

Lecture 26

Union-Find

15-122: Principles of Imperative Computation (Fall 2024)
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Kruskal's algorithm for minimum weight spanning trees starts with a collection of single-node trees and adds edges until it has constructed a spanning tree. At each step, it must decide if adding the edge under consideration would create a cycle. If so, the edge shall not be added to the spanning tree; if not, it will.

In this lecture we will consider an efficient data structure for checking if adding an edge to a partial spanning tree would create a cycle, a so-called *union-find* structure.

Additional Resources

- [Review slides](https://cs.cmu.edu/~15122/handouts/slides/review/26-unionfind.pdf) (<https://cs.cmu.edu/~15122/handouts/slides/review/26-unionfind.pdf>)
- [Code for this lecture](https://cs.cmu.edu/~15122/handouts/code/26-unionfind.tgz) (<https://cs.cmu.edu/~15122/handouts/code/26-unionfind.tgz>)

This lecture fits our learning goals as follows.

Computational Thinking: We complete our overview of graphs with union-find, a data structure often used when working with graphs.

Algorithms and Data Structures: Union-find is an important data structure beyond graphs as it allows to work efficiently with equivalence classes whenever we can easily designate an element as a canonical representative of the class.

Programming: We leave it to the reader to study the code that implements union-find.

1 Maintaining Equivalence Classes

The basic idea behind the union-find data structure is to maintain equivalence classes of nodes, efficiently. An equivalence class is a set of elements

related by an *equivalence relation*, which must be *reflexive*, *symmetric*, and *transitive*. In our case, this equivalence relation is defined on nodes in a partial spanning tree, where two nodes are related if there is a path between them. This is reflexive, because from each node u we can reach u by a path of length 0. It is symmetric because we are working with undirected graphs, so if u is connected to w , then w is also connected to u . It is transitive because if there is a path from u to v and one from v to w , then the concatenation is a path from u to w .

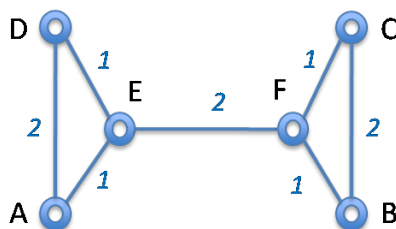
Initially in Kruskal's algorithm, each node is in its own equivalence class. When we connect two trees with an edge, we have to form the *union* of the two equivalence classes, because each node in either of the two trees is now connected to all nodes in both trees.

When we have to decide if adding an edge between two nodes u and w would create a cycle, we have to *find* out if u and w belong to the same equivalence class. If so, then there is already a path between u and w ; adding the edge would create a cycle. If not, then there is not already such a path, and adding the edge would therefore not create a cycle.

The union-find data structure maintains a so-called *canonical representative* for each equivalence class, which can be computed efficiently from any element in the class. We then determine if two nodes u and w are in the same class by computing the canonical representatives of u and w , respectively, and comparing them. If they are equal, they must be in the same class, otherwise they are in two different classes.

2 An Example

In order to motivate how the union-find data structure works, we consider an example of Kruskal's algorithm. We have the following graph, with the indicated edge weights.

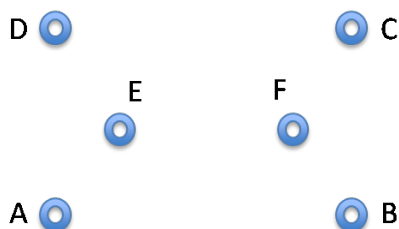


We have to consider the edges in increasing order, so let's fix the order AE , ED , FB , CF , AD , EF , CB . We represent the nodes A – F as integers 0–5.

How shall we keep track of equivalence classes? In its simplest form, the *union-find structure* is just an array with as many positions as there are

vertices in the graph (or elements in the set), so that each index represents a vertex. The contents of the array at position i is the index of the canonical representative of vertex i (or of another vertex in the same connected component — see below). In particular, if i is a canonical representative, position i will contain i itself.

