Contracts

A Mystery Function

The Story

Your first task at your new job is to debug this code written by your predecessor, who was fired for being a poor programmer.

```
int f(int x, int y) {
  int r = 1;
  while (y > 1) {
    if (y % 2 == 1) {
      r = x * r;
    }
    x = x * x;
    y = y / 2;
  }
  return r * x;
}
```

This is all you are given

How do you go about this “friendly” challenge?

The Language

- This code is written in **C0**
  - The language we will use for most of this course
- This is also valid **C** code
  - For the most part, C0 programs are valid C programs
  - We will use C0 as a gentler language to
    - learn to write complex code that is correct
    - learn to write code in C itself
- *But what does this function do?*

```
int f(int x, int y) {
  int r = 1;
  while (y > 1) {
    if (y % 2 == 1) {
      r = x * r;
    }
    x = x * x;
    y = y / 2;
  }
  return r * x;
}
```

The Programmer

- Is this good code?
  - there are no comments
  - the names are non-descript
    - the function is called f
    - the variables are called x, y, r

No! ✘

- No wonder your predecessor was fired as a bad programmer!

- *But what does this function do?*

```
int f(int x, int y) {
  int r = 1;
  while (y > 1) {
    if (y % 2 == 1) {
      r = x * r;
    }
    x = x * x;
    y = y / 2;
  }
  return r * x;
}
```

The Function

- *But what does this function do?*
- We can run *experiments*
  - call f with various inputs and observe the outputs
- We do so by loading it in the **C0 interpreter** – coin

The command for the C0 interpreter

The file where we saved the function

Linux Terminal

```
# coin mystery.c0
C0 interpreter (coin) 0.3.3 'Nickel' (r590, Mon Aug 29 12:04:13 UTC 2016)
Type `#help' for help or `#quit' to exit.
-->
```

The coin prompt

Running Experiments

- *Call f with various inputs and observe the outputs*

Linux Terminal

```
# coin mystery.c0
C0 interpreter (coin) …
…
--> f(7, 12);
956385313 (int)
--> f(3, 17);
129140163 (int)
-->
```

We are calling f with inputs 7 and 12

The result is 956385313

The result has type int

- These are not very good experiments
  - they don't help us understand what f does

Running Experiments

- *Call f with various inputs and observe the outputs*
  - we are better off calling f with small inputs
  - and vary them by just a little bit so we can spot a pattern

Linux Terminal

```
--> f(2, 3);
8 (int)
--> f(2, 4);
16 (int)
--> f(2, 5);
32 (int)
--> f(2, 6);
64 (int)
-->
```

Much better!

- It looks like f(x, y) computes x^y
- Let's confirm with more experiments

Confirming the Hypothesis

- *It looks like f(x, y) computes x^y*
- *Let's confirm with more experiments*

Linux Terminal

```
--> f(2, 2);
4 (int)
--> f(3, 2);
9 (int)
--> f(4, 2);
16 (int)
--> f(5, 2);
25 (int)
-->
```

Yep! That's x^y

- We find a secret memo in a hidden drawer

Power not working.
Fix by tonight or you're out

Not the friendliest of work places!

- Let's run a few more experiments to identify the problem

Discovering the Bug

- f(x, y) is *meant to* computes x^y
  - but it doesn't
- Let's find where it fails with more experiments

Linux Terminal

```
--> f(-2, 3);
-8 (int)
--> f(-2, 2);
4(int)
--> f(2, 1);
2 (int)
--> f(2, 0);
2 (int)
--> f(2, -1);
2 (int)
-->
```

It seems to work for negative values of x

That's not 2^0

That's definitely not 2^{-1}

- Now we have something to chew on

Preconditions

The Power Function

- What does it mean to be the power function x^y ?
  - $\underbrace{x * \ldots * x}_{y \text{ times}}$
    - Yes, but that's not very precise

• *y* times what? *x*? *?
• What if *y* is 0?

- Let's write a *mathematical* definition
  - $$\begin{cases} x^0 = 1 \\ x^y = x^{y-1} * x \end{cases}$$

This is a *recursive* definition

and this is its base case

The Power Function

- *What does it mean to be the power function x^y ?*

$$\begin{cases} x^{\mathbf{0}} = 1 \\ x^{\mathbf{y}} = x^{\mathbf{y-1}} * x \end{cases}$$

  - What happens if y is negative?
    - we never reach the base case …

- The power function x^y on integers is **undefined** if $y < 0$

$$\begin{cases} x^{\mathbf{0}} = 1 \\ x^{\mathbf{y}} = x^{\mathbf{y-1}} * x \quad \text{if } y > 0 \end{cases}$$

This defines x^y for $y \geq 0$ **only**

The Power Function

```
int f(int x, int y) {
  int r = 1;
  while (y > 1) {
    if (y % 2 == 1) {
      r = x * r;
    }
    x = x * x;
    y = y / 2;
  }
  return r * x;
}
```

- *What does it mean to be the power function x^y?*

$$
\begin{cases} x^{0} = 1 & \\ x^{y} = x^{y-1} * x & \text{if } y > 0 \end{cases}
$$

- To implement the power function, f must disallow negative exponents
  - It can raise an error
  - It can tell the caller that the exponent should be ≥ 0

We need to test y.
This would slow f down a bit.

Better!
no need to test y

Preconditions

- Disallow negative exponents
  - by telling the caller that the exponent should be ≥ 0
- A restriction on the admissible inputs to a function is called a **precondition**
  - We need to impose a precondition on f
- In most languages, we are limited to writing a comment
  - and hope the caller reads it

This is how we would write a precondition in C

```
// y must be greater than or equal to 0
int f(int x, int y) {
 int r = 1;
  while (y > 1) {
    if (y % 2 == 1) {
      r = x * r;
    }
    x = x * x;
    y = y / 2;
  }
 return r * x;
}
```

Preconditions in C0

- *We need to impose a precondition on f*
  - *to tell the caller that y should be ≥ 0*
- In C0 we can write an **executable contract directive**

  //@requires y >= 0;

  C0 keyword to specify a precondition
  • written between the function header and the body
  • before the first "{"

  - We check contracts by invoking coin with the **-d** flag
    - "dynamic checking"
      - but everybody understands it as *debug mode*
  - without the **-d** flag, contracts are treated as comments

```
int f(int x, int y)
//@requires y >= 0;
{
 int r = 1;
  while (y > 1) {
    if (y % 2 == 1) {
      r = x * r;
    }
    x = x * x;
    y = y / 2;
  }
 return r * x;
}
```

Using Contract

Running with contracts disabled

Linux Terminal

```
# coin mystery.c0
C0 interpreter (coin) …
--> f(2, 3);
8 (int)
--> f(2, -1);
2 (int)
-->
```

Contracts are treated as comments

Running with contracts enabled

Linux Terminal

```
# coin -d mystery.c0
C0 interpreter (coin) …
--> f(2, 3);
8 (int)
--> f(2, -1);
mystery.c0:2.4-2.20: @requires annotation failed

Last position: mystery.c0:2.4-2.20
           f from <stdio>:1.1-1.9
-->
```

Line number where contract failed

File where contract failed

Contracts are executed
- if **true**, execution proceeds normally
- if **false**, execution aborts

cc0, the C0 compiler, works the same way

Safety

- If we call f(x,y) with a negative y
  - with **-d**, execution aborts
  - without **-d**, f can return an arbitrary result
    - there is **no** right value it could return

- Calling a function with inputs that cause a precondition to fail is **unsafe**
  - execution will never do the right thing
    - either abort
    - or compute a wrong result

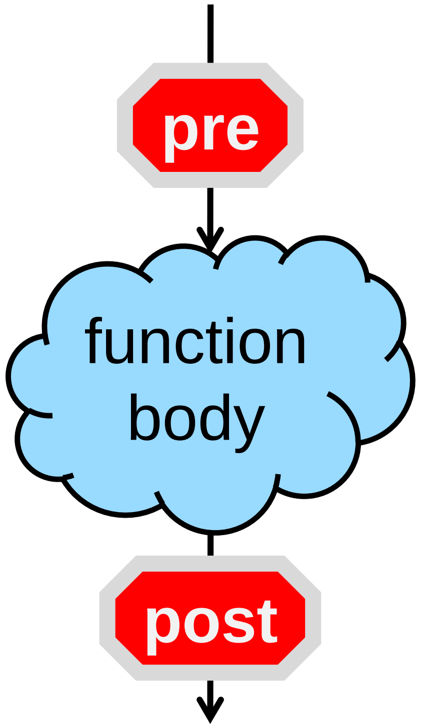
- The caller must make sure that the call is **safe**
    - that $y \geq 0$

