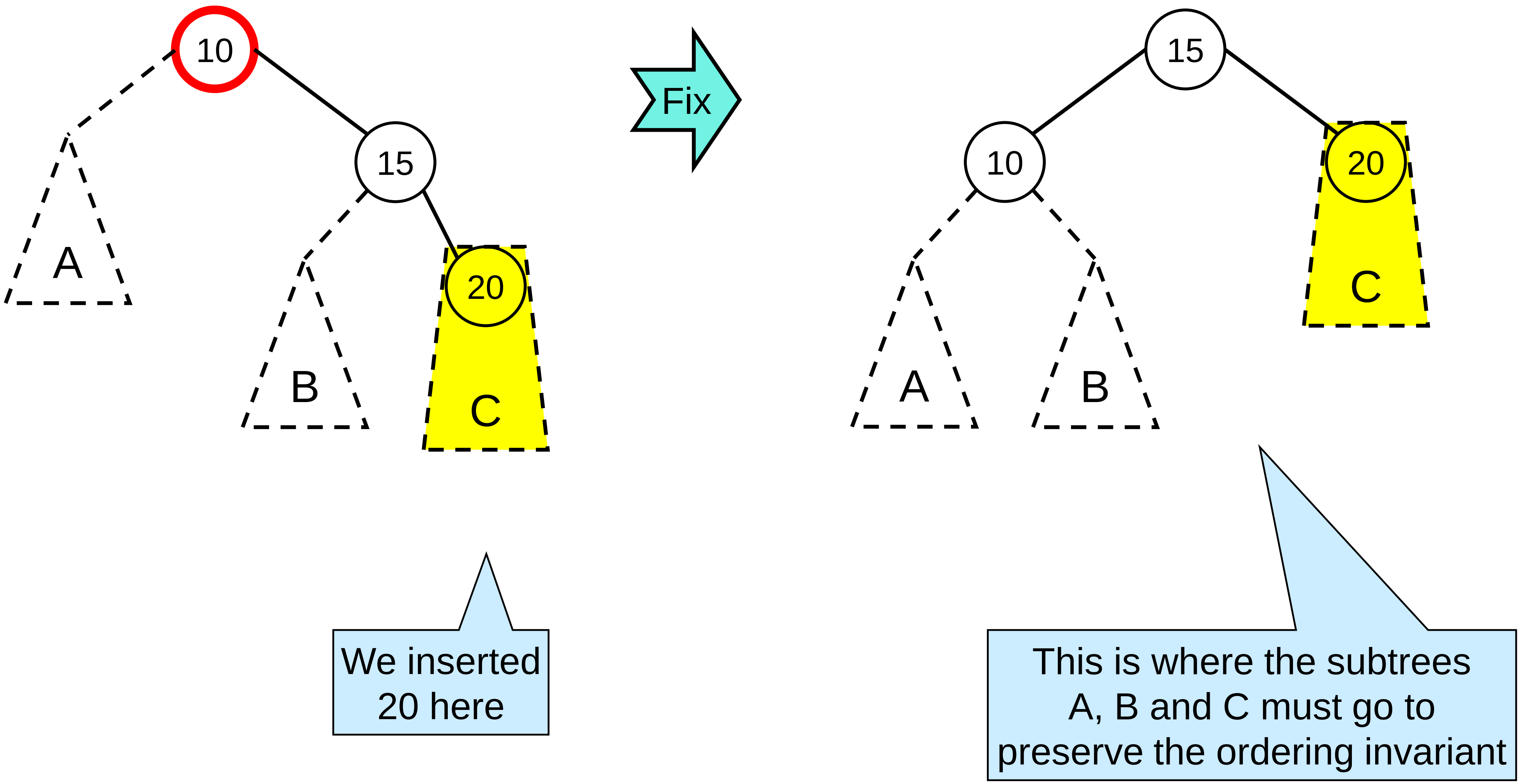


Inserting 25 **as in a BST** causes a violation at node 10

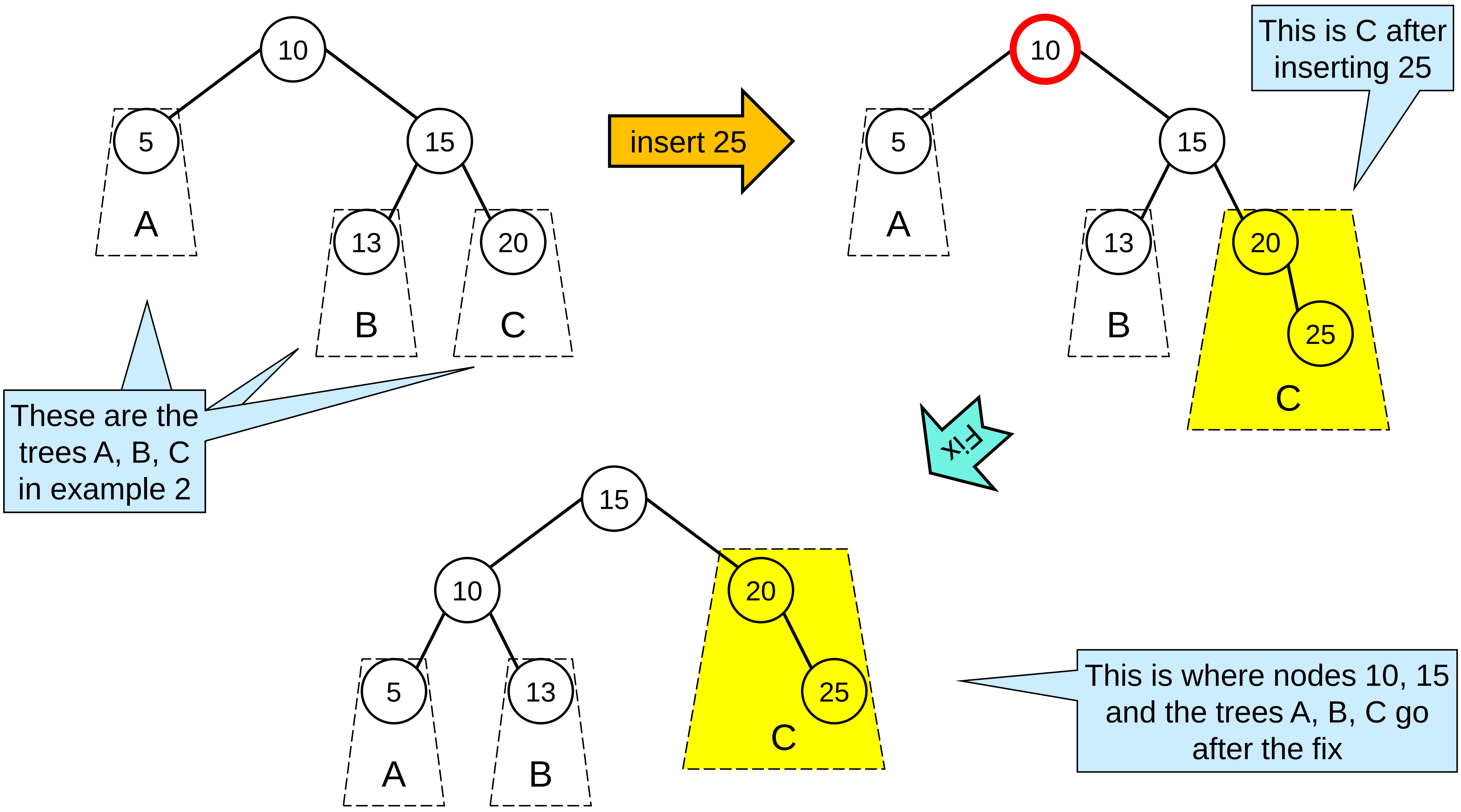
There are a lot of AVL trees with these elements: *which one to pick?*

Example 1 Revisited

- If this example was part of a bigger tree, what would it look like?

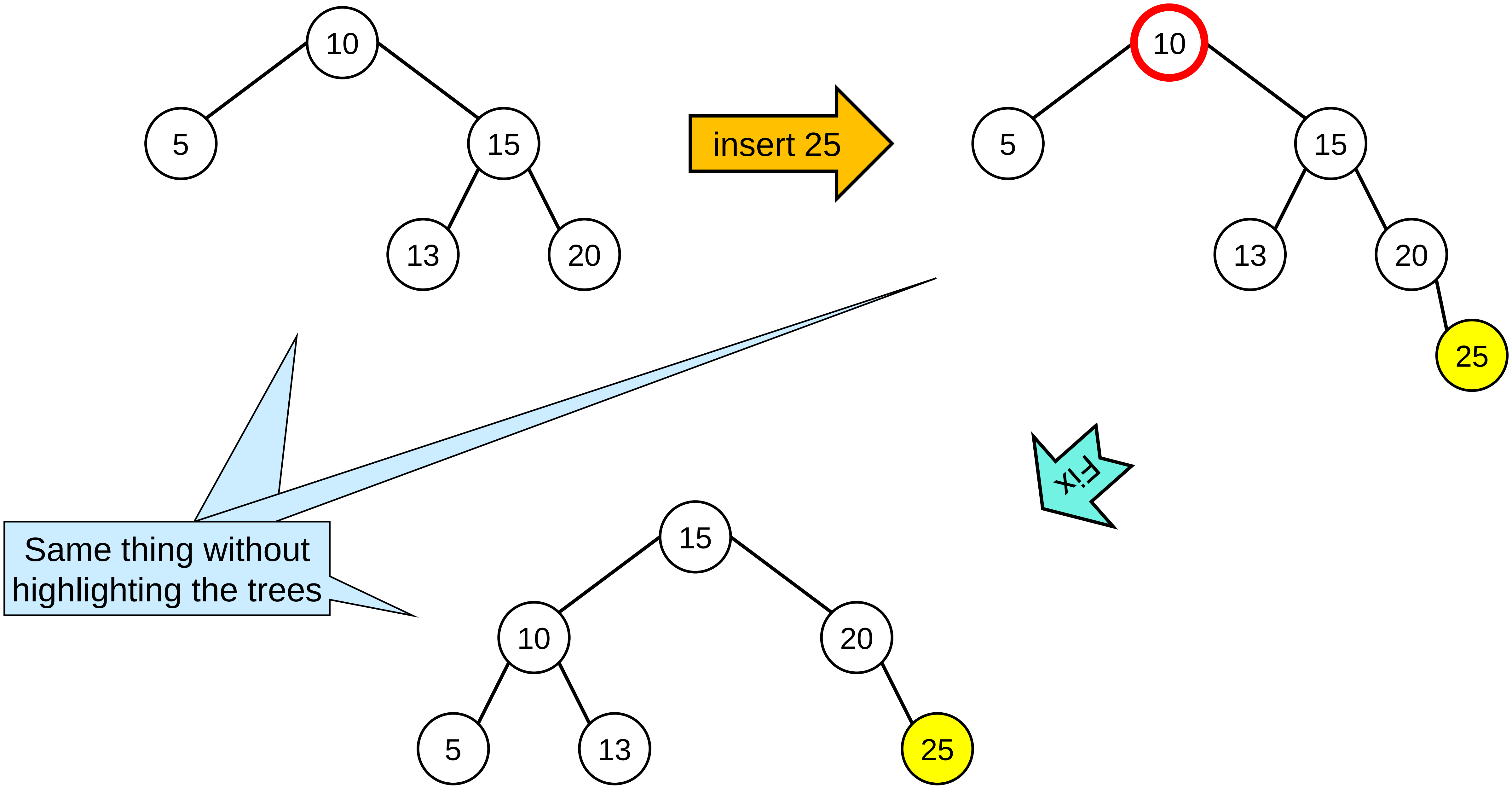


Example 2



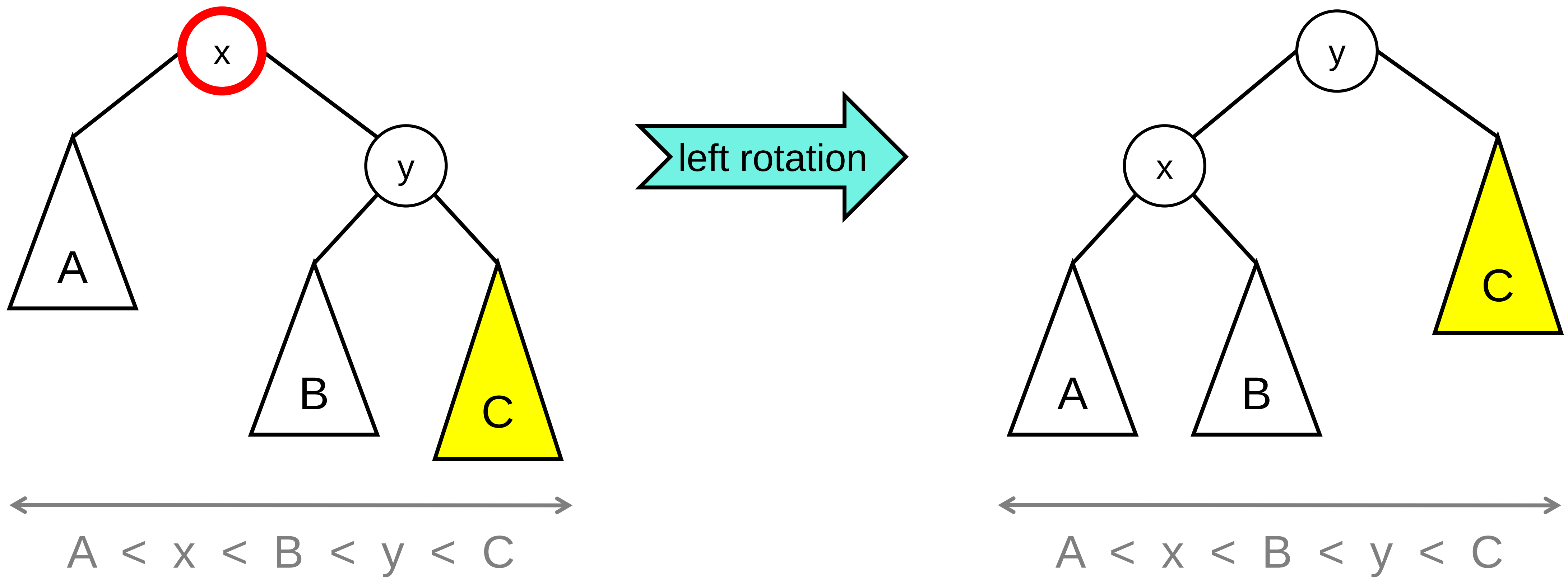
10
5
15
A
13
20
B
C
insert 25
10
5
15
A
13
20
25
B
C
This is C after inserting 25
Fix
These are the trees A, B, C in example 2
15
10
20
5
13
25
A
B
C
This is where nodes 10, 15 and the trees A, B, C go after the fix