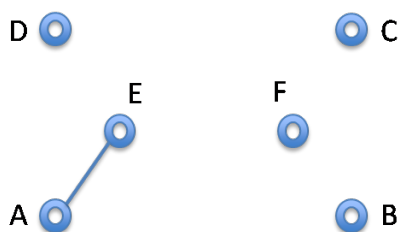
Initially, each node is in its own equivalence class.



In the union-find array, UF , is in the following state

A	B	C	D	E	F
0	1	2	3	4	5
0	1	2	3	4	5

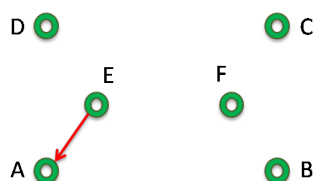
We begin by considering the edge AE . We see that vertices A (index 0) and E (index 4) are in two different equivalence classes because $UF[0] = 0$ and $UF[4] = 4$, and $0 \neq 4$. This means we have to add an edge between A and E .



In the array of canonical representatives, we either have to set $UF[0] = 4$ or $UF[4] = 0$, depending on whether we choose 4 or 0 as the representative the new class containing A and E . Let's assume it's 0. The array then would be the following:

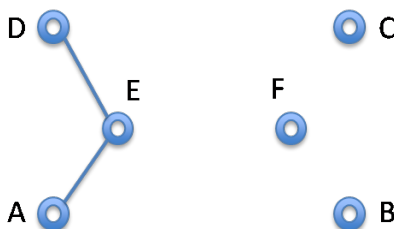
A	B	C	D	E	F
0	1	2	3	4	5
0	1	2	3	0	5

where we highlighted the change in red. It is convenient to visualize the contents of the union-find structure as a *directed* graph with the same vertices as our original graph, and an edge from i to j exactly when $UF[i]$ contains j . We display this new graph in green to distinguish it from both the original graph and the spanning tree we are constructing.

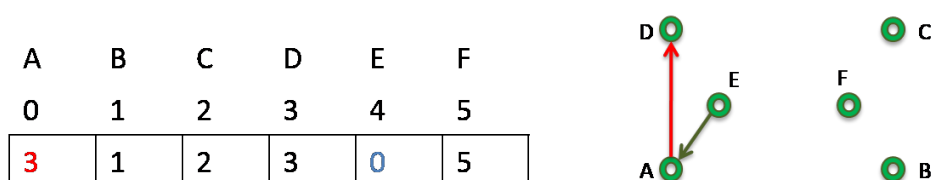


The sole purpose of this graph is to make it easier for us humans to visualize the contents of the union-find structure as we simulate the union-find algorithm. An implementation would operate exclusively on the array. From now on, we will display the two representations side by side. They carry the same information — you can follow the discussion on the one that works best for you.

Next we consider ED . Again, this edge should be added because $UF[4] = 0 \neq 3 = UF[3]$.

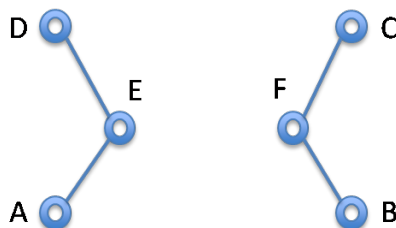


The union-find structure tells us that A is the canonical representative of the tree with vertices A and E , and D is the canonical representative of the singleton tree containing just D . Which vertex should we appoint as the canonical representative of the combined tree? E is not a good candidate because that would involve making two changes in the array (both $UF[0]$ and $UF[3]$ would need to be set to 4). In general, we want to appoint one of the existing canonical representatives as the canonical representative of the combined equivalence class: in this way a single array cell needs to be modified. As we are learning our way around union-find, we will pick D to be the new representative, although we will see later that this may not be the best choice. Therefore, we change $UF[0]$ to 3. The union-find structure now looks as follows:

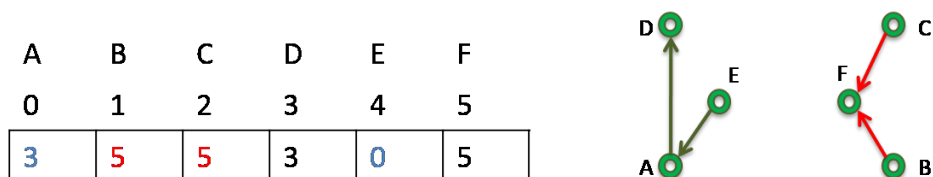


Notice that now $UF[4]$ does not contain any more the canonical representative of E : it points to a vertex from where we can find its canonical representative. In general, going through the array and repointing all vertices to their canonical representative would be very expensive, although we partially revisit this idea later on. Observe also that the edge added to the (directed) union-find visualization graph is different from the edge inserted in the spanning tree: these two graphs do not carry the same information.

We now combine two more steps, because they are analogous to the above, adding edges FB and CF .

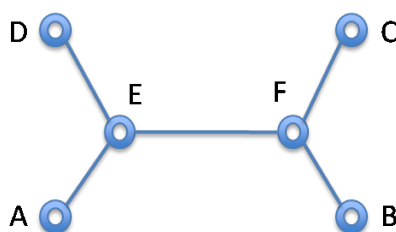


Again we have some choices. Let's say the union-find structure becomes:



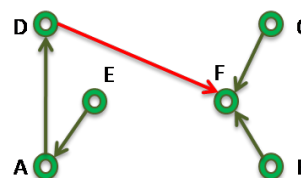
Next was the edge AD . In the array we have that $UF[0] = 3 = UF[3]$, so A and D belong to the same equivalence class. Adding the edge would create a cycle, so we ignore it and move on.

The next edge to consider is EF . Since $UF[4] = 0$ and $UF[0] = 3 \neq 5 = UF[5]$ they are in different equivalence classes. We shall appoint one among D and F as the canonical representative of the combined class. We arbitrarily choose F . Taking this step we now have the tree



and a union-find structure which is as follows:

A	B	C	D	E	F
0	1	2	3	4	5
3	5	5	5	0	5



Observe that the edge added to the visualization graph is not in the original graph.

At this point we can stop, and we don't even need to consider the last edge BC . That's because we have already added $5 = v - 1$ edges (where v is the number of nodes), so we must have a spanning tree at this stage.

How does union-find affect the cost of determining whether two vertices are connected in a graph? We first need to find their canonical representatives. Doing so has a maximum cost of $O(v)$ for each of them, since we may need to chase (nearly) every vertex in the graph to reach its canonical representative. Therefore, checking whether these vertices are connected costs $O(v)$, which is what we were able to achieve using BFS (recall that we checked connectivity on the spanning tree under construction, not on the original graph). If they are not connected, updating the union-find structure through a union operation has cost $O(1)$ — we are just modifying one value in the array.

Thus, adopting union-find as part of Kruskal's algorithm does not change the complexity, which remains $O(e \log e + ev)$ or, simplifying, $O(ev)$.

3 Height Tracking

Can we do better?

The cost of the find operation is given by how many vertices we need to examine to get to the canonical representative of a node. The fewer number of intermediate vertices the faster this will be.

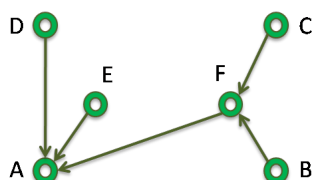
It is useful to look at the graphical representation of the union-find structure as a set of trees, each with its canonical representative as its root.

Then, the find operation amounts to following edges to the root of a tree. Its cost is therefore given by the height of the tree, i.e., the number of vertices on the longest path from a leaf to the root.