Postconditions

Contracts about Function Outcomes

- Preconditions are checked *before* the function starts executing

- A contract that is checked *after* it is done executing could tell us if the function did the right thing
  - ➢ check that the output is what we expect
  - ○ This is a **postcondition**



Postconditions in C0

- In C0, the contract directive

  //@ensures *<some_condition>* ;

  > C0 keyword to specify a postcondition
  > - written between the function header and the body
  >   - after the preconditions (by convention)
  > - before the first "{"

  allows us to write a postcondition

  - <some_condition> can mention the contract-only variable \result
    - what the function returns
    - can only be used with //@ensures

```
int f(int x, int y)
//@requires y >= 0;
//@ensures …;
{
  int r = 1;
  while (y > 1) {
    if (y % 2 == 1) {
      r = x * r;
    }
    x = x * x;
    y = y / 2;
  }
  return r * x;
}
```

Writing a Postcondition

- The postcondition we want to write is

```
//@ensures \result == x**y;
```

That's how we write x^y in Python

  - but x**y is not defined in C0
    - C0 has no primitive power function!

- What do we do?
  - transcribe the mathematical definition into a C0 function

$$\begin{cases} x^0 = 1 & \\ x^y = x^{y-1} * x & \text{if } y > 0 \end{cases}$$

```
int POW(int x, int y)
//@requires y >= 0;
{
  if (y == 0) return 1;
  return POW(x, y-1) * x;
}
```

Writing a Postcondition

- Then our postcondition is

  //@ensures \result == POW(x, y);

  right? … almost

```
Linux Terminal
# coin -d mystery.c0
mystery.c0:18.5-18.6:error:cannot assign to
   variable 'x' used in @ensures annotation
    x = x * x;
    ~
Unable to load files, exiting...
```

  - The function modifies x (and y)
    - Which values of x and y should C0 evaluate the postcondition with?
      - We want the initial values, but it is checked when returning …
  - To avoid confusion, C0 disallows modified variables in postconditions

```
int POW(int x, int y)
//@requires y >= 0;
{
  if (y == 0) return 1;
  return POW(x, y-1) * x;
}

int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
 int r = 1;
  while (y > 1) {
    if (y % 2 == 1) {
      r = x * r;
    }
    x = x * x;
    y = y / 2;
  }
 return r * y;
}
```

Writing a Postcondition

- *C0 disallows modified variables in postconditions*
  - Make copies x and y and modify those

Linux Terminal

```
# coin -d mystery.c0
C0 interpreter (coin) …
--> f(2, 3);
8 (int)
--> f(2, 0);
mystery.c0:11.4-11.33: @ensures annotation failed
Last position: mystery.c0:11.4-11.33
           f from <stdio>:1.1-1.8
```

Line number where contract failed

  - We're good

```
int POW(int x, int y)
//@requires y >= 0;
{
  if (y == 0) return 1;
  return POW(x, y-1) * x;
}

int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
 int b = x;
 int e = y;
 int r = 1;
  while (e > 1) {
   if (e % 2 == 1) {
     r = b * r;
   }
   b = b * b;
   e = e / 2;
  }
 return r * b;
}
```

Recall Safety

- In the postcondition of f, we are making a call to POW
  - **Is it safe?**
    - We need to show that y >= 0
    - ➢ The precondition tells us that y >= 0 ✓
- The body of POW makes a call to POW
  - **Is it safe?**
    - We need to show that y-1 >= 0
    - i.e., y >= 1
    - ➢ The precondition tells us that y >= 0
    - ➢ Since we don't return on the if, y ≠ 0 ✓
    - ➢ So y >= 1 by math

This should always be on our mind

- These are examples of **point-to reasoning**
  - We justify something by pointing to lines of code that supports it

```
int POW(int x, int y)
//@requires y >= 0;
{
  if (y == 0) return 1;
  return POW(x, y-1) * x;
}

int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

The Power Function

- But wait!
  - f was meant to implement the power function
  - … but POW **is** the power function!
- Let’s use it!
  - There may be benefits to fixing f instead
    - it may be more efficient than POW
  - Keep reading …

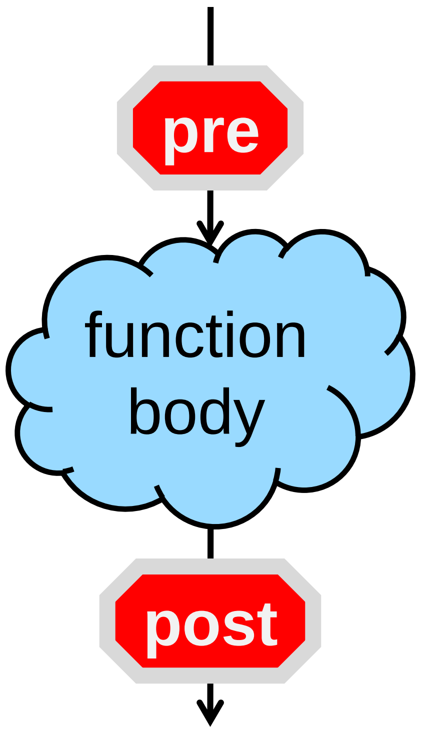
```
int POW(int x, int y)
//@requires y >= 0;
{
  if (y == 0) return 1;
  return POW(x, y-1) * x;
}

int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

Correctness

- If a call violates a function's postconditions
  (assuming its preconditions were met so it actually ran)

  the function is doing something wrong
  - the function has a **bug**



- The function is **incorrect**
  - Our mystery function f is incorrect

- When writing a function, we must make sure that it is **correct**
  - i.e., that its postconditions will be satisfied for any safe input

Blame

- If a function preconditions fail, it’s the caller’s fault
  - the caller passed invalid inputs
  - the call is **unsafe**

- If its postconditions fail, it’s the implementation’s fault
  - the function code does the wrong thing
  - the function is **incorrect**

We will develop methods to make sure that the code we write is **safe** and **correct**

How to Use Contracts

- Contract-checking helps us write code that works as expected
  - Use **-d** while writing our code
  - At this stage, this is **development code**
    - bugs are likely
- Once we are confident our code works, compile it without **-d**
  - The code can be used in its intended application
  - At this stage, this is **production code**
    - there should be no bugs
- Why not use **-d** always?
  - it slows down execution