Example 2


10
5
15
13
20
insert 25
10
5
15
13
20
25
Fix
15
10
20
5
13
25
Same thing without
highlighting the trees

Left Rotation

- This transformation is called a **left rotation**

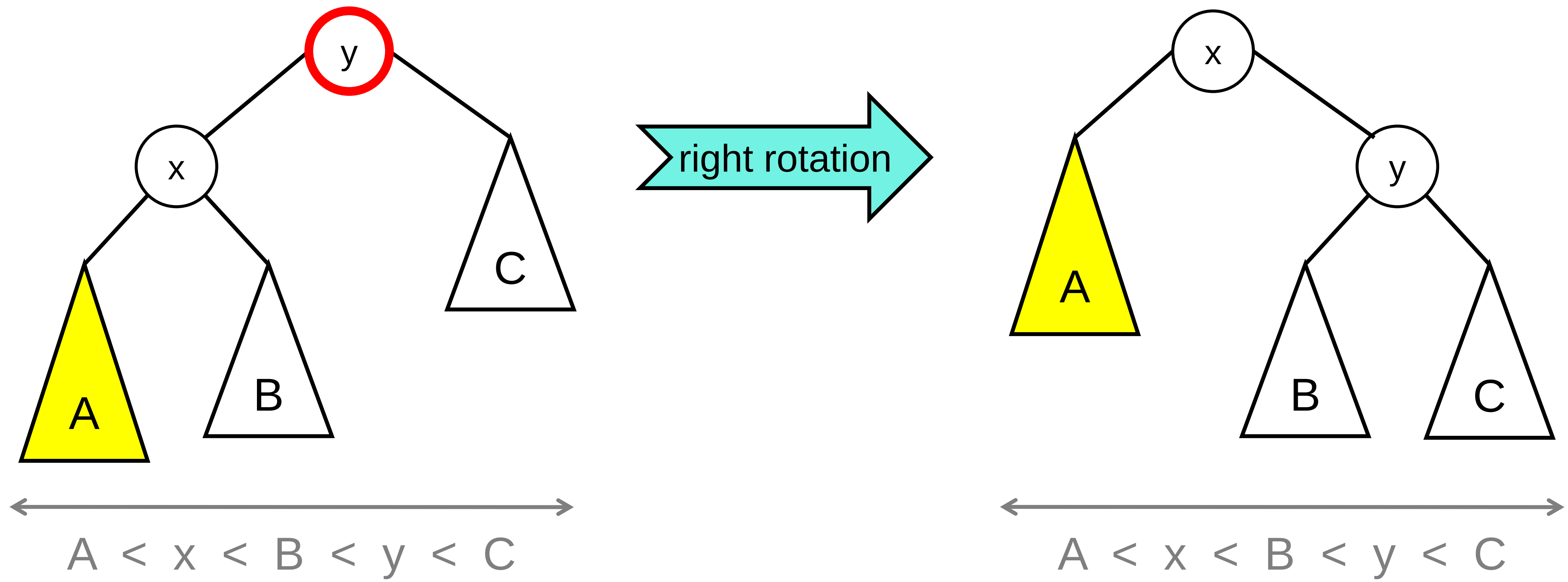


  - Note that it maintains the ordering invariant

- We do a left rotation when C has become too tall after an insertion

Right Rotation

- The symmetric situation is called a **right rotation**

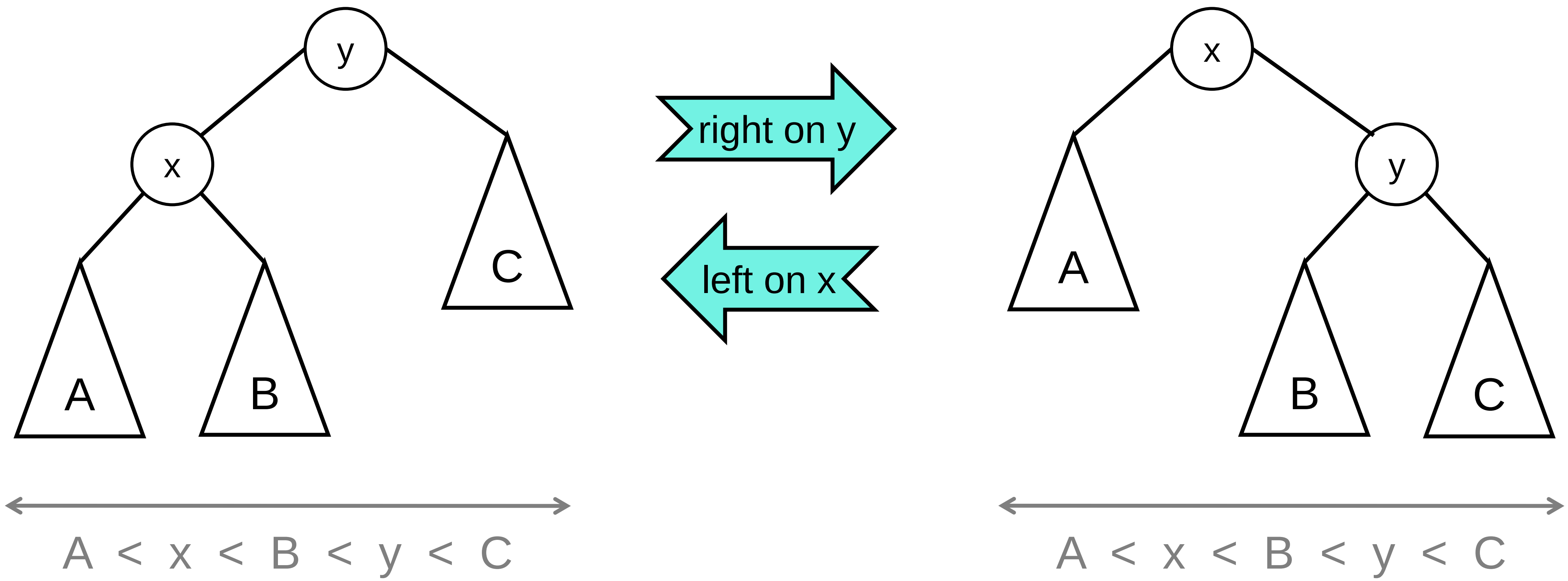


  - It too maintains the ordering invariant

- We do a right rotation when A has become too tall after an insertion

Single Rotations Summary

- Right and left rotations are **single rotations**



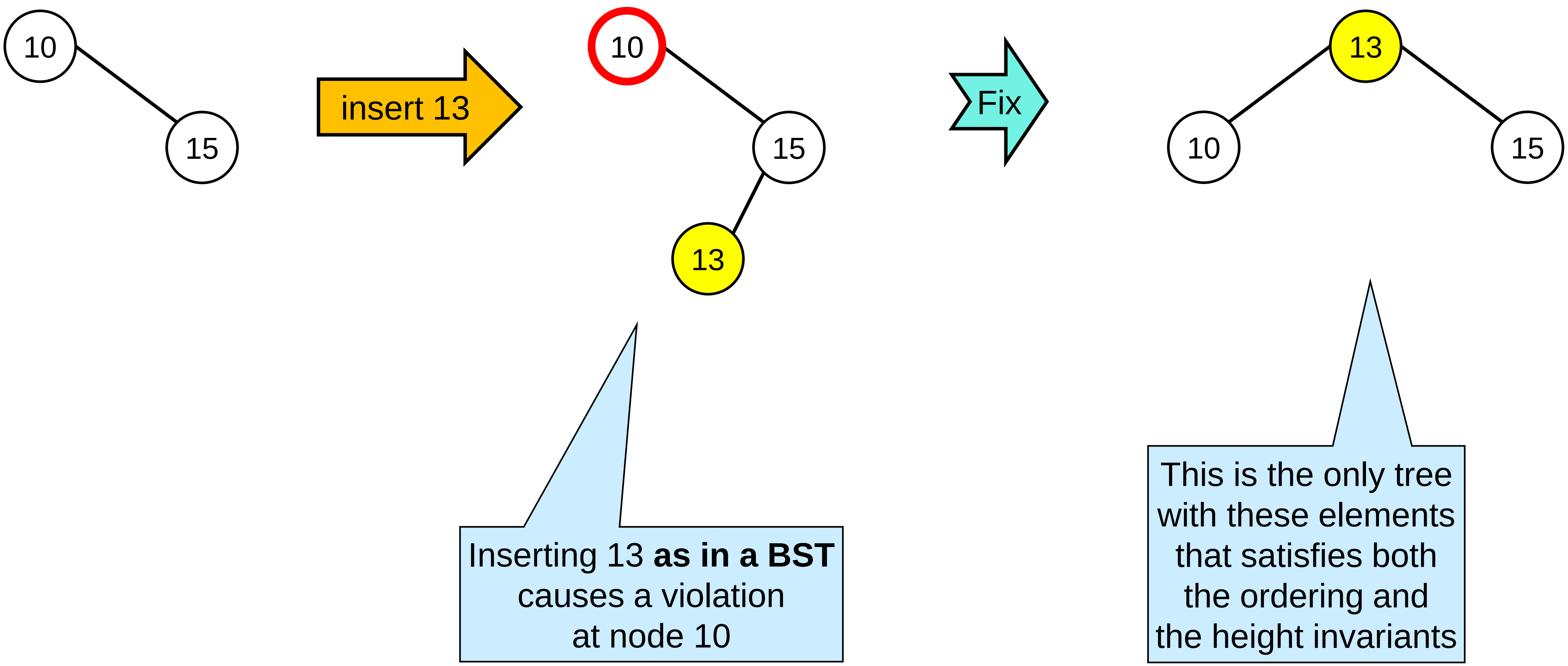
  - They maintain the ordering invariant

- We do one of them when
  - the **lowest violation** is at the root
  - one of the **outer subtrees** has become too tall

