Lecture 25
Spanning Trees

15-122: Principles of Imperative Computation (Fall 2024)
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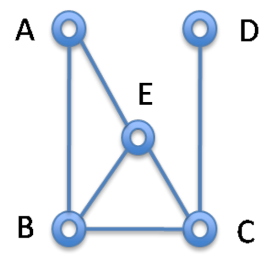
The following is a simple example of a connected, undirected graph with 5 vertices (A, B, C, D, E) and 6 edges (AB, BC, CD, AE, BE, CE).



In this lecture we are particularly interested in the problem of computing a *spanning tree* for a connected graph. What is a tree here? They are a bit different than the binary search trees we considered earlier in the course. One simple definition is that a *tree* is a *connected graph with no simple cycles*, where a simple cycle is a path that lets you go from a node to itself without repeating an edge. A *spanning tree* for a connected graph G is a tree containing all the vertices of G and a subset of the edges of G.

Additional Resources

- Review slides (https://cs.cmu.edu/~15122/handouts/slides/review/25-spanning.pdf)

The corresponding learning goals are as follows:

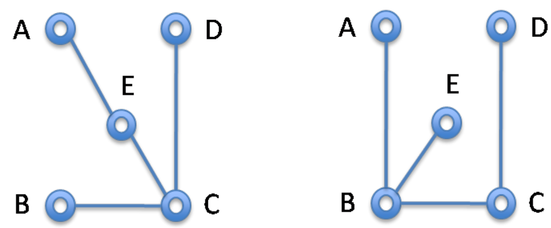
Computational Thinking: We continue our introduction to graphs by defining spanning trees as well as minimum spanning trees for graphs with weighted edges.

Algorithms and Data Structures: We examine two ways to compute a spanning tree, and introduce Kruskal's algorithm, a classical method for calculating a minimum spanning tree.

Programming: We leave the implementation of these algorithms as exercises to the reader.

1 Spanning Trees

Below are two spanning trees for our original example graph — there are more.



When dealing with a new kind of data structure, it is a good strategy to try to think of as many different characterizations as we can. This is somewhat similar to the problem of coming up with good representations of the data; different ones may be appropriate for different purposes. Here are some alternative characterizations:

1. Connected graph with no cycle (original).

2. Connected graph where no two neighbors are otherwise connected. *Neighbors* are vertices connected directly by an edge, *otherwise connected* means connected without the connecting edge.

3. Two trees connected by a single edge. This is a recursive characterization. The base case is a single node, with the empty tree (no vertices) as a possible special case.

4. A vertex connected to a tree by a single edge. The base case is again a single vertex. This is another recursive characterization.

5. A connected graph with exactly $v - 1$ edges, where v is the number of vertices.

6. A graph with exactly one path between any two distinct vertices, where a path is a sequence of distinct vertices where each is connected to the next by an edge. (For paths in a tree to be distinct, we have to disallow paths that double back on themselves).

We call a collection of trees a *forest*. Naturally, for a graph with more than one connected component, we will want to compute a spanning forest consisting of a spanning tree for each connected component.

2 Computing a Spanning Tree

There are many algorithms to compute a spanning tree for a connected graph. We will look at two of them.

2.1 Edge-centric Algorithm

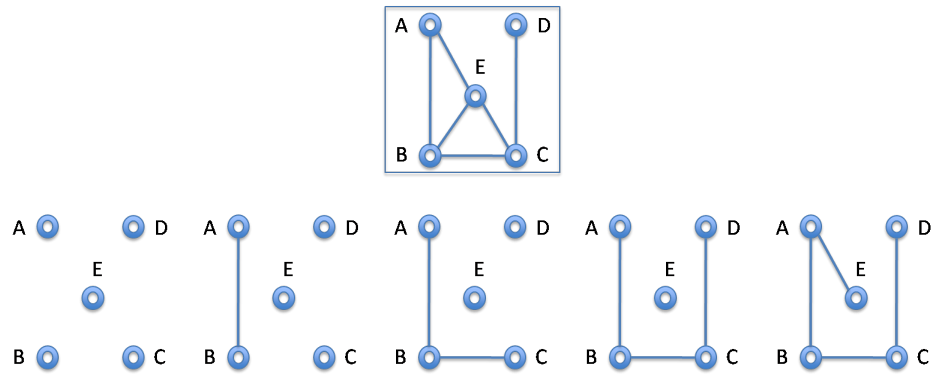
The first is an example of an *edge-centric* algorithm. It leverages definition (3) of trees in the last section. It proceeds as follows:

1. Start with the collection of singleton trees, each with exactly one node.
2. As long as we have more than one tree, connect two trees together with an edge in the graph.