Specification Functions

- POW is used only in contracts
  - It is not executed when contract-checking is disabled
    - without **-d**
- Functions used only in contracts are called **specification functions**
  - They help us state what the code should do
  - They are critical to writing good code

```
int POW(int x, int y)
//@requires y >= 0;
{
  if (y == 0) return 1;
  return POW(x, y-1) * x;
}

int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

Function Contracts

Where are we?

- We have learned a lot about f
  - the preconditions describe what valid inputs are
  - the postconditions describe what it is supposed to do
    - on valid inputs
- We have a fully documented function
- We have not looked at all at its body
    - but we know there is a bug in there
    - it is incorrect

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

The Caller’s Perspective

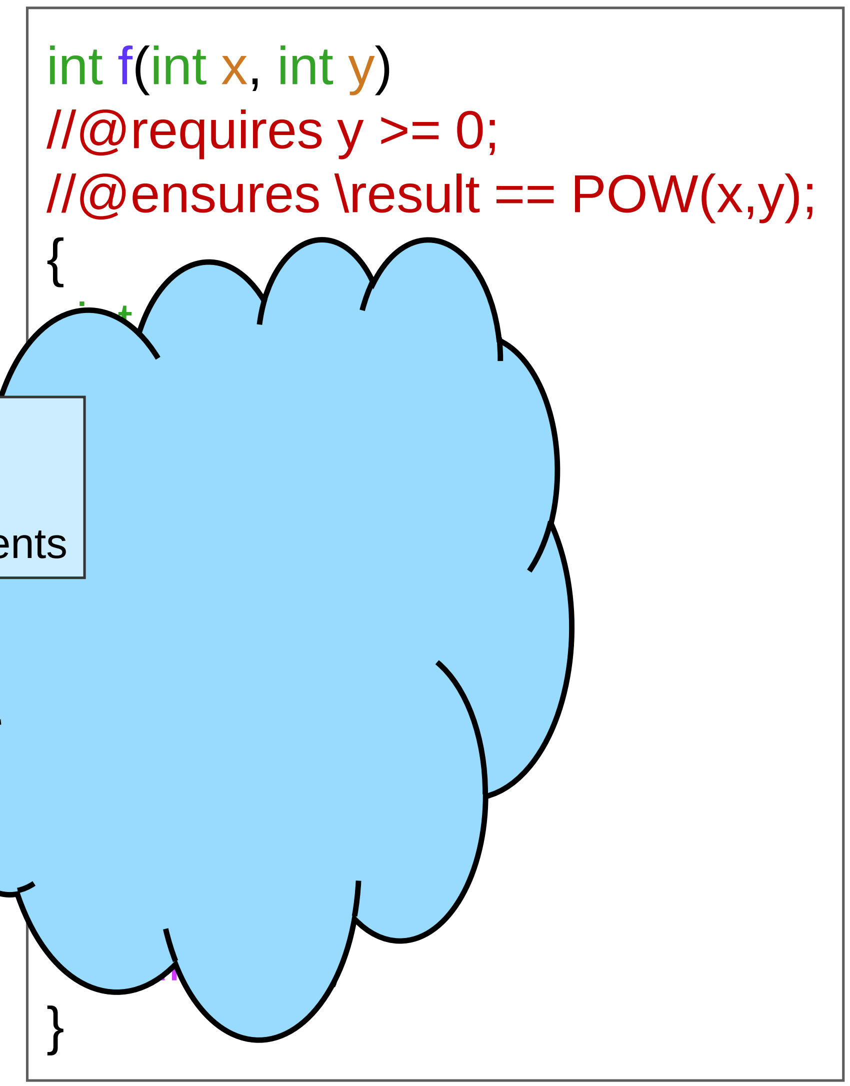
Preconditions describe valid inputs
Postconditions describe what it does

- That’s what the **caller** needs to know to use the function

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
```

Header:
- function name
- number and type of its arguments

Contracts:
- pre- and post-conditions



- The caller should be able to use it without knowing anything about how it is implemented
    - The implementation details are **abstracted away**

Abstraction

- Split a complex system into **small** chunks that can be understood **independently**

Bother with as few details as possible at any time

- Computer science is all about abstraction

The Function’s Perspective

Preconditions describe valid inputs
Postconditions describe what it does

- That’s what the implementation is to do
  - guidelines to write the body of the function
- How to write good code
  - **First write the contracts**
  - and then the body
    - in this way, you always know what you are aiming for

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

Now, we need to look at the body of f to find the bug

Loop Invariants

Diving In

- We need to look at the body of f
  - The complicated part is the **loop**
    - the values of the variables change at each iteration
    - it's unclear how many iterations there are
  - If we understand the loop, we understand the function
- How to go about that?

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

Abstraction

- *If we understand the loop,*
  we understand the function

- How to go about that?
  - Contracts summarize what a function does so we don't need to bother with the details of its implementation
    - An abstraction over functions
  - Come up with a summary of the loop so we don't need to bother with the details of its implementation
    - **An abstraction over loops!**