That’s either y or x

That’s either A or C respectively

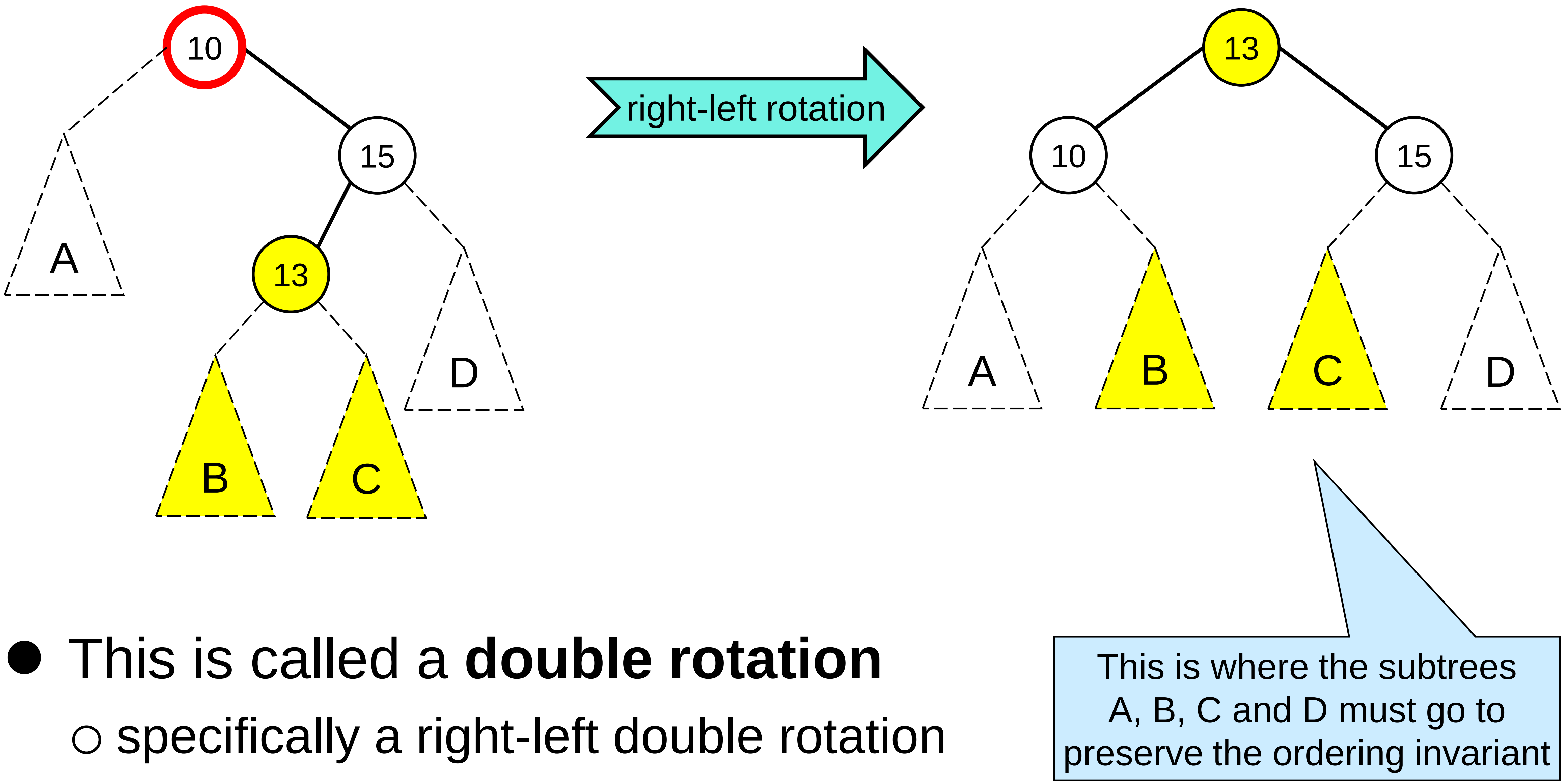
Example 3



- The fix is **not** a single rotation at 10

Double Rotations

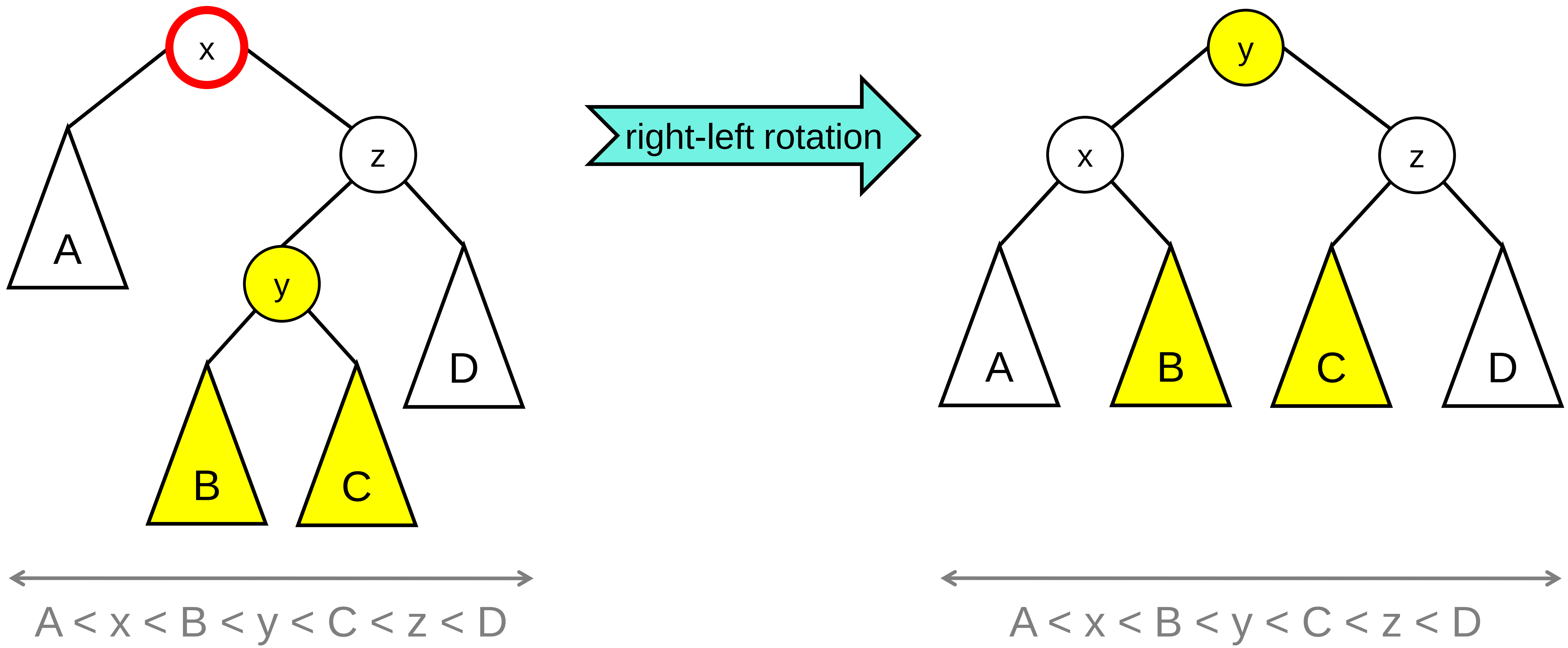
- We can generalize this example to the case where the nodes have subtrees



- This is called a **double rotation**
  - specifically a right-left double rotation

Right-left Double Rotation

- Here's the general pattern

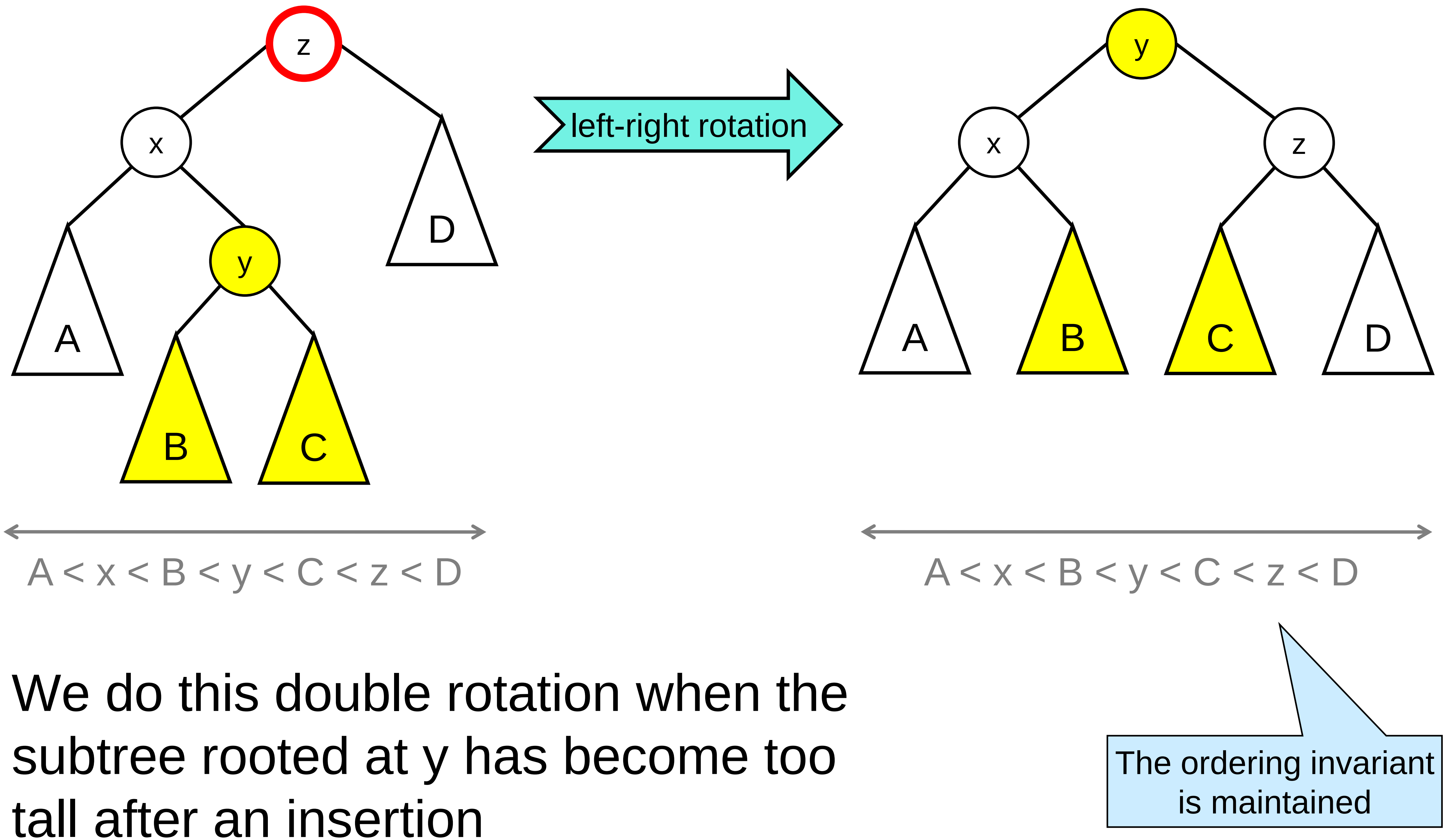


- We do this double rotation when the subtree rooted at y has become too tall after an insertion

The ordering invariant is maintained

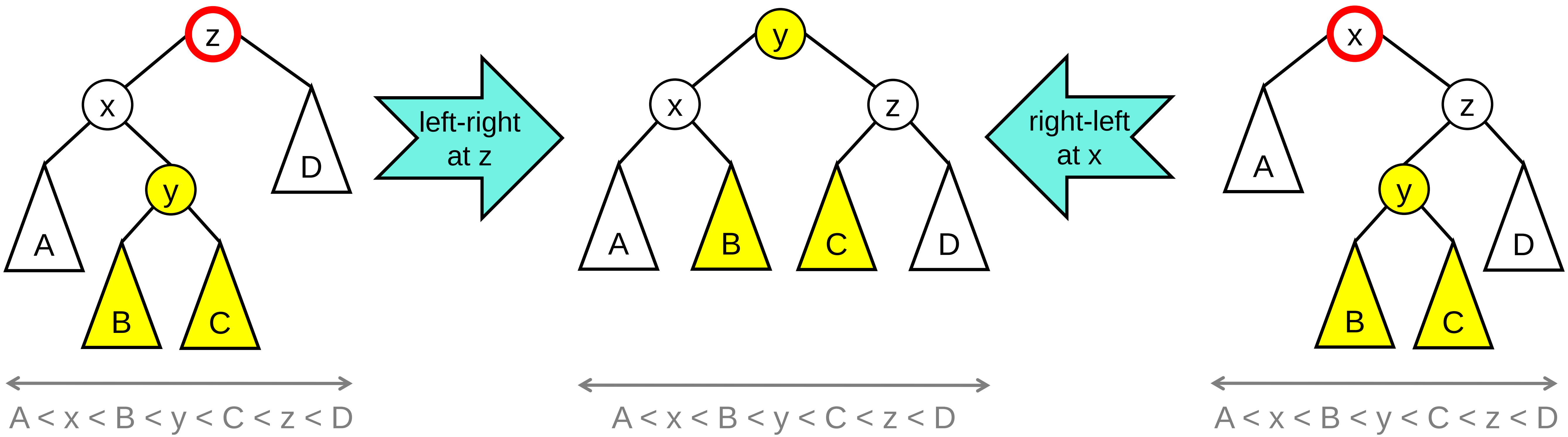
Left-right Double Rotation

- The symmetric transformation is a left-right double rotation



- We do this double rotation when the subtree rooted at y has become too tall after an insertion

Double Rotations Summary



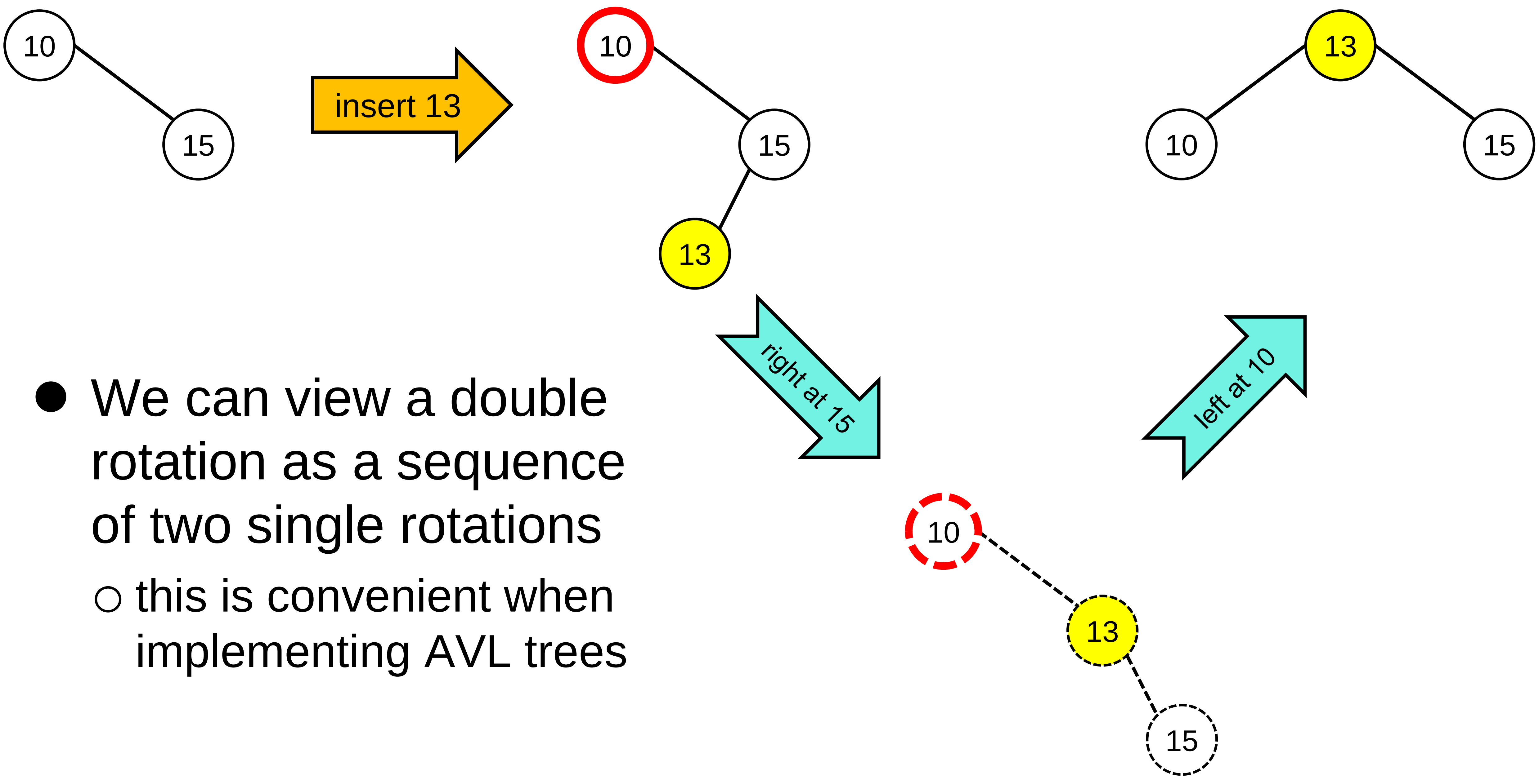
- ○ Double rotations maintain the ordering invariant

- We do one of them when
  - ○ the **lowest violation** is at the root
  - ○ one of the **inner subtrees** has become too tall