The union operation merges one tree into another. Key to an efficient find operation is therefore to keep the height of the merged tree as small as possible. This insight determines an improved strategy for merging two trees, T_1 of height h_1 and T_2 of height h_2 :

- If $h_1 > h_2$, merge T_2 into T_1 by appointing the root of T_1 as the canonical representative of the combined tree. The resulting tree will have height h_1 .
- If $h_1 < h_2$, proceed the opposite way and pick the root of T_2 as the root of the merged tree, which will have height h_2 .
- If $h_1 = h_2$, it doesn't matter which tree we merge into which: the combined tree will have height $h_1 + 1$.

Let's apply this strategy on our ongoing example. When adding edge ED on the second step, we would appoint A rather than D as the representative of the new tree, which would give it height 2 instead of 3. These were the kind of choices we made when adding edges FB and CF . On the final step, when adding edge EF , we would have to merge two trees of height 2, and so it doesn't matter which way we go: the resulting tree will have height 3. Merging the tree rooted at F into the tree rooted at A , the final union-find structure is as follows:



To implement this strategy, we need to record the height of each tree in the union-find structure. We can still store a single number in each position of the array by observing that we need to know the height of a tree only when reaching its root (so that we can decide which way to merge two trees) and that recording the fact that i is the root of a tree by storing i in $UF[i]$ is overkill: a simple flag is enough. Whenever i is the root of a tree, this suggests setting $UF[i] = -h$ where h is the height of this tree — we use the sign bit as our flag. Positions that do not correspond to roots store indices of their parent in the tree, as we did in our first version of union-find.

Applying this idea to our ongoing example, the UF array evolves as follows:

	A	B	C	D	E	F
	0	1	2	3	4	5
<i>Initially</i>	-1	-1	-1	-1	-1	-1
Add AE	-2	-1	-1	-1	0	-1
Add ED	-2	-1	-1	0	0	-1
Add FB	-2	5	-1	0	0	-2
Add CF	-2	5	5	0	0	-2
Add EF	-3	5	5	0	0	0

Here, we highlight in red the changed at each step and use bold to further emphasize canonical representatives.

Does using this strategy, known as *height tracking*, lower the cost of the find operation? To figure this out, we need to understand what the maximum height of a tree with v nodes can be. The following property gives us a way to answer this question:

A tree of height h has at least 2^{h-1} nodes.

The proof proceeds by a simple induction on h :

Base case: $h = 1$ A tree of height 1 has exactly one node. And indeed $2^{1-1} = 2^0 = 1$.

Inductive case: $h > 1$ We need to distinguish subcases on how this tree came about. This tree was constructed by merging two trees, T_1 of height h_1 and T_2 of height h_2 .

- If $h_1 > h_2$, then $h = h_1$. We know that T_1 contained at least 2^{h-1} nodes and therefore the combined tree contains all these 2^{h-1} nodes plus the nodes of T_2 .
- If $h_1 < h_2$, the argument is symmetric.
- If $h_1 = h_2$, then $h = h_1 + 1$. Thus the merged tree contains at least $2^{h-2} + 2^{h-2} = 2^{h-1}$ nodes.

If a tree of height h contains at least 2^{h-1} vertices, then a tree with v nodes has height at most $\log v$. If we have a forest with v vertices and e_T edges, the tallest tree has height at most $\log(\min(e_T + 1, v))$ because in this case $e_T \leq v - 1$.

Consequently, the find operation of union-find has cost $O(\log(\min(e + 1, v)))$ where e is the number of edges in the original graph: that's because the number of edges that end up in the spanning tree (e_T above) will be at most e .