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {

  }
  return r * b;
}
```

Loop Invariants

The values of the variables change at each iteration

- One valuable abstraction is what does **not** change
  - This is called a **loop invariant**
    - a quantity that remains constant at each iteration of the loop
      - a quantity may be an expression, not just a variable

We will see what makes some loop invariants **really valuable** shortly

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

Tracing Code

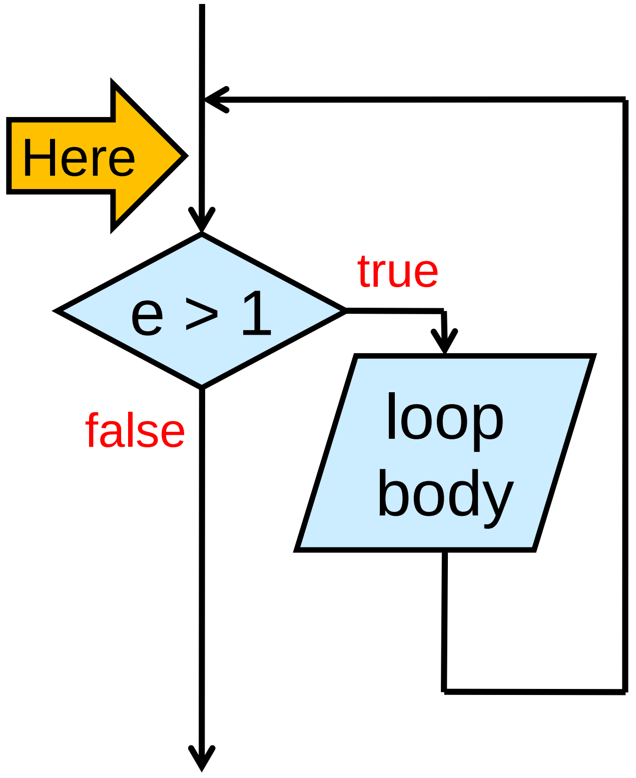
- How to find a **loop invariant**?
  - *a quantity that remains constant at each iteration of the loop*
- Run the function on sample inputs
- Track the value of the variables that change
  - b, e, r
    - no need to bother with x and y since they don't change
  - just before the loop guard is tested
    - That's e > 1

This is called **tracing** an execution

- Look for patterns

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

Loop guard



Tracing Code

- Run the function on sample inputs and track the value of the variables
  - Let's try with f(2,8)

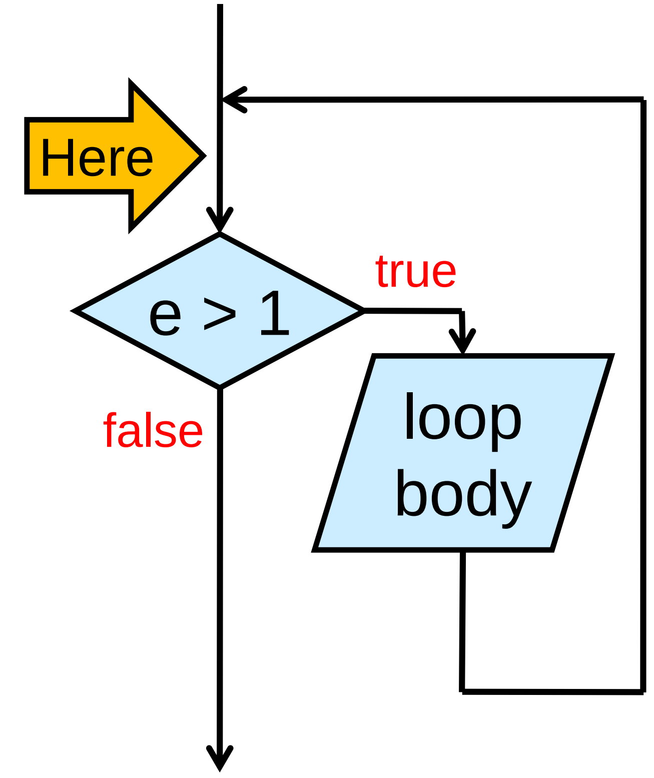
b	e	r
2	8	1
4	4	1
16	2	1
256	1	1

At this point we exit the loop

  - Can we spot a quantity that doesn't change?

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

This checks if e is odd



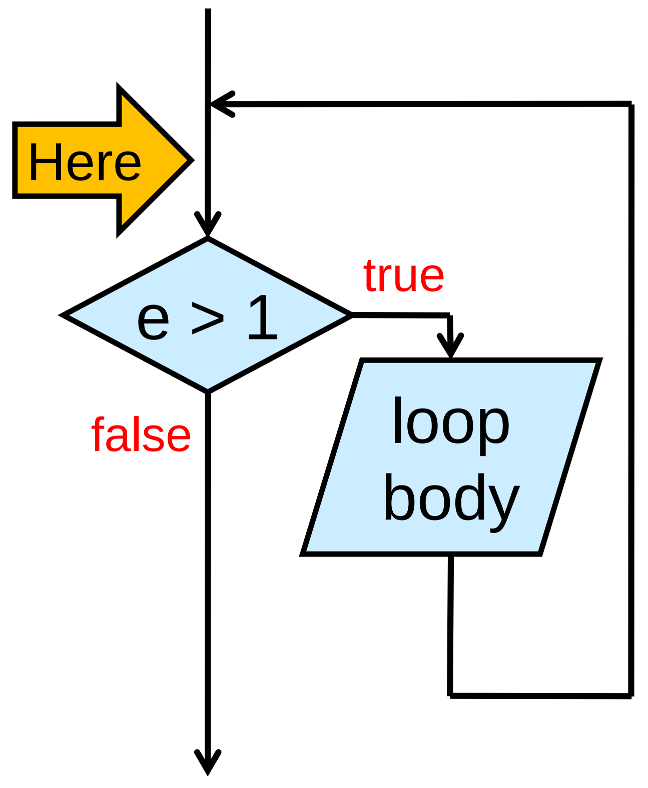
Tracing Code

- Trying with f(2,8)
  - *Can we spot a quantity that doesn't change?*
  - $\mathbf{b^e}$ is always 256

b	e	r	$\mathbf{b^e}$
2	8	1	**256**
4	4	1	**256**
16	2	1	**256**
256	1	1	**256**

  - This is a **candidate loop invariant**
    - $\mathbf{b^e}$ is constant on one set of inputs
    - a loop invariant must stay constant on all inputs

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```



Tracing Code

- b^e is a *candidate* loop invariant
- Let's try with f(2,7)

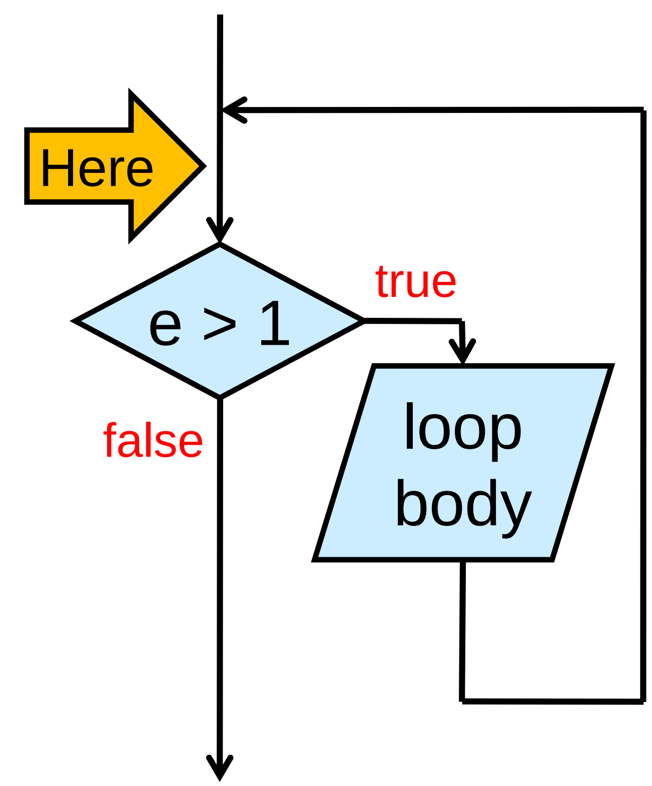
b	e	r	b^e
2	7	1	**128**
4	3	2	**64**
16	1	8	**16**

Not constant on these inputs

  - b^e is **not** invariant on these inputs!
    - It was a candidate that didn't pan out

- Can we spot another quantity that doesn't change?

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```



Tracing Code

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

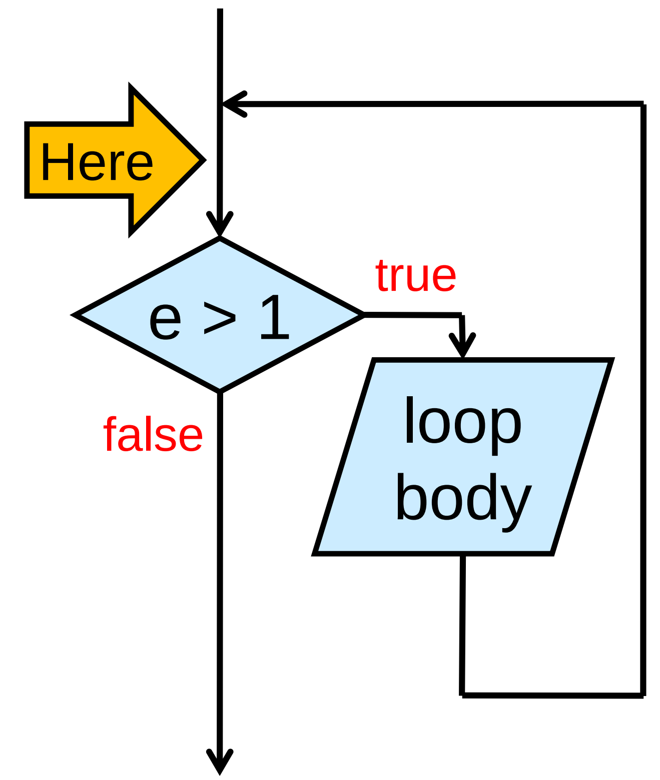
- *Trying with f(2,7)*
  - *Can we spot a quantity that doesn't change?*
  - $b^e * r$ is always 128

b	e	r	b^e	$b^e * r$
2	7	1	128	**128**
4	3	2	64	**128**
16	1	8	16	**128**

- This is another candidate loop invariant
  - Let's test it on f(3,5)

b	e	r	$b^e * r$
3	5	1	**243**
9	2	3	**243**
81	1	3	**243**

  - This seems to work



A Candidate Loop Invariant

- $b^e * r$ is a promising candidate loop invariant
  - It works on *three* inputs!
- How do we know it works in general?
  - We can’t test it on all inputs
  - We need to provide a **proof**
- *But first, let’s add it to our code*

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

Loop Invariants in C0

- In C0, we use the directive

  //@loop_invariant

  to specify a loop invariant

C0 keyword to specify a loop invariant
• written between the loop guard and the loop body

- Then, simply write

  //@loop_invariant POW(b, e) * r;

  - … this won't work
    - C0 would need to keep track of the values of this expression across all iterations of the loop
    - also, what if the loop runs 0 times?

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1)
  //@loop_invariant  … ;
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

- In C0, loop invariants must be **boolean expressions**
  - **true** means it was satisfied in the current iteration
  - **false** means it wasn't

Loop Invariants in C0

- They are boolean expressions
  - **true** means satisfied
- What can we use?

b	e	r	**$b^e * r$**
2	7	1	**128**
4	3	2	**128**
16	1	8	**128**

  - As we enter the loop, b is x and e is y
    - so $\mathbf{x^y}$ is 128 too
    - thus, $\mathbf{b^e * r} = \mathbf{x^y}$
- Then, we can write

  //@loop_invariant POW(b, e) * r == POW(x, y);

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1)
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

Execution will abort when ran with **-d** if LI is ever **false**

Safety

```
1.  int f(int x, int y)
2.  //@requires y >= 0;
3.  //@ensures \result == POW(x,y);
4.  {
5.    int b = x;
6.    int e = y;
7.    int r = 1;
8.    while (e > 1)
9.    //@loop_invariant POW(b,e) * r == POW(x,y);
10.   {
11.     if (e % 2 == 1) {
12.       r = b * r;
13.     }
14.     b = b * b;
15.     e = e / 2;
16.   }
17.   return r * b;
18. }
```

?

We have two new calls to POW

- Are they safe?

- **POW(x, y)**
  - **To show:** y >= 0
  - y >= 0 by line 2 (precondition of f) ✓

- **POW(b, e)**
  - **To show:** e >= 0
  - “e is *initially* equal to y which is >= 0 and it is halved at *each* iteration of the loop so e is *always* >= 0”
  - This is an example of **operational reasoning**
    - The justification relies on what is happening in all the iterations of the loop
      - This is error-prone
    - We will disallow safety proofs based on operational reasoning on loops ✗

Safety