That's either z or x

That's the subtree rooted at y

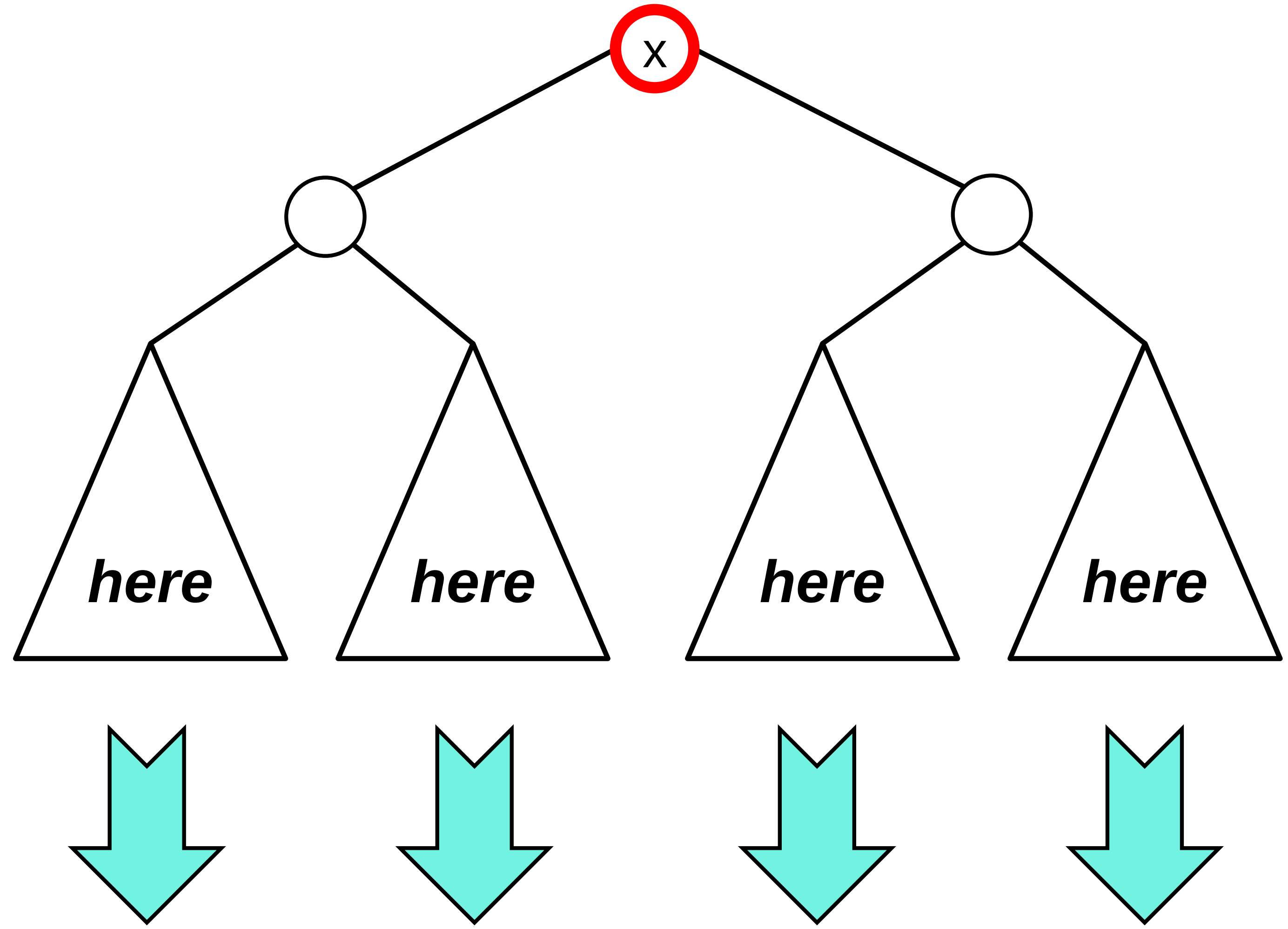
Why is it Called a ***Double*** Rotation?



- We can view a double rotation as a sequence of two single rotations
  - this is convenient when implementing AVL trees

AVL Rotation When-to

If the insertion
that caused the lowest
violation x happened …



… **then** do a …

right single rotation at x	**left/right double rotation at x**	**right/left double rotation at x**	**left single rotation at x**

Self-balancing Requirements

- Does the height constraint satisfy our requirements?

1. It guarantees that $h \in O(\log n)$ ✓ Left as exercise

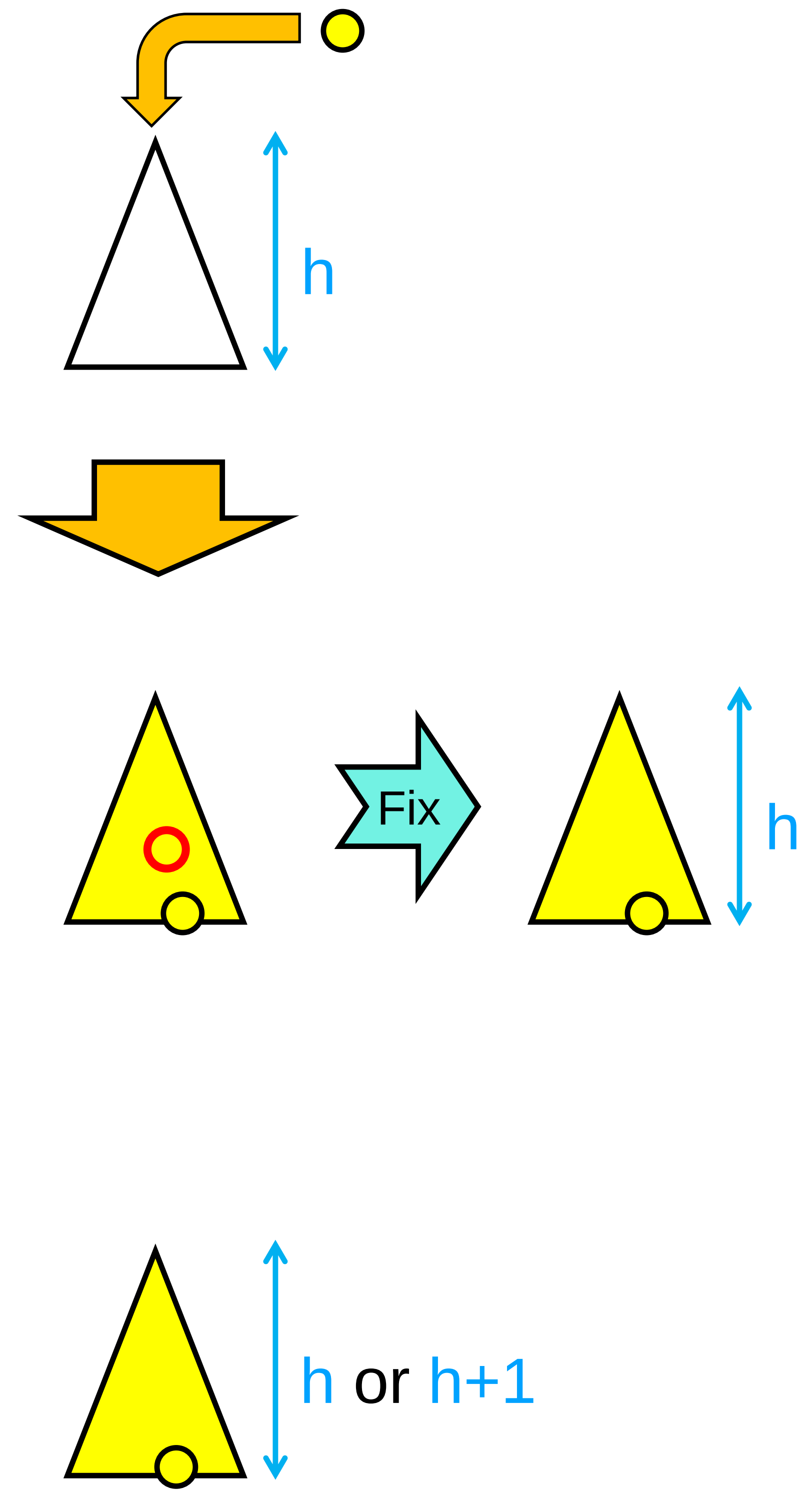
2. It is cheap to maintain — at most $O(\log n)$
   - ➢ each type of rotation costs O(1)
   - ➢ at most one rotation is needed for each insertion We will see why next

   So, maintaining the height invariant costs O(1) ✓

Height Analysis

Insertion into an AVL Tree

- Assume we are inserting a node into an AVL tree of height h



One of two things can happen:

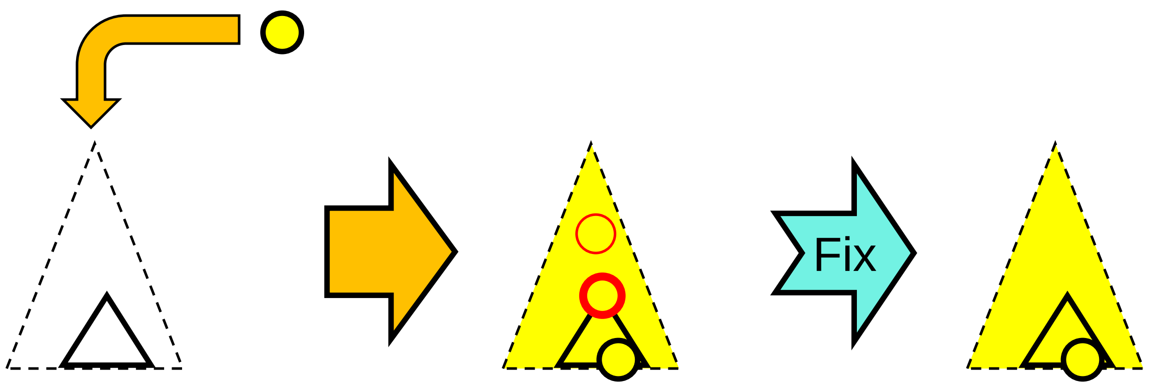
1. This causes a height violation
   - we fix it with a rotation
     - the resulting tree is a valid AVL tree
   - the fixed tree still has height h
     - the tree does not grow

Let's see why

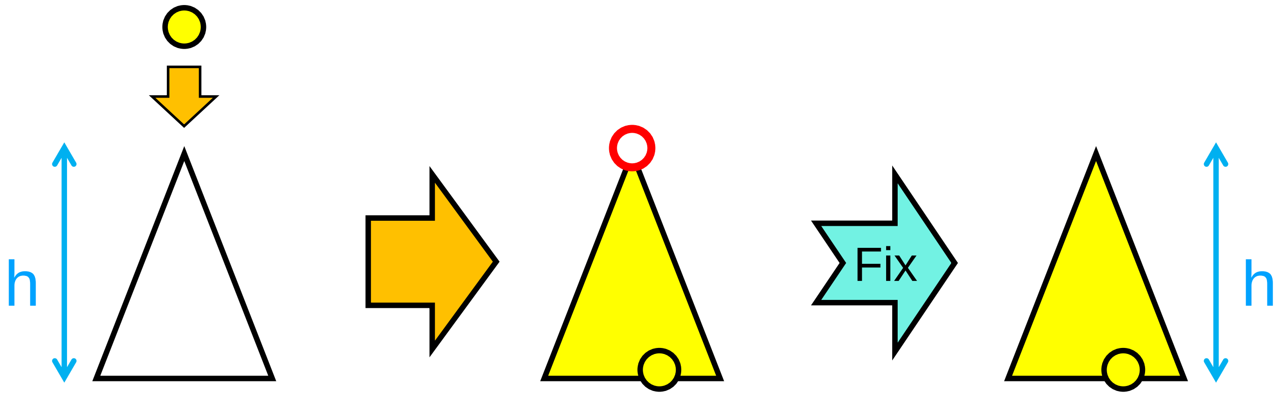
2. This does not cause a violation
   - the resulting tree has height h or h+1
     - the tree may grow only when there is no violation

Fixing the Lowest Violation

- Assume an insertion causes a violation
  - possibly more than one

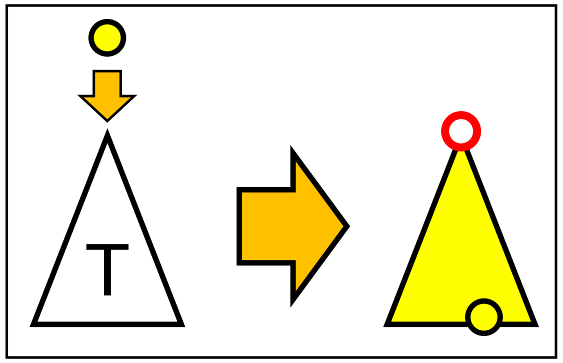


- We will focus on the subtree under the **lowest** violation
  - We will find that fixing it yields a subtree with the **same height** h as the original subtree
  - This necessarily resolves all violations above it
    - because the height of this subtree has not changed
    - if it satisfied the height invariant for the nodes above it before, it still satisfies it after



Fixing the lowest violation fixes the whole tree

The Lowest Violation



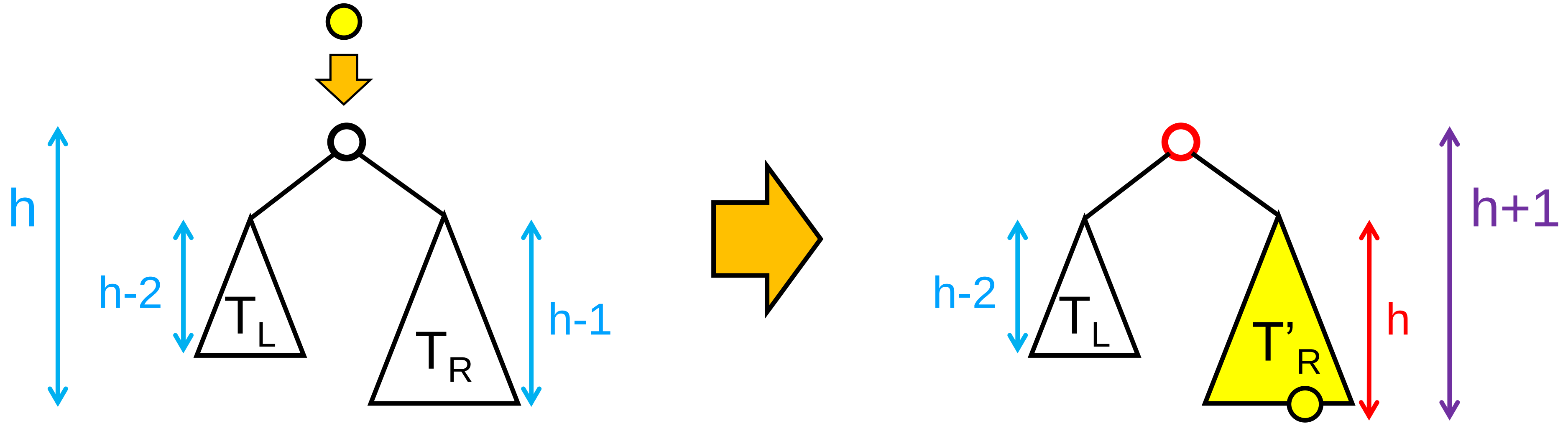
- Let's expand the tree
  - T cannot be empty