In the second step, we repeatedly examine one of the original graph edges and determine whether it spans two disconnected trees. We can naively do so by using DFS or BFS to check if its endpoints are already connected in our spanning forest — if so we discard the edge, if not we add it to the spanning tree. Recall that, with an adjacency list representation, the complexity of DFS and BFS is bounded by the number of edges. The cost of this test is then $O(v)$ because we carry it out *on the trees* which contain at most $v - 1$ edges at any time — not on the original graph. How many edges will we need to examine? We know by definition (5) of a tree that, if the graph has v vertices, we will end up adding $v - 1$ edges. However, not all tests are successful! In the worst case, we will need to examine all edges in the original graph, i.e., perform the test e times. This give this version of the edge-centric algorithm an $O(ev)$ complexity.

Can we do better? The efficiency of this algorithm is greatly affected by how quickly we can tell if an edge would connect two trees or would connect two nodes already in the same tree. Using DFS or BFS to answer this question seems overkill because it does not account for any information about which node is in which tree — something we can track since *we* put them in there. We will come back to this question in the next lecture.

Let's try this algorithm on our first graph, considering edges in the listed order: (AB, BC, CD, AE, BE, CE).



The given graph is highlighted on top. The completely disconnected graph on the left is the starting point for this algorithm. At the far right, we have computed a spanning tree, which we know because we have added $v - 1 = 4$ edges. If we tried to continue, the next edge BE could not be added because it does not connect two trees, and neither can CE. The spanning tree is complete.

2.2 Vertex-centric Algorithm

The second algorithm is *vertex-centric*. It is based on definition (4) of a tree and proceeds as follow:

1. Pick an arbitrary node and mark it as being *in the tree*.

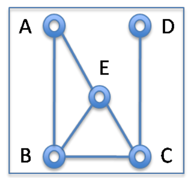
2. Repeat until all nodes are marked as *in the tree*:

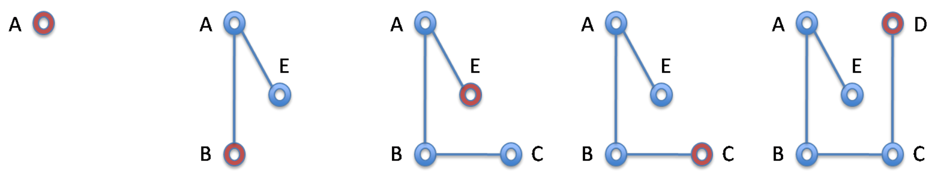
   > Pick an arbitrary node u in the tree with an edge e to a node w not in the tree. Add e to the spanning tree and mark w as in the tree.

We can implement this by modifying BFS or other algorithm to check connectivity, where we use a work list (a queue if adapting BFS) to remember vertices to expand next in step 2. Specifically, step 1 will pick an arbitrary vertex and insert it in the queue. Step 2 will repeatedly pick a vertex v from the queue and replace it with every neighbor w that has never been encountered before (which can be tracked using an array of marks). At the same time, it will add the edge (v, w) to the spanning tree. This will continue as long as there are vertices in the queue (were the original graph to be disconnected, we can pick an unmarked node and repeat the algorithm starting from it, thereby building a spanning forest).

If the original graph is connected, this algorithm has cost $O(e)$ by an analysis that is identical as that of DFS in the last chapter: for each vertex it checks whether its neighbors have been visited, which amounts to two checks for each edge in the graph (one from each endpoint). If the original graph is not connected, we will repeat this procedure for all connected components. In particular, we will end up visiting all v vertices — and nothing else in the degenerate case of a graph with no edges. Therefore, this algorithm has cost $O(\max(v, e))$. This is better than the edge-centric algorithm we saw earlier.

Let's play it out on our running example, starting with vertex A and enqueuing new vertices in alphabetical order:





At each step, the vertex highlighted in red is the node we are visiting, after dequeuing it but before examining its neighbors. It is a coincidence that the resulting spanning tree is identical to the one we obtained by using the edge-centric algorithm.

3 Creating a Random Maze

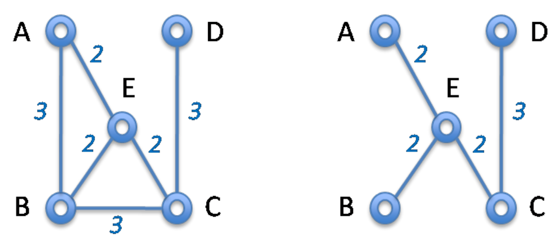
We can use the algorithm to compute a spanning tree for creating a random maze. We start with the graph where the vertices are the cells and the edges represent the neighbors we can move to in the maze. In the graph, all potential neighbors are connected. A spanning tree will be defined by a subset of the edges in which all cells in the maze are still connected by some (unique) path. Because a spanning tree connects all cells, we can arbitrarily decide on the starting point and end point after we have computed it.

How would we ensure that the maze is random? The idea is to generate a random permutation (see Exercise 1) of the edges and then consider the edges in the fixed order. Each edge is either added (if it connects two disconnected parts of the maze) or not (if the two vertices are already connected). But, of course, we need an efficient way to determine if the two vertices are already connected. The best we can do so far is $O(v)$; we will see in the next lecture how to do better.