```
1.  int f(int x, int y)
2.  //@requires y >= 0;
3.  //@ensures \result == POW(x,y);
4.  {
5.    int b = x;
6.    int e = y;
7.    int r = 1;
8.    while (e > 1)
9.    //@loop_invariant e >= 0;
10.   //@loop_invariant POW(b,e) * r == POW(x,y);
11.   {
12.     if (e % 2 == 1) {
13.       r = b * r;
14.     }
15.     b = b * b;
16.     e = e / 2;
17.   }
18.   return r * b;
19. }
```

POW(b, e)

- ➢ **To show:** e >= 0
- ○ We can sort of do it with operational reasoning
  - ➢ error prone!
- ○ but we really want to prove it using point-to reasoning

- We do believe that e >= 0 at every iteration of the loop
  - ○ Turn it into a candidate loop invariant!

    //@loop_invariant e >= 0;

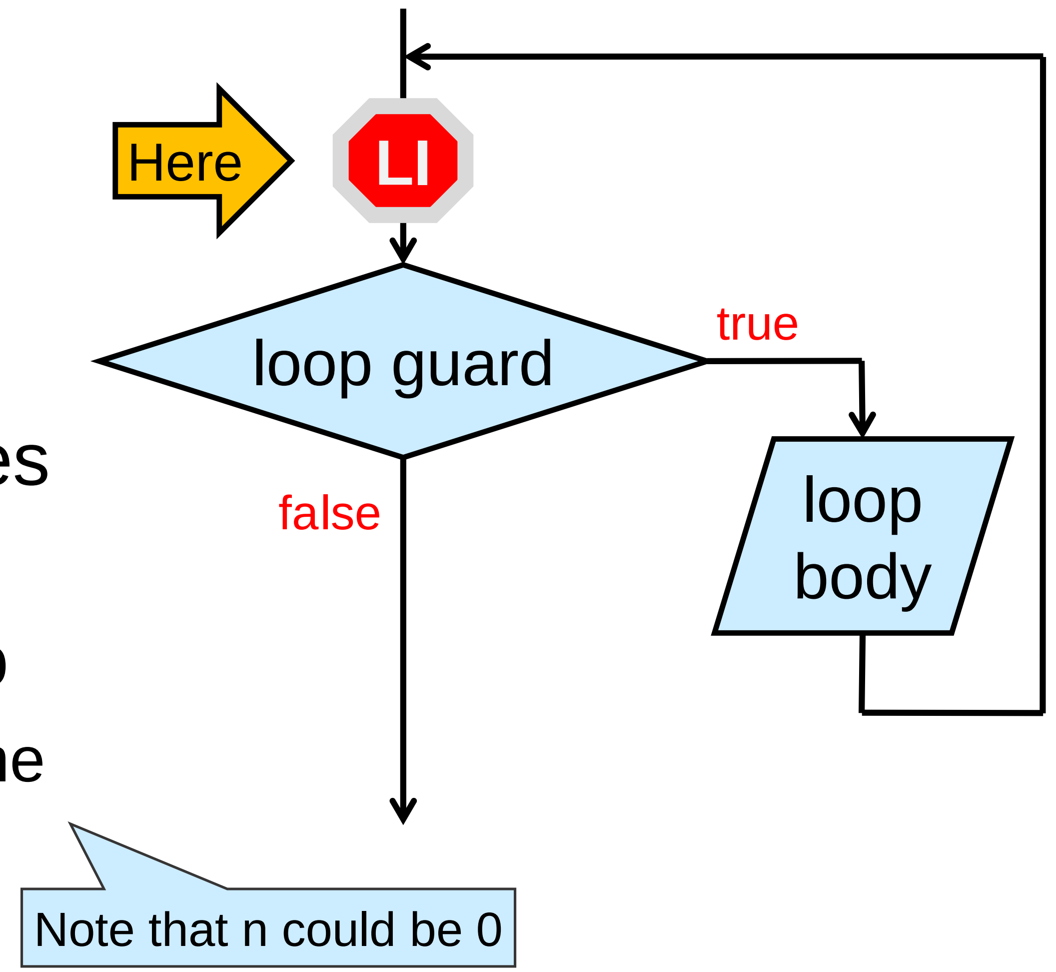
    - ➢ We will need to prove later that it is valid
  - ○ Then we prove that POW(b, e) is safe by pointing to line 9 ✓

> An operational hunch is often a good candidate loop invariant

How Loop Invariants Work

- Loop invariants are checked **just before** the loop guard is tested
- If the loop body runs n times,
  - the loop invariant is checked n+1 times
    - must be **true** all n+1 times
  - the loop guard is tested n+1 times too
    - **true** the first n times and **false** the last time
- When we exit the loop
  - the loop invariant is **true**
  - the loop guard **false**

Important!



Validating Loop Invariants

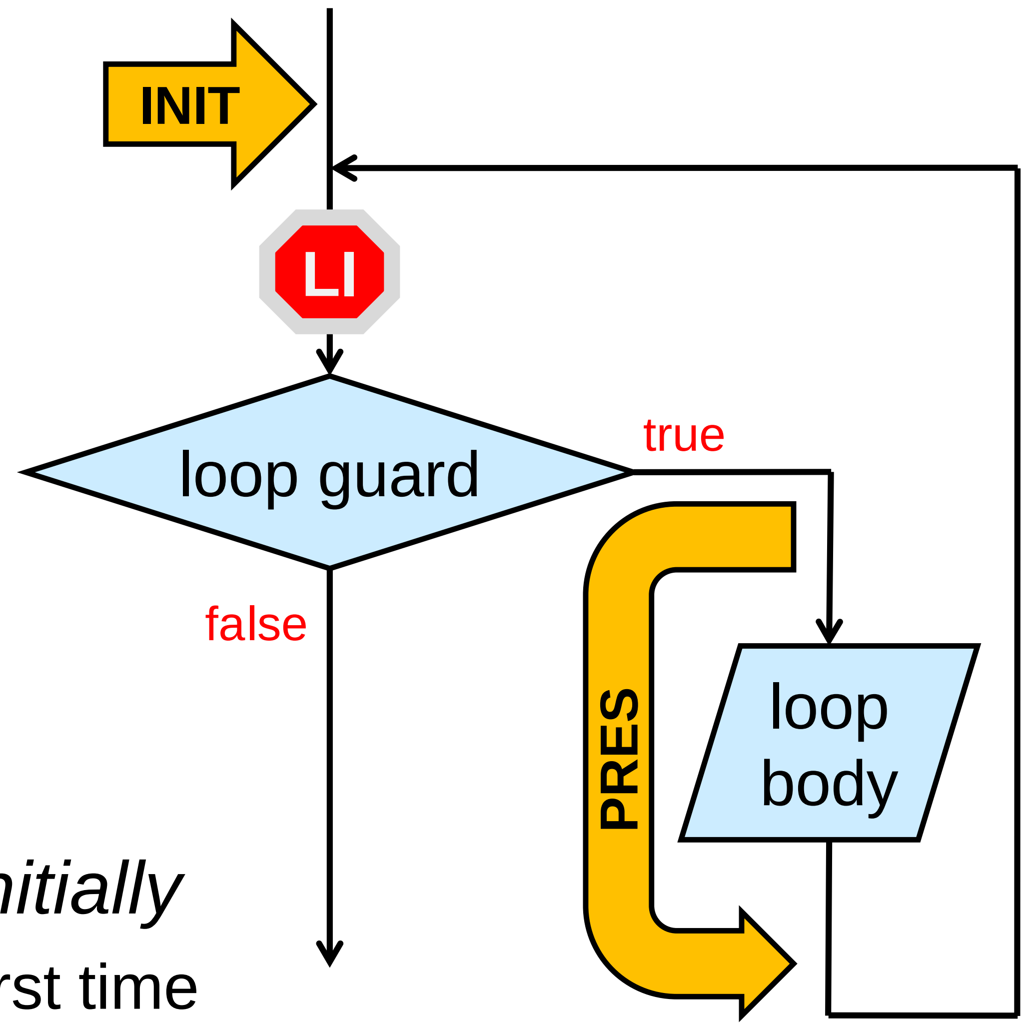
Where are we?

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

- We have learned even more about f
  - The contracts tell us what it is meant to do
  - The loop invariants give us useful information about how the loop works
    - but these are **candidate** loop invariants
    - we need to prove that they are valid
- We have started learning about proving things about code
    - just safety so far
  - point-to reasoning: good
  - operational reasoning: error prone

Proving a Loop Invariant Valid

- We cannot show a loop invariant is valid by running it on all possible inputs
  - We need to supply a proof
    - using point-to reasoning



- Two steps

  INIT: show that the loop invariant is true *initially*
    - just before we test the loop guard the very first time

  PRES: show that the loop invariant is **preserved** by the loop
    - if it is true at the beginning of an **arbitrary iteration** of the loop,
    - then it is also true at the end of this iteration

But it may become false temporarily in the middle of the loop body

Validity of $e \geq 0$

We use math notation for brevity