No violation possible

  - the new node can have been inserted in its left or right subtree

- Let's consider insertion in T_R

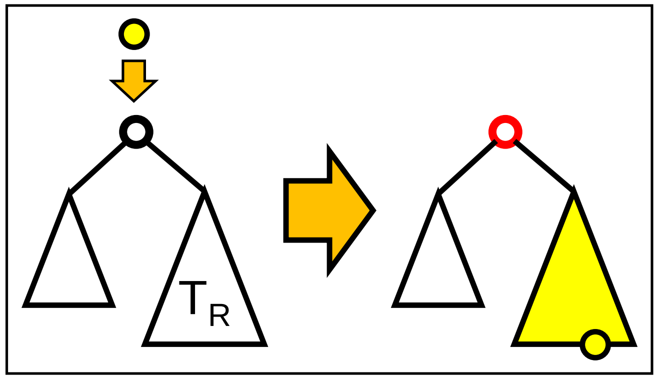
Insertion in T_L is symmetric



The right subtree has become too tall

  - To have a violation
    - T_R must be taller than T_L
      - h-1 vs. h-2
    - T_R must have grown after the insertion
      - from h-1 to h

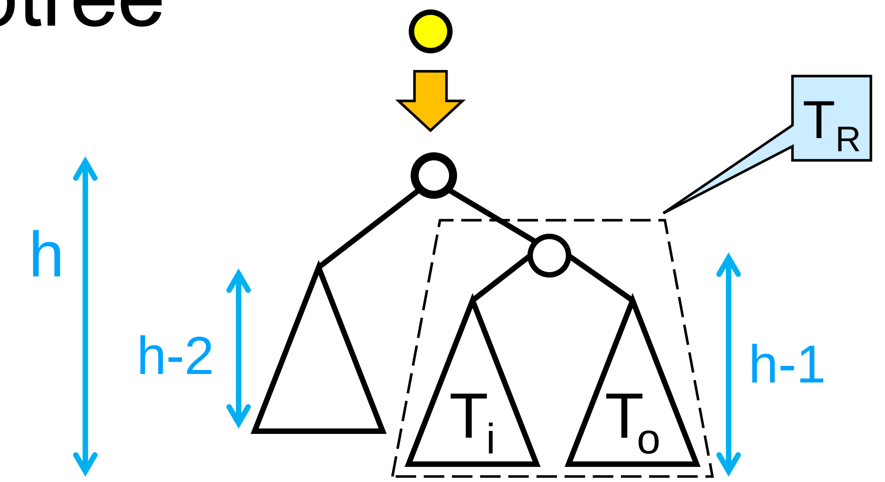
The Lowest Violation

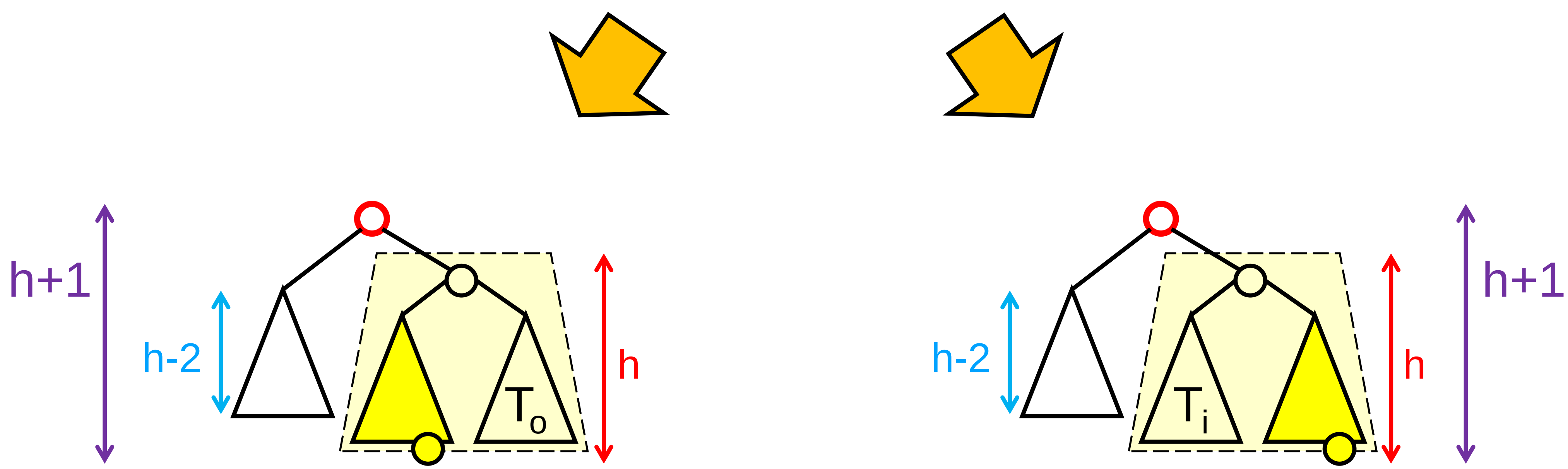


- Let's expand the right subtree
  - T_R cannot be empty

No violation possible

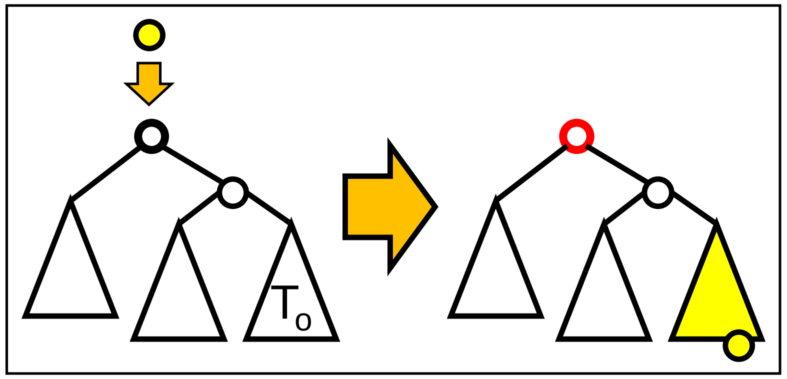
  - the new node can have been inserted in its left or right subtree



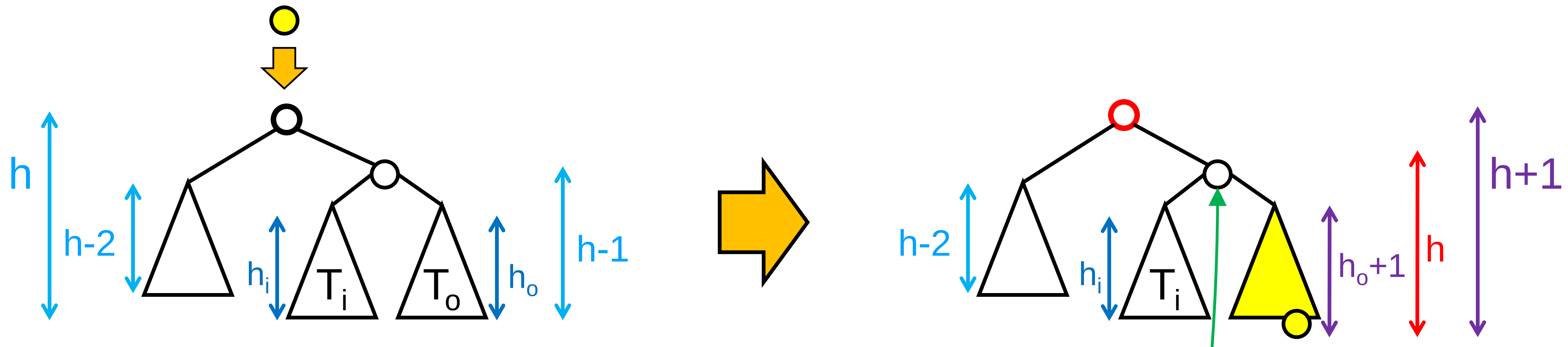


  - *Let's examine each case in turn*

Insertion in the Outer Subtree



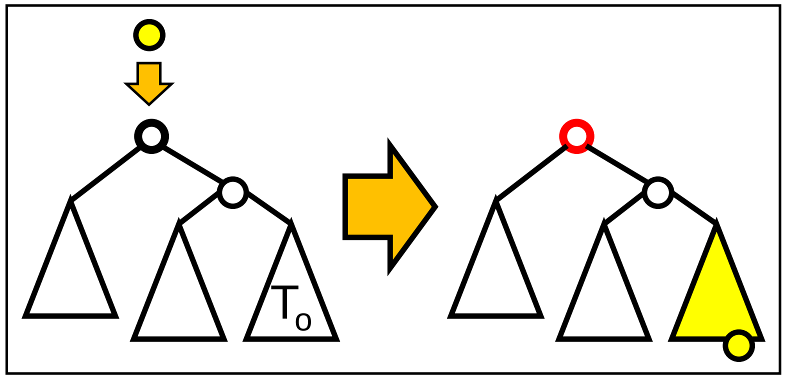
- How tall are T_i and T_o?



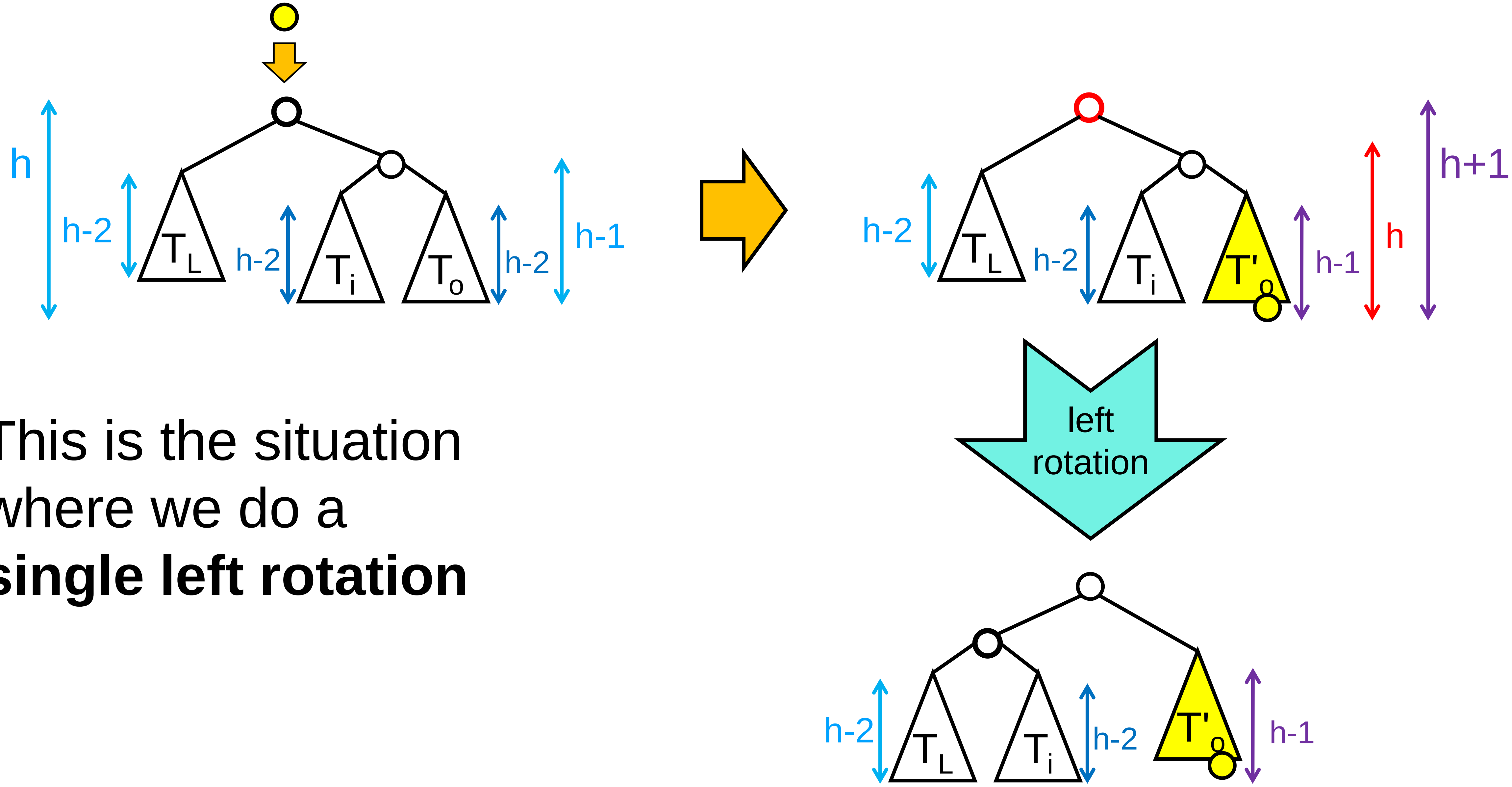
  - h_o = h-2
    - T_o needs to be as tall as possible to causes the violation
  - h_i = h_o = h-2
    - h_i may be either h-2 or h-3
    - but if h_i were h-3, the lowest violation would be here