4 Minimum Weight Spanning Trees

In many applications of graphs, there is some measure associated with the edges. For example, when the vertices are locations then the edge weights could be distances. We might then be interested in not any spanning tree, but one whose total edge weight is minimal among all the possible spanning trees, a so-called *minimum weight spanning tree* (MST). An MST is not necessarily unique. For example, all the edge weights could be identical in which case any spanning tree will be minimal.

We annotate the edges in our running example with edge weights as shown on the left below. On the right is the minimum weight spanning tree, which has weight 9.



Before we develop a refinement of our edge-centric algorithm for spanning trees to take edge weights into account, we discuss a basic property it is based on.

Cycle Property.

> Let C be a simple cycle in graph G, and e be an edge of maximal weight in C. Then there is some MST of G that does not contain e.

How do we convince ourselves of this property? Assume we have a minimum spanning tree T, and edge e from the cycle property connects vertices u and w. If e is not in T, then, indeed, we don't need it. If e is in T, we will construct another spanning tree without e of weight less than or equal to T's weight. Removing edge e splits T into two subtrees. There must be another edge e' from C that is not in T which also connects the two subtrees. Removing e and adding e' instead yields another spanning tree, T', which does not contain e. T' has equal or lower weight to T, since e' must have weight less than or equal to e.

The cycle property is the basis for *Kruskal's algorithm*.

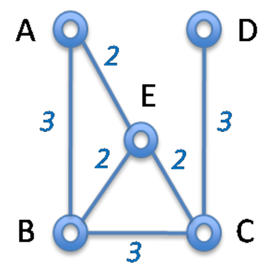
1. Sort all edges in increasing weight order.

2. Consider the edges in order. If the edge does not create a cycle, add it to the spanning tree. Otherwise discard it. Stop when $v - 1$ edges have been added, because then we must have a spanning tree.

Why does this create a minimum-weight spanning tree? It is a straightforward application of the cycle property (see Exercise 2).

Sorting the edges will take $O(e \log e)$ steps with most appropriate sorting algorithms. The complexity of the second part of the algorithm depends on how efficiently we can check if adding an edge will create a cycle or not. So far, the best we can do is $O(ev)$.

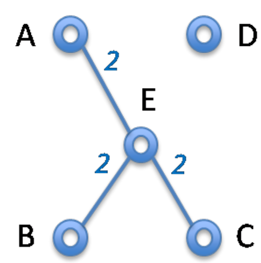
Illustrating the algorithm on our example



we first sort the edges. There is some ambiguity — say we obtain the following list

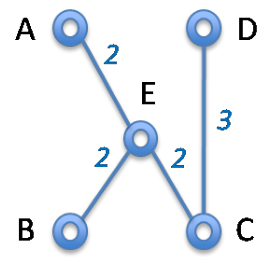
AE	2
BE	2
CE	2
BC	3
CD	3
AB	3

We now add the edges in order, making sure we do not create a cycle. After AE, BE, CE, we have



At this point we consider BC. However, this edge would create a cycle BCE since it connects two vertices in the same tree instead of two different trees. We therefore do not add it to the spanning tree. Next we consider

CD, which does connect two trees. At this point we have a minimum spanning tree



We do *not* consider the last edge, AB, because we have already added $v - 1 = 4$ edges.

In the next lecture we will analyze the problem of incrementally adding edges to a tree in a way that allows us to quickly determine if an edge would create a cycle.

Kruskal's algorithm is nothing more than the edge-centric algorithm examined in Section 2, preceded by the additional step of sorting the edges by increasing weight (and examining edges on the basis of that order). The vertex-centric algorithm can similarly be adapted to compute a minimum spanning tree of a weighted graph. At each step, of all edges between vertices in the tree and vertices outside the tree, we will add an edge of minimal weight. To this end, when visiting a vertex v, we cannot put down the edge (v, w) to an unvisited neighbor w since there may be a cheaper way to get to w. Instead, we will record these edges (rather than just the vertices) not in a queue but in a *priority queue* with lighter edges having priority over heavier edges. A step now consists in retrieving an edge of minimal weight from the priority queue: if its endpoint is already in the spanning tree we discard it (we have already found a cheaper way to get to that vertex), otherwise we add it (and insert its unvisited neighbors in the priority queue). The resulting procedure is known as *Prim's algorithm* and its run time complexity is dominated by the cost of inserting edges in the priority queue. This cost is $O(e \log e)$ since we may be inserting all edges in the priority queue.

5 Exercises

Exercise 1 (Randomizing an Array)**.** *Write a function to generate a random permutation of a given array, using a random number generator with the interface in the standard* `rand` *library. What is the asymptotic complexity of your function?*

Exercise 2 (Proving Correctness)**.** *Prove that the cycle property implies the correctness of Kruskal's algorithm.*