```
1. int f(int x, int y)
2. //@requires y >= 0;
3. //@ensures \result == POW(x,y);
4. {
5.   int b = x;
6.   int e = y;
7.   int r = 1;
8.   while (e > 1)
9.   //@loop_invariant e >= 0;
10.  //@loop_invariant POW(b,e) * r == POW(x,y);
11.  {
12.    if (e % 2 == 1) {
13.      r = b * r;
14.    }
15.    b = b * b;
16.    e = e / 2;
17.  }
18.  return r * b;
19. }
```

INIT:

- **To show:** $e \geq 0$ initially

A. $y \geq 0$ by line 2

B. $e = y$ by line 6

C. $e \geq 0$ by math on A and B ✓

This is a typical proof format in this course

PRES:

LI at **start** of current iteration

LI at **end** of current iteration

- **To show:** if $e \geq 0$, then $e \geq 0$

But isn't this trivially true?

- The value of e changes in the body of the loop
- We need a way to distinguish the value at the start and end of the current iteration
  - e ⇦ value of e at the **start** of the current iteration
  - e' ⇦ value of e at the **end** of the current iteration

Validity of e ≥ 0

INIT: e ≥ 0 initially ✓

LI at **start** of current iteration

LI at **end** of current iteration

PRES:

- **To show:** if e ≥ 0, then e' ≥ 0

A. e' = e/2 by line 16
B. e ≥ 0 by assumption
C. e/2 ≥ 0 by math on B
D. e' ≥ 0 by A and C ✓

Both INIT and PRES were proved by point-to reasoning

```
1.  int f(int x, int y)
2.  //@requires y >= 0;
3.  //@ensures \result == POW(x,y);
4.  {
5.    int b = x;
6.    int e = y;
7.    int r = 1;
8.    while (e > 1)
9.    //@loop_invariant e >= 0;
10.   //@loop_invariant POW(b,e) * r == POW(x,y);
11.   {
12.     if (e % 2 == 1) {
13.       r = b * r;
14.     }
15.     b = b * b;
16.     e = e / 2;
17.   }
18.   return r * b;
19. }
```

Validity of $b^e\ r = x^y$

```
1. int f(int x, int y)
2. //@requires y >= 0;
3. //@ensures \result == POW(x,y);
4. {
5.   int b = x;
6.   int e = y;
7.   int r = 1;
8.   while (e > 1)
9.   //@loop_invariant e >= 0;
10.  //@loop_invariant POW(b,e) * r == POW(x,y);
11.  {
12.    if (e % 2 == 1) {
13.      r = b * r;
14.    }
15.    b = b * b;
16.    e = e / 2;
17.  }
18.  return r * b;
19. }
```

INIT:

- **To show:** $b^e\ r = x^y$ initially

A. $b = x$ by line 5

B. $e = y$ by line 6

C. $r = 1$ by line 7

D. $b^e\ r = x^y$ by math on A, B, C ✓

PRES:

LI at **start** of current iteration

LI at **end** of current iteration

x and y don't change in the loop

- **To show:** if $b^e\ r = x^y$, then $b'^{e'}\ r' = x^y$

○ We need to distinguish 2 cases based on the test e % 2 == 1

- **e % 2 == 1** is **true** — *e is odd*
- **e % 2 == 1** is **false** — *e is even*

Validity of $b^e\ r = x^y$

```
1.  int f(int x, int y)
2.  //@requires y >= 0;
3.  //@ensures \result == POW(x,y);
4.  {
5.    int b = x;
6.    int e = y;
7.    int r = 1;
8.    while (e > 1)
9.    //@loop_invariant e >= 0;
10.   //@loop_invariant POW(b,e) * r == POW(x,y);
11.   {
12.     if (e % 2 == 1) {
13.       r = b * r;
14.     }
15.     b = b * b;
16.     e = e / 2;
17.   }
18.   return r * b;
19. }
```

PRES:

- **To show:** if $b^e\ r = x^y$, then $b'^{e'}\ r' = x^y$
- Case **e is odd** (e % 2 == 1)
  - Then e = 2n+1 for some n

A.	$b' = b*b$	by line 15
B.	$e' = e/2$	by line 16
C.	$= n$	by case assumption and math
D.	$r' = b * r$	by line 13
E.	$b'^{e'}\ r' = (b*b)^n\ b*r$	by A, B, C, D
F.	$= b(b^2)^n\ \ r$	by math
G.	$= b^{2n+1}\ r$	by math
H.	$= b^e\ r$	by case assumption
I.	$= x^y$	by assumption

This is one of the most complex proofs in this course

- This proves the first case

Validity of $b^e r = x^y$

```
1. int f(int x, int y)
2. //@requires y >= 0;
3. //@ensures \result == POW(x,y);
4. {
5.   int b = x;
6.   int e = y;
7.   int r = 1;
8.   while (e > 1)
9.   //@loop_invariant e >= 0;
10.  //@loop_invariant POW(b,e) * r == POW(x,y);
11.  {
12.    if (e % 2 == 1) {
13.      r = b * r;
14.    }
15.    b = b * b;
16.    e = e / 2;
17.  }
18.  return r * b;
19. }
```

PRES:

- **To show:** if $b^e r = x^y$, then $b'^{e'} r' = x^y$
- Case **e is even** (e % 2 == 0)
  - Then e = 2n for some n

A.	$b' = b*b$	by line 15
B.	$e' = e/2$	by line 16
C.	$= n$	by case assumption and math
D.	$r' = r$	since r is unchanged
E.	$b'^{e'} r' = (b*b)^n r$	by A, B, C, D
F.	$= (b^2)^n r$	by math
G.	$= b^{2n} r$	by math
H.	$= b^e r$	by case assumption
I.	$= x^y$	by assumption

- This proves the second case too

PRES holds for $b^e r = x^y$ ✓

Loop Invariants

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

- $e \geq 0$ is valid ✓
  - it holds **INIT**ially
  - it is **PRES**erved by an arbitrary iteration of the loop
    - if $e \geq 0$, then $e' \geq 0$
- $b^e\ r = x^y$ is valid ✓
  - it holds **INIT**ially
  - it is **PRES**erved by an arbitrary iteration of the loop
    - if $b^e\ r = x^y$, then $b'^{e'}\ r' = x^y$
- This shows that both are **genuine loop invariants**
  - not just candidates
  - we can forget about the body of the loop when reasoning about this function

Proof-directed Debugging

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {

  }
  return r * b;
}
```

Where are we?