T_i and T_o have the same height

Insertion in the Outer Subtree

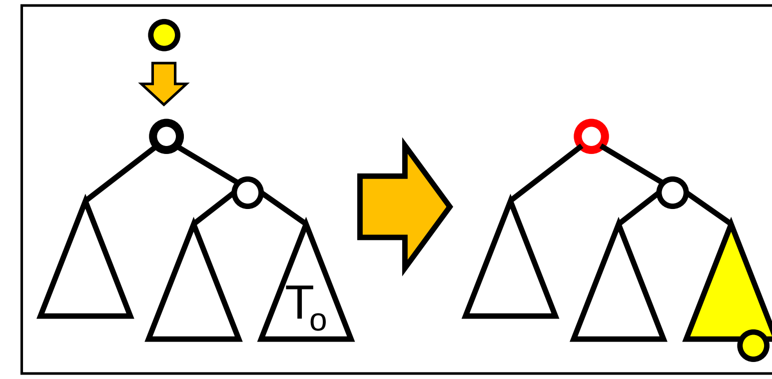


- T_i and T_o have height h-2

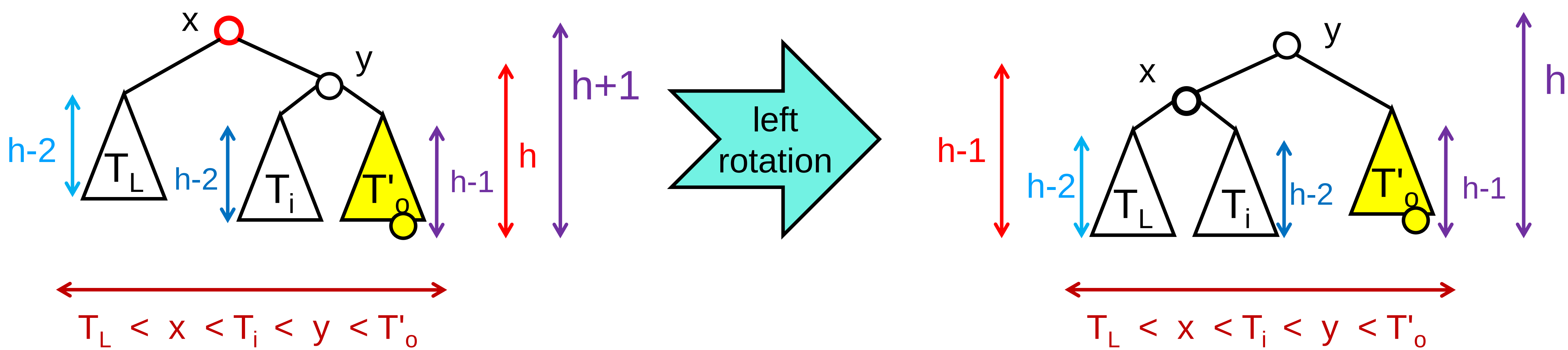


- This is the situation where we do a **single left rotation**
  - *Is this an AVL tree?*

Insertion in the Outer Subtree



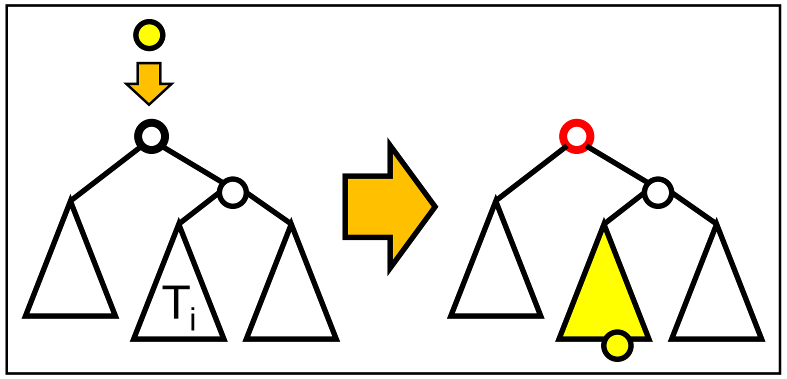
- Is this an AVL tree?



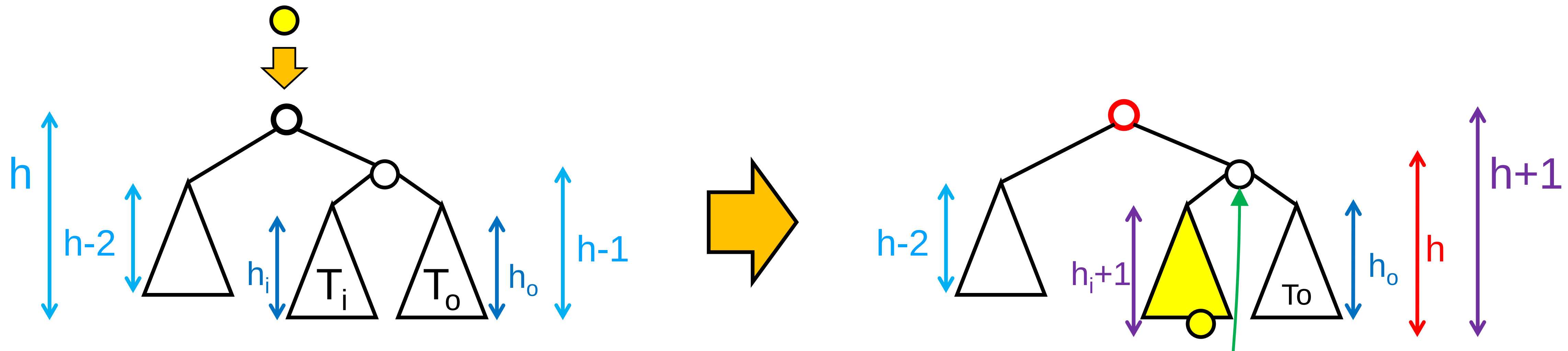
- BST insertion and the rotations maintains the ordering invariant
- T_L, T_i and T'_o are AVL trees
  - because x was the lowest violation
- T_L–x–T_i is an AVL tree of height h-1
  - because both T_L and T_i have height h-2
- (T_L–x–T_i)–y–T'_o is an AVL tree of height h
  - because T'_o also has height h-1

The height invariant is restored

Insertion in the Inner Subtree



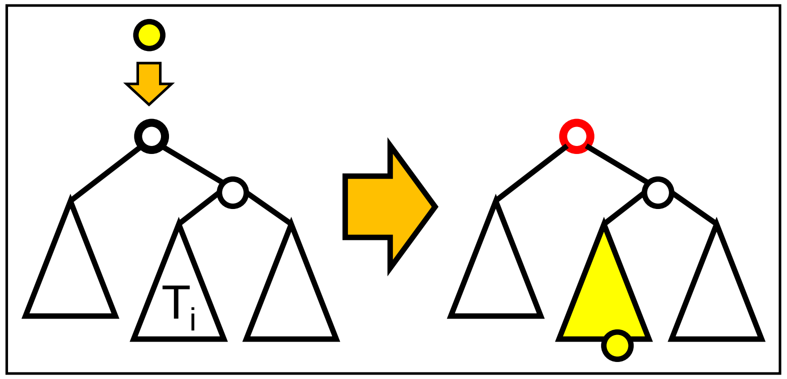
- How tall are T_i and T_o?



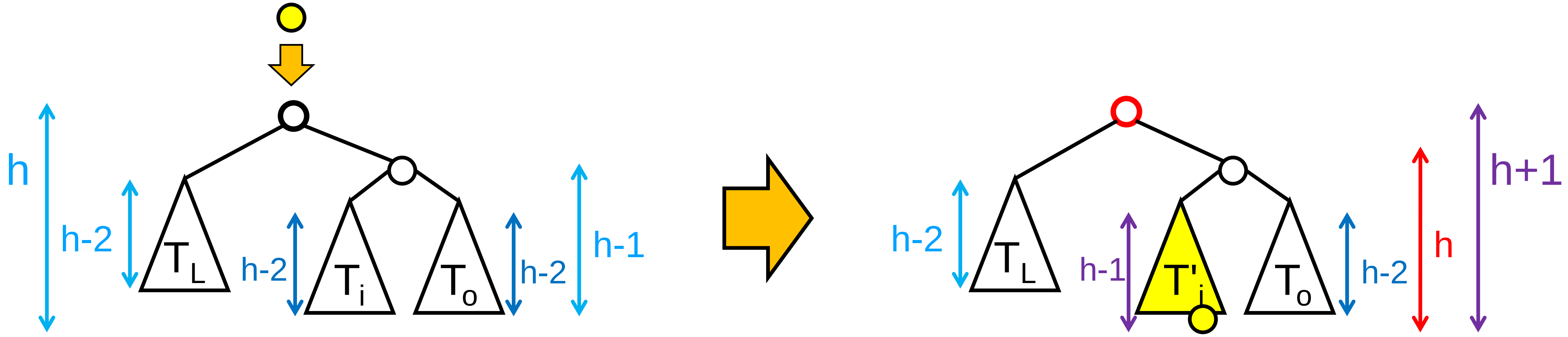
- h_i = h-2
  - T_i needs to be as tall as possible to causes the violation
- h_o = h_i = h-2
  - h_o may be either h-2 or h-3
  - but if h_o were h-3, the lowest violation would be here

T_i and T_o have the same height

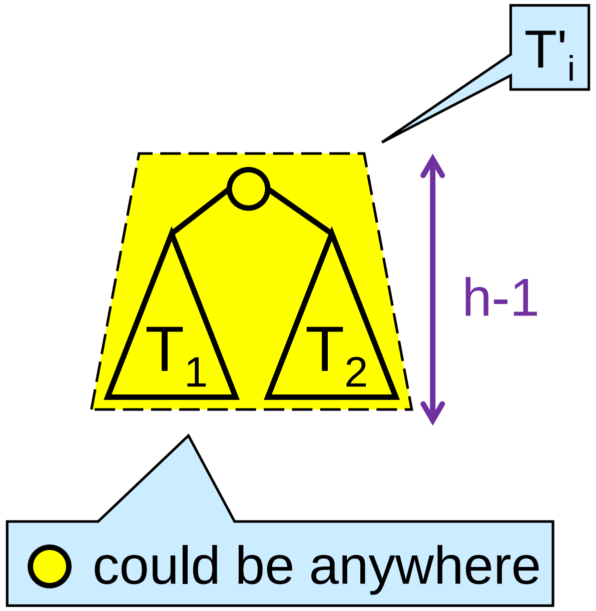
Insertion in the Inner Subtree



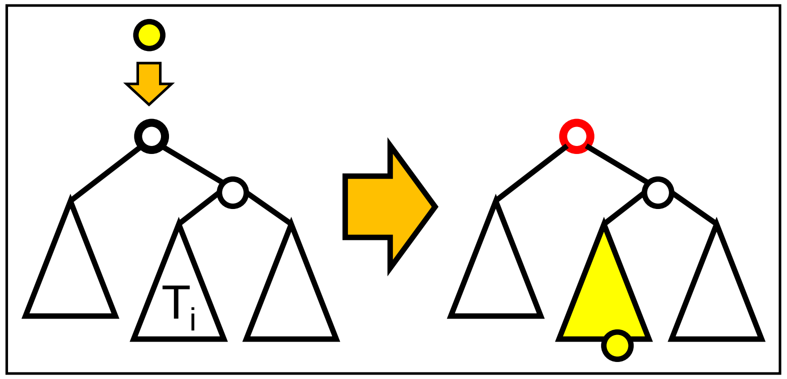
- T_i and T_o have height h-2

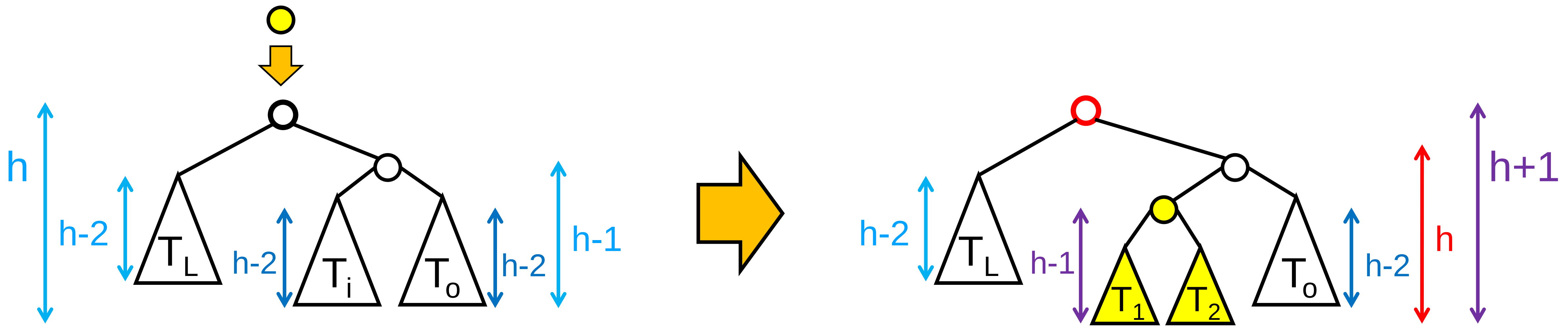


- T'_i contains at least the inserted node
  - let's expand it
  - T_1 and T_2 have height h-2 or h-3
    - one of them has height h-2
  - the inserted node could be
    - the root – if T_1 and T_2 are empty
    - in T_1
    - in T_2