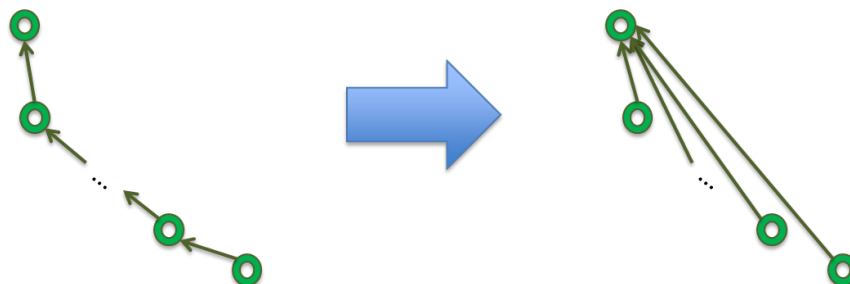
Thus, using union-find with height tracking lowers the cost of computing a minimum spanning tree using Kruskal's algorithm to $O(e \log e + e \log(\min(e + 1, v)))$. If $e < v$, this trivially reduces to $O(e \log e)$. If $e \geq v$, this simplifies to $O(e \log e + e \log v)$, but since $e \log v \leq e \log e$ in this case, it too reduces to $O(e \log e)$.

This shows that the overall cost of our algorithm is $O(e \log e)$, which is an upper bound on the cost of sorting the edges of the original graph.

4 Path Compression

Can we do even better?

A further optimization comes from the observation that when finding the canonical representative of a node, we have the opportunity to update every node in the path to point directly to the root of the tree:



In this way, any future find operation that involves one of these nodes will be one hop away from the root. This will also often reduce the height of the tree, sometimes dramatically.

This optimization, known as *path compression*, gives the find operation a nearly-constant amortized complexity. For a graph with v vertices, the amortized cost of union-find with height tracking and path compression is in $O(1 + A^{-1}(v))$, the inverse of the function $A(v)$ which is equal to $A(n) = \text{Ack}(n, n)$. The function $\text{Ack}(m, n)$ is known as the *Ackermann function*, defined as follows:

$$\text{Ack}(m, n) = \begin{cases} n + 1 & \text{if } m = 0 \\ \text{Ack}(m - 1, 1) & \text{if } m > 0 \text{ and } n = 0 \\ \text{Ack}(m - 1, \text{Ack}(m, n - 1)) & \text{if } m > 0 \text{ and } n > 0 \end{cases}$$

This function grows very fast: $Ack(0, 0) = 1$, $Ack(1, 1) = 3$, $Ack(2, 2) = 7$, $Ack(3, 3) = 61$, $Ack(4, 4)$ is larger than the number of atoms in the universe. Therefore, $1 + Ack^{-1}(n, n)$ is nearly 1 for all practical purposes.

5 An Implementation

Instead of developing the implementation here, we refer the reader to the code on the course web site.

A first implementation, `unionfind-lin.c`, does not track the height of the trees, and is therefore linear in the worst case. It does perform a weak form of path compression: a postcondition of `ufs_find(eqs, i)` is that `eqs->A[i] == ufs_find(eqs, i)`. That is, before returning the representative for `i`, the implementation stores that representative at `A[i]`. This shortens the search time for subsequent find operations on `i`. (See the exercises for strong path compression.)

A second implementation, `unionfind-log.c`, changes the representation to use height tracking as discussed above. This allows us to make a quick decision how to pick a representative for the union.

6 Exercises

Exercise 1. *Prove that after n union operations, the longest chain from an element to its representative is $O(\log n)$ if we always take care to have the class with longer chains be the canonical representative of the union. This is without any form of path compression. Since n is bounded by the number v of vertices of the graph, the length of this chain is $O(\log v)$.*

Exercise 2. *Modify the simple implementation in `unionfind-lin.c` so it does strong path compression, which means that on every find operation, every intermediate node will be redirected to point directly to its canonical representative.*

Exercise 3. *Modify the more efficient implementation at `unionfind-log.c` to do path compression. Note that this may require loosening the invariants, since in the straightforward implementation the stored number is only a bound on the longest path and may not be exact (since the path may be compressed).*