- The contracts tell us what the function is *meant to* do
  - but we know there is a bug in there
- The loop invariants abstract away the details of the loop

But what to do with them is still a bit mysterious

- *Let's find the bug!*

After the Loop

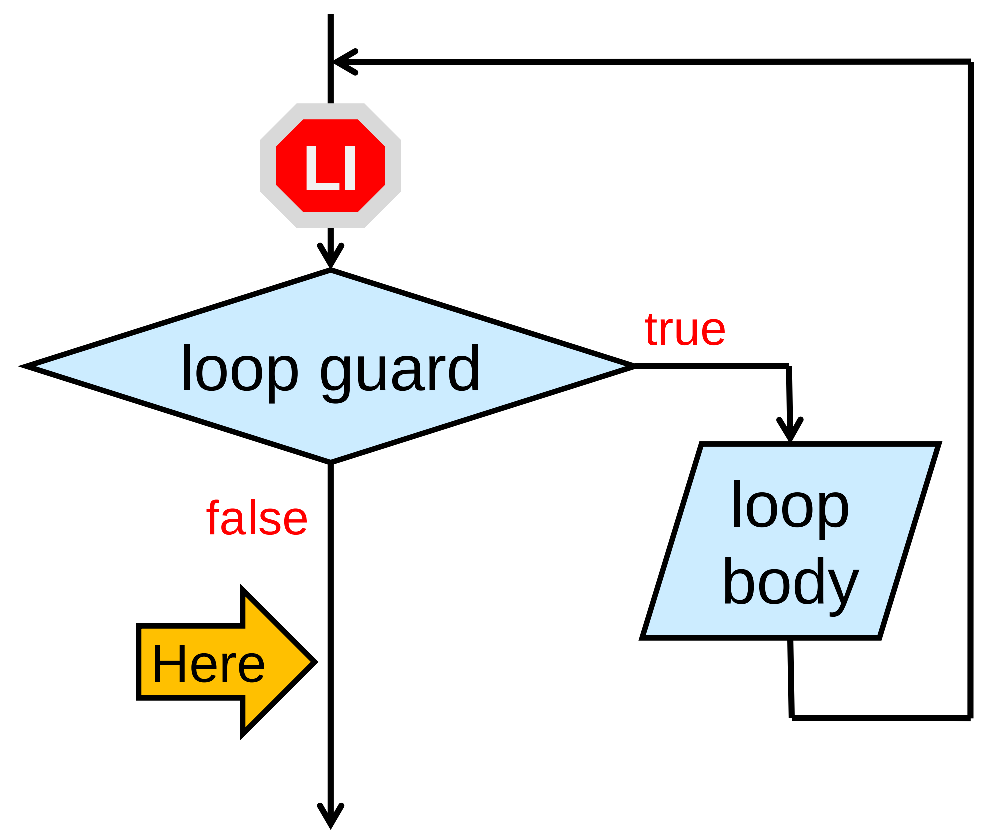
- What do we know when execution exits the loop?
  - the loop guard is **false**
    - $e \leq 1$
  - the loop invariants are **true**
    - $e \geq 0$
    - $b^e\ r = x^y$
- Knowing this will
  - enable us to prove correctness
  - or expose a bug

Since f is incorrect, this should happen

Here

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {

  }
  return r * b;
}
```



After the Loop

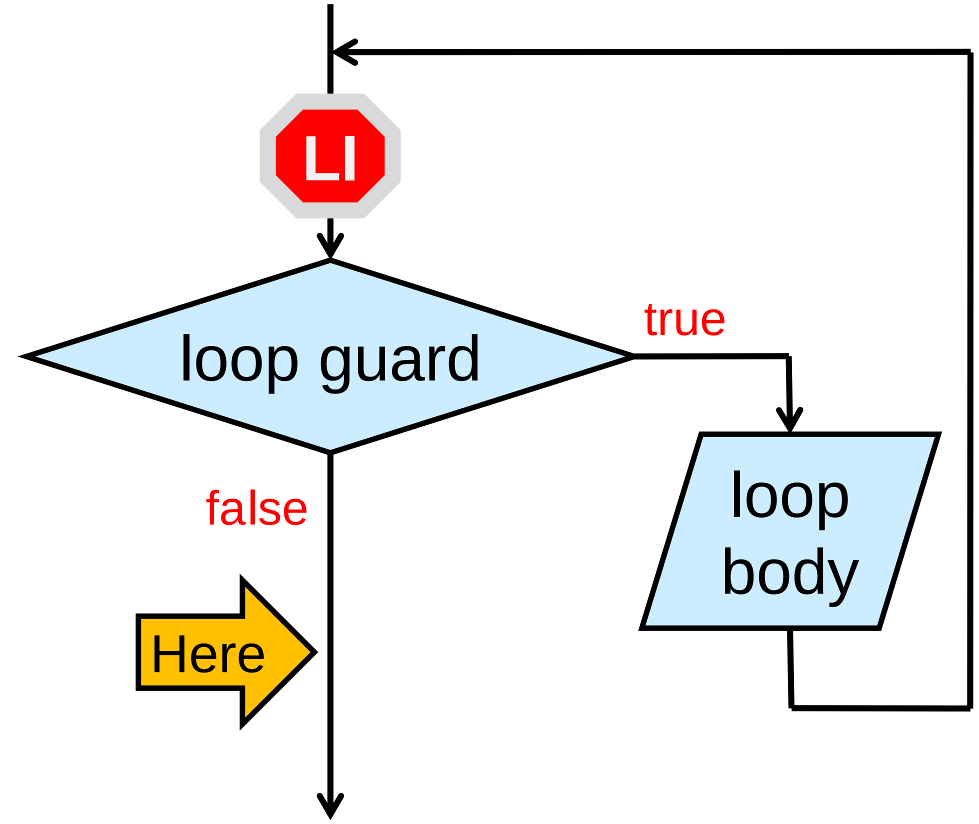
- What do we know when execution exits the loop?
  - the loop guard is **false**
    - $e \leq 1$
  - the loop invariants are **true**
    - $e \geq 0$
    - $b^e\ r = x^y$
- From $e \leq 1$ and $e \geq 0$, we have that
  - either $e = 0$
  - or $e = 1$

  as we exit the loop

Recall that e has type int

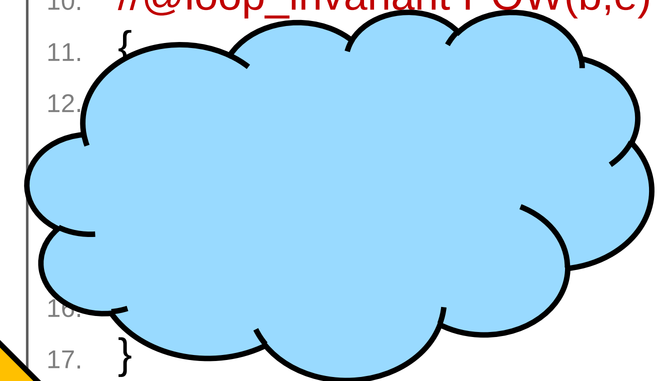
```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
  }
  return r * b;
}
```

Here



After the Loop

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
```



```
  }
  return r * b;
}
```

- Either $e = 0$ or $e = 1$
  - Let's plug these values in the other loop invariant, $b^e\ r = x^y$

➔ If $e = 1$, then $x^y = b^e\ r = b^1\ r = r\ b$
  - Thus, $x^y = r\ b$ in this case

This is exactly what f returns. ✓

➔ if $e = 0$, then $x^y = b^e\ r = b^0\ r = r$
  - Thus, $x^y = r$ in this case
    - $x^y \neq r\ b$

This is **not** what f returns.

This is the bug! ✗

Tracking the Bug

- The bug is when e = 0 as we exit the loop
- This can happen **only** if f is called with 0 as y
  - if e = 1, the loop doesn't run and e stays 1
  - if e > 1 at the start of an iteration, then e' ≥ 1 as we end it

Here

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

Fixing the Bug

Idea #1: return 1 if y = 0

- This works but it introduces a **special case** in the code
- Special cases leads to contrived, unmaintainable code
  - sometimes unavoidable
  - but let's see if we can do better

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  if (y == 0) return 1;
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

Fixing the Bug

Idea #2: change the precondition
to y > 0

- This forces the caller to have special cases in their code!
  - calls to f need to be **guarded**

```
int c = f(a, b)
```

➡

```
int c = 1;
if (b > 0) c = f(a, b);
```

```
int f(int x, int y)
//@requires y > 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

- This also means that f is not the power function any more
  - undefined when exponent is 0

- Not a great solution ✘

Fixing the Bug

Idea #3: forget about f and use POW instead

- Recall the trace of f(2,8)
  - the loop ran 4 times

b	e	r
2	8	1
4	4	1
16	2	1
256	1	1

- Trace POW(2, 8)
  - 9 recursive calls

x	y
2	8
2	7
2	6
2	5
2	4
2	3
2	2
2	1
2	0

- f is much more efficient

✘

```
int POW(int x, int y)
//@requires y >= 0;
{
  if (y == 0) return 1;
  return POW(x, y-1) * x;
}

int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

Fixing the Bug

Observations: with this body,

- if e == 1, then
  - e/2 == 0
  - r becomes b*r by line 13

Idea #4: make f return only when e = 0

- change the loop guard to e > 0
  - the loop always end with e = 0
- return r instead of r * b
  - that's what we had to return when e = 0

```
1.  int f(int x, int y)
2.  //@requires y >= 0;
3.  //@ensures \result == POW(x,y);
4.  {
5.    int b = x;
6.    int e = y;
7.    int r = 1;
8.    while (e > 0)
9.    //@loop_invariant e >= 0;
10.   //@loop_invariant POW(b,e) * r == POW(x,y);
11.   {
12.     if (e % 2 == 1) {
13.       r = b * r;
14.     }
15.     b = b * b;
16.     e = e / 2;
17.   }
18.   return r;
19. }
```

No special cases!