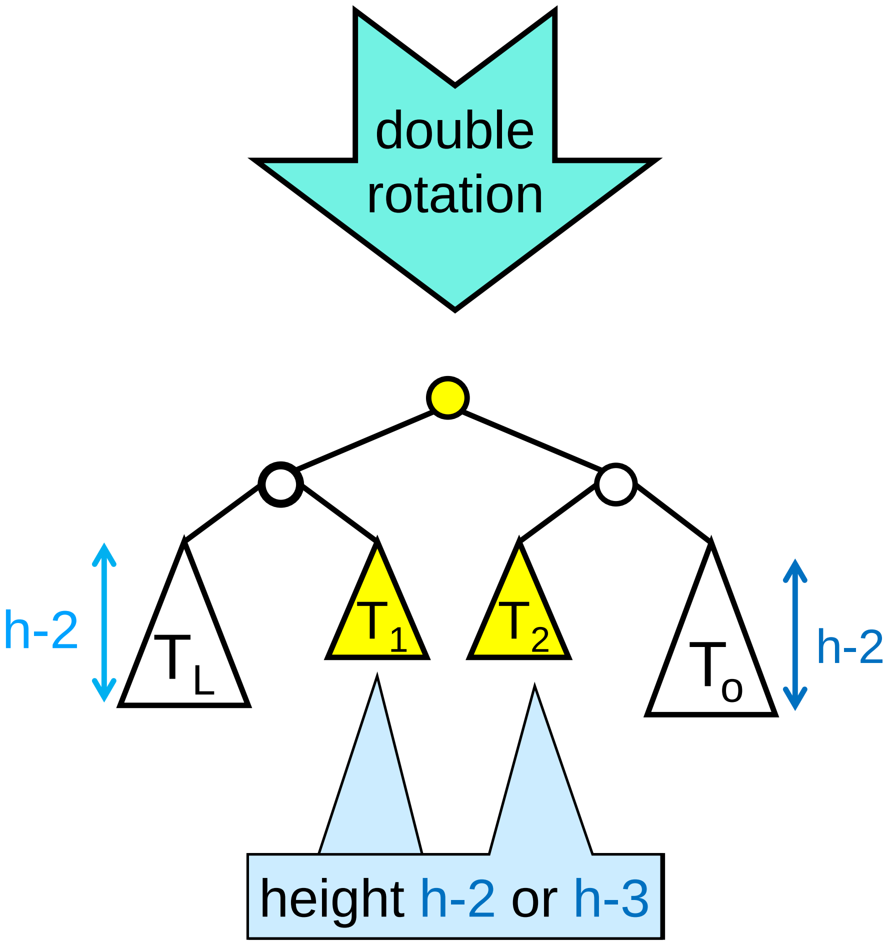
Insertion in the Inner Subtree



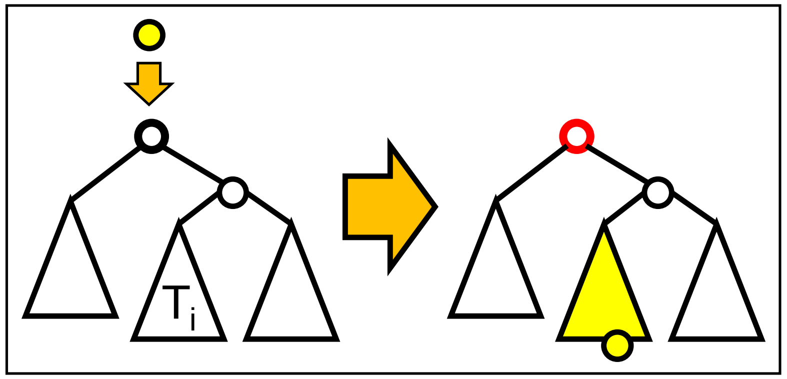


- This is the situation where we do a **double right/left rotation**

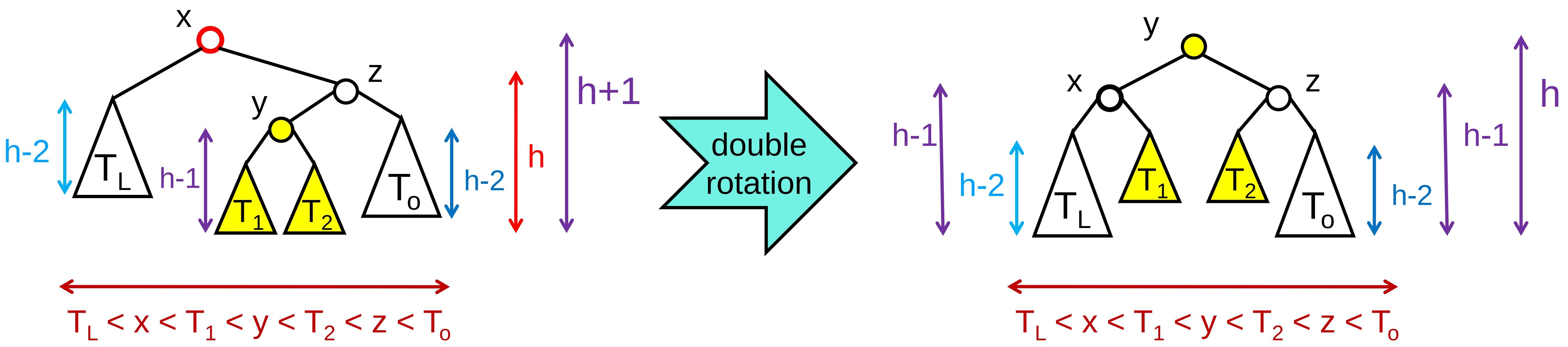
  - *Is this an AVL tree?*



Insertion in the Inner Subtree

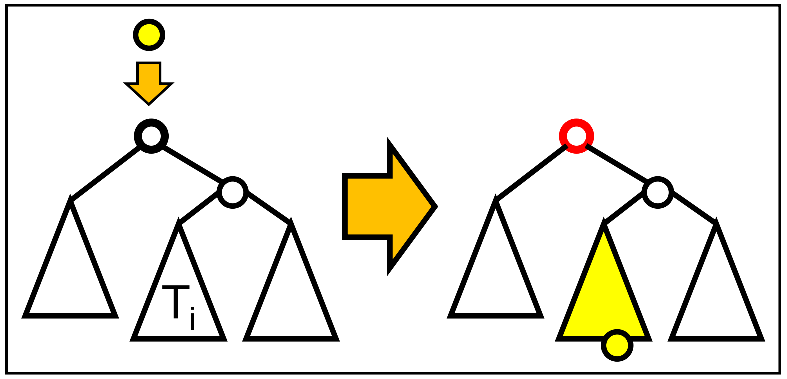


- Is this an AVL tree?

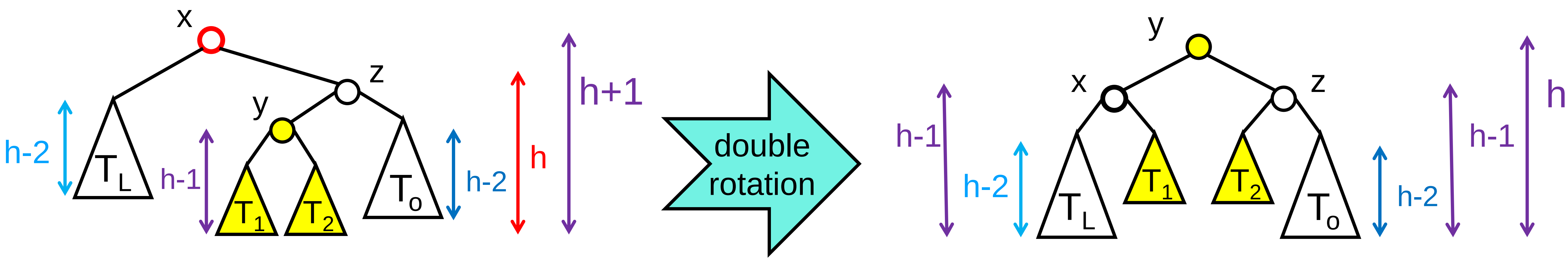


  - BST insertion and the rotations maintains the ordering invariant

Insertion in the Inner Subtree



- Is this an AVL tree?



- T_L, T_1, T_2 and T_o are AVL trees
  - because x was the lowest violation
- T_L–x–T_1 is an AVL tree of height h-1
  - because T_L has height h-2 and
  - T_1 has height either h-2 or h-3
- T_2–z–T_o is an AVL tree of height h-1
  - because T_2 has height either h-2 or h-3
  - T_o has height h-2 and
- (T_L–x–T_i)–y–(T_2–z–T_o) is an AVL tree of height h

The height invariant is restored

✓

Summary

- When inserting into an AVL tree of height *h*
  - If there is no violation, the tree height remains *h* or grows to *h+1*
  - If there is a violation, the tree height remains *h*

- To fix a violation
  - perform a rotation on the **lowest violation**
    - a **single rotation** if the node was inserted in its **outer subtree**
    - a **double rotation** if the node was inserted in its **inner subtree**

- **One** rotation fixes the whole tree
  - The resulting tree is again an AVL tree
  - lookup, insert and find_min cost *O(log n)* in it
    - where *n* is the number of nodes

Implementation

The AVL Dictionary Interface

- This is exactly the same interface we had for BST dictionaries
  - the client can't tell the difference

except that it's much faster

Library Interface

```
// typedef ______* dict_t;

dict_t  dict_new()
  /*@ensures \result != NULL;                             @*/ ;

entry dict_lookup(dict_t D, key k)
  /*@requires D != NULL;                                  @*/
  /*@ensures \result == NULL
               || key_compare(entry_key(\result), k) == 0;  @*/ ;

void dict_insert(dict_t D, entry e)
  /*@requires D != NULL && e != NULL;                     @*/
  /*@ensures dict_lookup(D, entry_key(e)) == e;           @*/ ;

entry dict_min(dict_t D,)
  /*@requires D != NULL;                                  @*/ ;
```

Client Interface

```
// typedef ______* entry;
// typedef ______ key;

key entry_key(entry  e)
  /*@requires e != NULL;                 @*/ ;

int key_compare(key k1, key k2) ;
```

  - We modify the BST *implementation* to use AVL trees

The AVL Dictionary Implementation

- We make surgical changes to the BST dictionary implementation
  - because AVL trees are BSTs
    and the BST implementation *mostly* works

- Specifically,
  - we extend the representation invariant to account the height invariant of AVL trees 3
  - insert now needs to perform rotations to rebalance the tree when needed 2
  - lookup and find_min remains unchanged 1
    - because an AVL tree is a special case of a BST

Order in which we will examine them

avl_lookup

1

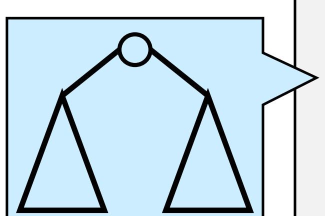
- The implementation remains unchanged
  - but we rename all the …bst… functions …avl…

```
entry avl_lookup(tree* T, key k)
//@requires is_avl(T);
//@ensures \result == NULL
            || key_compare(entry_key(\result), k) == 0;
{
  // Code for empty tree
  if (T == NULL) return NULL;

  // Code for non-empty tree
  int cmp = key_compare(k, entry_key(T->data));
  if (cmp == 0) return T->data;
  if (cmp < 0) return avl_lookup(T->left, k);
  //@assert cmp > 0;
  return avl_lookup(T->right, k);
}
```

We will implement it later

EMPTY



- If T is an AVL tree with *n* nodes, then
  - it has height *O(log n)*
  - so avl_lookup costs *O(log n)*

- find_min stays the same too
  - it now costs *O(log n)*

Inserting into an AVL Tree

```
tree* avl_insert(tree* T, entry e)
//@requires is_avl(T) && e != NULL;
//@ensures is_avl(\result) && \result != NULL;
//@ensures avl_lookup(\result, entry_key(e)) == e;
{
  // Code for empty tree
  if (T == NULL) return leaf(e);

  // Code for non-empty tree
  int cmp = key_compare(entry_key(e), entry_key(T->data));
  if (cmp == 0) T->data = e;
  else if (cmp < 0) {
    T->left = avl_insert(T->left, e);
    T = rebalance_left(T);
  else { //@assert cmp > 0;
    T->right = avl_insert(T->right, e);
    T = rebalance_right(T);
  }
  return T;
}
```

The tree layout does not change

added

added

- **After** each recursive call, we rebalance the tree
  - rebalance_left after an insertion in the **left** subtree
  - rebalance_right after an insertion in the **right** subtree
  - This guarantees we fix the **lowest violation**
- For insert to cost *O(log n)*
  - rebalance_left/right must cost *O(1)*

Let's look at one of them

rebalance_right

- We call it right after an insertion in the right subtree

```
tree* rebalance_right(tree* T)
//@requires T != NULL && T->right != NULL;
{
  if (height(T->right) - height(T->left) == 2) {   // violation!
    if (height(T->right->right) > height(T->right->left)) {
      // Single rotation
      T = rotate_left(T);
    } else
    { //@assert height(T->right->left) > height(T->right->right);
      // Double rotation
      T->right = rotate_right(T->right);
      T = rotate_left(T);
    }
  }
  return T;
}
```

The height invariant doesn't hold

Just return T if it holds

The insertion was in T->right

The insertion was in the **outer subtree**

we perform a **single rotation**

The insertion was in the **inner subtree**

we perform a **double rotation**

  - rebalance_right must have cost O(1)

rebalance_right

- We use the height of various subtrees to determine
  - if there is a violation
  - if the insertion happened in the inner or outer subtree

```
tree* rebalance_right(tree* T)
//@requires T != NULL && T->right != NULL;
{
  if (height(T->right) - height(T->left) == 2) {   // violation!
    if (height(T->right->right) > height(T->right->left)) {
      // Single rotation
      T = rotate_left(T);
    } else
    { //@assert height(T->right->left) > height(T->right->right);
      // Double rotation
      T->right = rotate_right(T->right);
      T = rotate_left(T);
    }
  }
  return T;
}
```

The height invariant doesn't hold

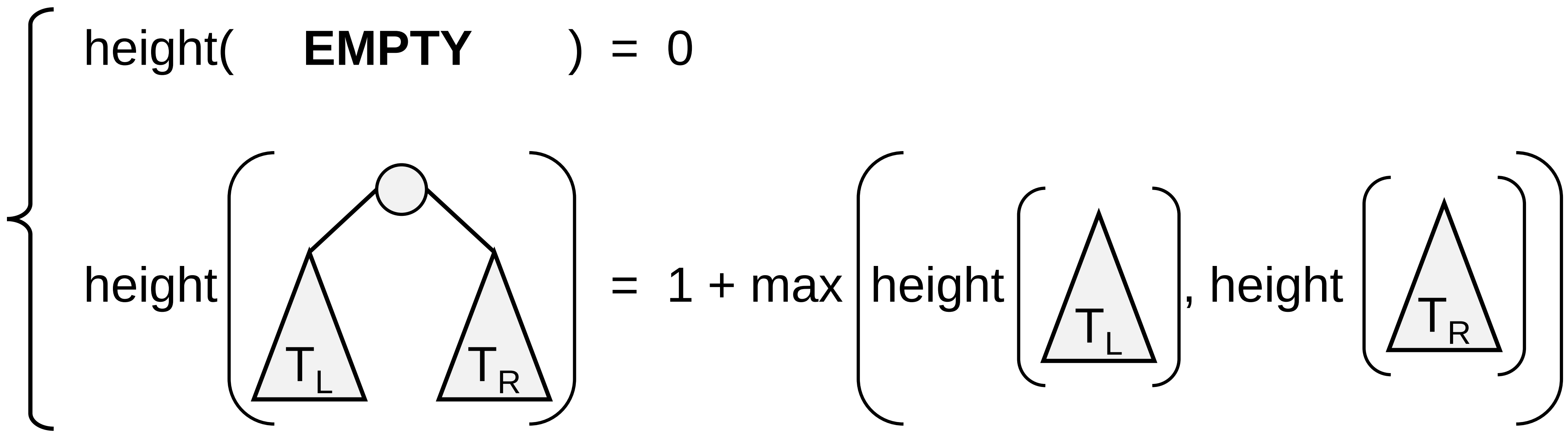
The insertion was in the **outer subtree**

The insertion was in the **inner subtree**

  - rebalance_right must have cost O(1)
    - so height, rotate_left and rotate_right must cost O(1)

height

- We can transcribe the mathematical definition



and get

```
int height(tree* T)
//@requires is_tree(T);
//@ensures \result >= 0;
{
  if (T == NULL) return 0;
  return 1 + max(height(T->left), height(T->right));
}
```

height

- By transcribing the mathematical definition, we get

```
int height(tree* T)
//@requires is_tree(T);
//@ensures \result >= 0;
{
  if (T == NULL) return 0;
  return 1 + max(height(T->left), height(T->right));
}
```

  - If T has *n* nodes, height(T) costs *O(n)*
    - it recursively goes over every node in T
- But we need height to cost *O(1)*
  - otherwise insert will cost more than *O(log n)*
- *What can we do?*

height

- Rather than computing the height of a tree by traversing it, we can **store** it
  - we add a **height field** in each node

```
typedef struct tree_node tree;
struct tree_node {
  tree* left;
  int data;
  tree* right;
  int height;  // >= 0
};
```

- Then, the function height simply returns the contents of this field
  - ➢ or 0 if T is NULL
  - Its cost is now *O(1)*

```
int height(tree* T)
//@requires is_tree(T);
//@ensures \result >= 0;
{
  return T == NULL ? 0 : T->height;
}
```

Return 0 if T is NULL and T->height otherwise

- This is a **space-time tradeoff**
  - we are using a bit of extra space to save a lot of time

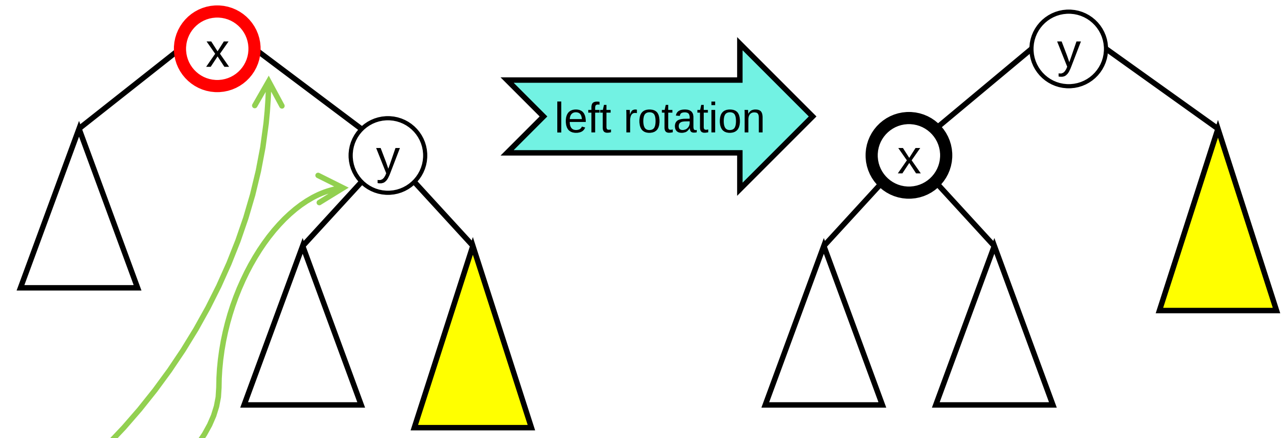
The new height field in the nodes

Computing the height of the tree over and over

Rotations

- We implement single rotations by transcribing the figure

```
tree* rotate_left(tree* T)
//@requires T != NULL && T->right != NULL;
{
  tree* temp = T->right;
  T->right = T->right->left;
  temp->left = T;
  return temp;
}
```



by updating two pointers

- ○ The cost is O(1)

- We implement double rotations as two single rotations
  - ○ The cost is O(1)

from rebalance_right

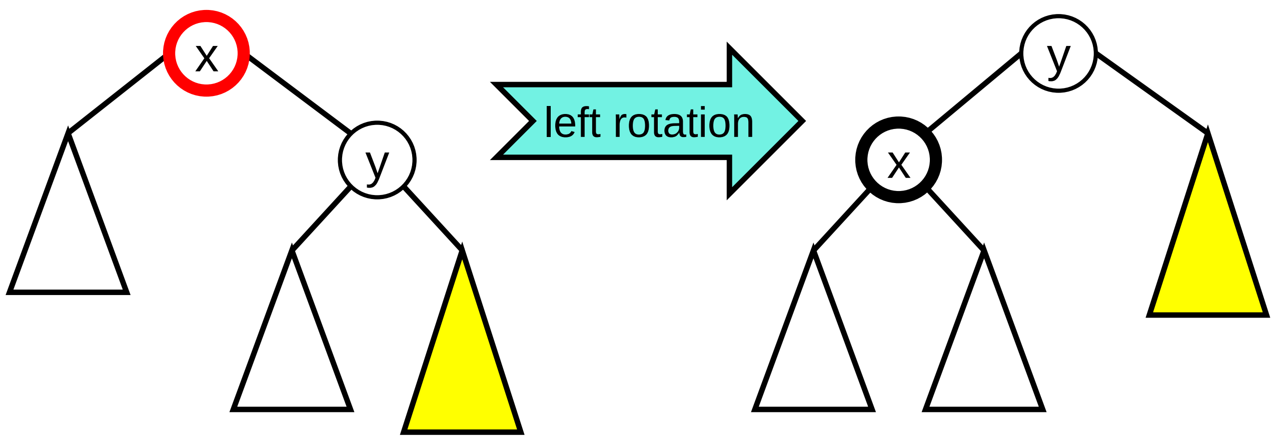
```
// Double rotation
T->right = rotate_right(T->right);
T = rotate_left(T);
```

- *Can it be this simple?*

Rotations

- *Can it be this simple?*

```
tree* rotate_left(tree* T)
//@requires T != NULL && T->right != NULL;
{
  tree* temp = T->right;
  T->right = T->right->left;
  temp->left = T;
  return temp;
}
```



- The height fields of nodes x and y are now wrong! ✘
  - We need to update them
  - We can do so based on the height of their subtrees

- Let's write a general function:
  - fix_height costs O(1)
    - because height costs O(1)

```
void fix_height(tree* T)
//@requires is_tree(T) && T != NULL;
{
  int hl = height(T->left);
  int hr = height(T->right);
  T->height = 1 + max(hl, hr);
}
```

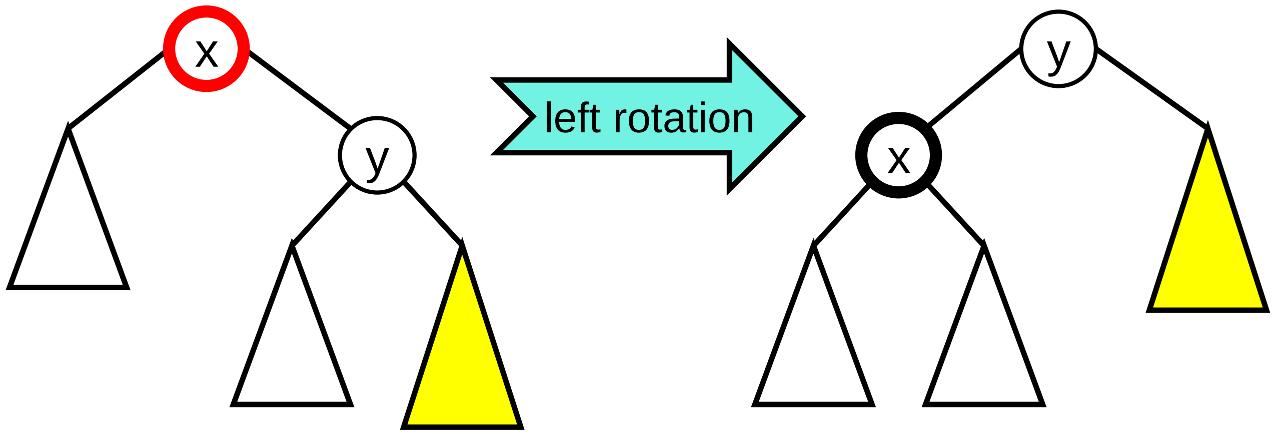
Rotations Revisited

- We implement single rotations by transcribing the figure

```
tree* rotate_left(tree* T)
//@requires T != NULL && T->right != NULL;
{
  tree* temp = T->right;
  T->right = T->right->left;
  temp->left = T;
  fix_height(T);
  fix_height(temp);
  return temp;
}
```

node x

node y



by updating two pointers

and then fixing the height of the affected nodes

- rotate_left costs O(1)

rebalance_right Revisited

- We also need to fix the height when there is no violation

```
tree* rebalance_right(tree* T)
// T must be immediate result of a right-insertion
//@requires T != NULL && T->right != NULL;
{
  if (height(T->right) - height(T->left) == 2) {   // violation!
    if (height(T->right->right) > height(T->right->left)) {
      // Single rotation
      T = rotate_left(T);
    } else {
      //@assert height(T->right->left) > height(T->right->right);
      // Double rotation
      T->right = rotate_right(T->right);
      T = rotate_left(T);
    }
  } else { // No rotation needed, but tree may have grown
    fix_height(T);
  }
  return T;
}
```

When we handle a violation, the rotations fix the heights

Fixes the heights when no rotation was performed

✓

```c
typedef struct tree_node tree;
struct tree_node {
  tree* left;
  int data;
  tree* right;
  int height;  // >= 0
};
```

New Leaves

- When insertion creates a new leaf, we need to set its height to 1

```c
tree* leaf(entry e)
//@requires e != NULL;
//@ensures is_avl(\result) && \result != NULL;
{
  tree* T = alloc(tree);
  T->data = e;
  T->left = NULL;     // not necessary
  T->right = NULL;   // not necessary
  T->height = 1;
  return T;
}
```

3

Representation Invariants

The AVL Representation Invariant

- An AVL tree is a BST that satisfies the height invariant
  - additionally, the height fields must all contain the true height

Checks that the height field in each node contains the true height of its subtree

Checks the height invariant

The AVL representation invariant

```
bool is_specified_height(tree* T)
//@requires is_tree(T);
{
  if (T == NULL) return true;
  return is_specified_height(T->left)       // height(T->left)  is correct
      && is_specified_height(T->right)      // height(T->right) is correct
      && T->height == max(height(T->left),
                          height(T->right)) + 1;   // height(T) is correct
}

bool is_balanced(tree* T)
//@requires is_tree(T);
{
  if (T == NULL) return true;
  return abs(height(T->left) - height(T->right)) <= 1
      && is_balanced(T->left)
      && is_balanced(T->right);
}

bool is_avl(tree* T) {
  return is_tree(T) && is_ordered(T, NULL, NULL)
      && is_specified_height(T)
      && is_balanced(T);
}
```

our old is_bst

checks the height

checks the height invariant

- We can use them to give precise contracts to all other functions

avl_insert Revisited

- We can track the representation invariants at each step of insertion

```
tree* avl_insert(tree* T, entry e)
//@requires is_avl(T) && e != NULL;
//@ensures is_avl(\result) && \result != NULL;
//@ensures avl_lookup(\result, entry_key(e)) == e;
{
  // Code for empty tree
  if (T == NULL) return leaf(e);

  // Code for non-empty tree
  //@assert is_avl(T->left) && is_avl(T->right);
  int cmp = key_compare(entry_key(e), entry_key(T->data));
  if (cmp == 0) T->data = e;
  else if (cmp < 0) {
    T->left = avl_insert(T->left, e);
    //@assert is_avl(T->left) && is_avl(T->right);
    T = rebalance_left(T);
     //@assert is_avl(T);
  else { //@assert cmp > 0;
    T->right = avl_insert(T->right, e);
    //@assert is_avl(T->left) && is_avl(T->right);
    T = rebalance_right(T);
     //@assert is_avl(T);
  }
  return T;
}
```

added

added

added

added

added

If T is an AVL tree, its subtrees are too

T->left is an AVL tree by the postcondition of avl_insert

T->right did not change

rebalance_left restores T into a valid AVL tree

Similar

rebalance_right Revisited

- rebalance_right
  - takes a tree whose two subtrees are AVL trees
    - but itself may not be a valid AVL tree
  - return an AVL tree

This is what we learned from avl_insert

```
tree* rebalance_right(tree* T)
// T must be immediate result of a right-insertion
//@requires T != NULL && T->right != NULL;
//@requires is_avl(T->left) && is_avl(T->right);
//@ensures is_avl(\result);
{
  if (height(T->right) - height(T->left) == 2) {   // violation!
    if (height(T->right->right) > height(T->right->left)) {
      // Single rotation
      T = rotate_left(T);
    } else {
      //@assert height(T->right->left) > height(T->right->right);
      // Double rotation
      T->right = rotate_right(T->right);
      T = rotate_left(T);
    }
  } else { // No rotation needed, but tree may have grown
    fix_height(T);
  }
   return T;
}
```

but T itself may not be an AVL tree

T may not be an AVL tree

T is again an AVL tree

T may not be an AVL tree

T is again an AVL tree

T may not be an AVL tree

T is again an AVL tree

✓

Rotations revisited

- We expect rotate_left to
  - takes a tree whose two subtrees are AVL trees
    - but itself may not be a valid AVL tree
  - return an AVL tree

```
tree* rotate_left(tree* T)
//@requires T != NULL && T->right != NULL;
//@requires is_avl(T->left) &&  is_avl(T->right);
//@ensures is_avl(\result);
{
  tree* temp = T->right;
  T->right = T->right->left;
  temp->left = T;
  fix_height(T);
  fix_height(temp);
  return temp;
}
```

but T itself may not be an AVL tree

- This would be true if used to implement single rotations **only**
- But we are also using it to implement double rotations
  - these contracts **do not hold** in this case

```
// Double rotation
T->right = rotate_right(T->right);
T = rotate_left(T);
```

Rotations revisited

- Because we implement double rotations using single rotations, we must deploy weaker contracts

```
tree* rotate_left(tree* T)
//@requires T != NULL && T->right != NULL;
//@requires is_specified_height(T->left);
//@requires is_specified_height(T->right);
//@ensures is_specified_height(\result);
{
  tree* temp = T->right;
  T->right = T->right->left;
  temp->left = T;
  fix_height(T);
  fix_height(temp);
  return temp;
}
```

This only says that the heights are right

✓

Maintaining the Height

```
typedef struct tree_node tree;
struct tree_node {
  tree* left;
  int data;
  tree* right;
  int height;  // >= 0
};
```

- We can use the same contracts in fix_height

```
void fix_height(tree* T)
//@requires is_tree(T) && T != NULL;
//@requires is_specified_height(T->left);
//@requires is_specified_height(T->right);
//@ensures is_specified_height(T);
{
  int hl = height(T->left);
  int hr = height(T->right);
  T->height = (hl > hr ? hl+1 : hr+1);
}
```

Assuming the subtrees have valid height fields, it will make the height field in the whole tree valid