✓

Rather than getting rid of the bad case (e = 0), we make it the good case and do away with the other case (e = 1)

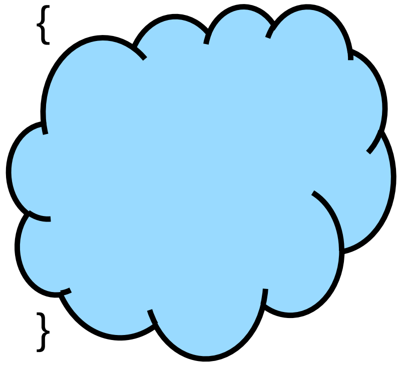
How's this for a movie plot?

Correctness

Did we Really Fix the Bug?

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 0)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {

  }
  return r;
}
```



- The loop invariants are still valid
  - we didn't change the body of the loop
  - we changed the loop guard
    - but we didn't use it in the validity proof

Go back and check

- Right after the loop, we know that
  - the loop guard is **false**: $e \leq 0$
  - the 1st loop invariant is **true**: $e \geq 0$

  so $e = 0$
  - the 2nd loop invariant is **true**: $b^e\, r = x^y$
    - so $x^y = b^e\, r = b^0\, r = r$

This is what f returns now

✓

Assertions

Right after the loop, we know that e = 0

- We can note this with the directive

  //@assert e == 0;

  - checked only when running with **-d**
  - aborts execution if the test is **false**

- //@assert is a great way to note
  - intermediate steps of reasoning
  - expectations about execution

- These are all the run-time directives of C0

  //@requires, //@ensures, //@loop_invariant, //@assert

  There are no others

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 0)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  //@assert e == 0;
  return r;
}
```

//@assert can appear anywhere a statement is expected

Is the Function Correct?

Correctness: for any safe input, the postconditions are true

- We just proved that, as we exit the loop, $r = x^y$
  - ➢ just before return r;
- This tells us that f will **never return the wrong result**

… but will it ***always return the right result***?

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 0)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  //@assert e == 0;
  return r;
}
```

Is the Function Correct?

Correctness: *for any safe input, the postconditions are true*

- Can a function **never return the wrong result** and yet not necessarily ***always return the right result*** ?
  - Let's empty out the loop body in our example

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 0)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  { }
  return r;
}
```

This is legal C0 code

The loop invariants are valid
- **INIT** is unchanged
- **PRES** holds trivially

If execution were to reach return r,
- e == 0 would have to be **true**
- r would have to contain x^y

But it never reaches return r!
So the postcondition will never be **true**

This code is **not** correct.

- … only if it never returns
  - if the loop runs for ever

Termination

- We need to have a reason to believe the loop terminates
  - ➢ it doesn't run for ever

- Here's a proof of termination
  - ○ *as the loop runs,*
    - ➢ *e gets strictly smaller at each iteration*
    - ➢ *it can never become smaller than 0*
    - ➢ *the loop guard is false when e = 0*
  - ○ so the loop must terminate

✓

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 0)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
 //@assert e == 0;
 return r;
}
```

This is an **operational** proof:
we are not pointing to anything

Termination

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 0)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
 //@assert e == 0;
 return r;
}
```

- Operational proof
  - *as the loop runs, e gets strictly smaller, it can never become smaller than 0, and the loop guard is false when e = 0*
  - so the loop must terminate
- Can we prove it using point-to reasoning?
  - Yes! Here's what we need to show
  - in an arbitrary iteration of the loop,
    - if $e \geq 0$, — 0 is a lower bound for e
    - then $e' < e$ — e is strictly decreasing
    - and $e' \geq 0$ — 0 stays a lower bound for e

if e starts >= 0,
it gets strictly smaller and
can never becomes smaller than 0

  - the loop guard is false when e = 0
    - $0 > 0$ is false

Termination

- Point-to-proof
  - **To show:** if e ≥ 0, then e’ < e and e’ ≥ 0

A. e > 0 by line 8 (loop guard)

B. e’ = e/2 by line 16

C. e’ < e by math

D. e’ ≥ 0 by math ✓

```
1.  int f(int x, int y)
2.  //@requires y >= 0;
3.  //@ensures \result == POW(x,y);
4.  {
5.    int b = x;
6.    int e = y;
7.    int r = 1;
8.    while (e > 0)
9.    //@loop_invariant e >= 0;
10.   //@loop_invariant POW(b,e) * r == POW(x,y);
11.   {
12.     if (e % 2 == 1) {
13.       r = b * r;
14.     }
15.     b = b * b;
16.     e = e / 2;
17.   }
18.  //@assert e == 0;
19.  return r;
20. }
```

However,
for termination proofs,
we will generally be Ok with an operational argument

Reasoning about Code

Reasoning about C0

- C0 programs have a precise behavior
  - we can reason about them mathematically

- We used two types of reasoning

  - **Operational reasoning:** drawing conclusions about how things <u>change</u> when certain lines of code are executed

  - **Point-to reasoning:** drawing conclusions about what we <u>know to be true</u> by pointing to specific lines of code that justify them
    - boolean expressions
    - basic mathematical properties
    - variable assignments

This is operational reasoning, but really simple

Operational Reasoning

- Examples
  - Value of variables right after an assignment ✓
  - Things happening in the body of a loop from outside this loop ✗
  - Things happening in the body of a function being called ✗
  - Previously true statement after variables in it have changed ✗

- Operational reasoning is hard to do right consistently
  - very error prone!
  - We want to stay away from anything beyond simple assignments
    - except in termination proofs

If a proof about loops uses words like "always", "never", "each", you are doing operational reasoning

But operational intuitions are a good way to form conjectures that we can then prove using point-to reasoning

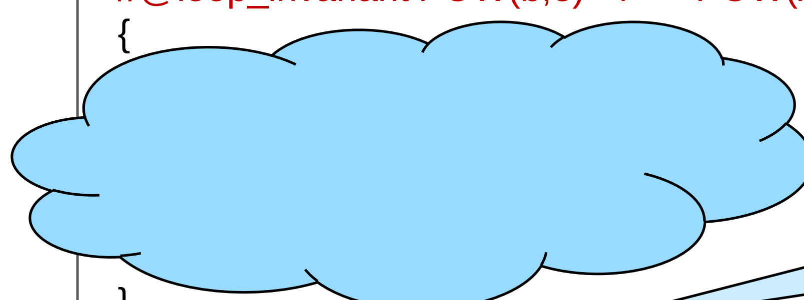
Point-to Reasoning

- Examples
  - Boolean conditions ✓
    - condition of an if statement in the “then” branch ✓
    - negation of the condition of an if statement in the “else” branch ✓
    - loop guard inside the body of a loop ✓
    - negation of the loop guard after the loop ✓
  - Contract annotations ✓
    - preconditions of the current function ✓
    - postconditions of a function just called ✓
    - loop invariant inside the loop body ✓
    - loop invariant after the loop ✓
    - earlier fully justified assertions ✓
  - Math ✓
    - laws of logic ✓
    - some laws of arithmetic ✓
  - Value of variables right after an assignment ✓

Point-to Reasoning: Tips and Tricks

- When reasoning about an earlier loop, **pretend the body of the loop is not there**
  - Only rely on the **loop guard** and **loop invariants**

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 0)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
```



```
  }
  //@assert r = POW(x,y);
  return r;
}
```

When reasoning about an earlier loop, pretend its body is not there

Point-to Reasoning: Tips and Tricks

- When reasoning about a function being called, **pretend the body of the function is not there**
  - ➢ unless it's a specification function
  - ○ Only rely on its **contracts**

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
}

int main() {
  int k = f(2,5);
  //@assert k >= 0;
  return 0;
}
```

When reasoning about the outcome of a function call, pretend its body is not there

Safety

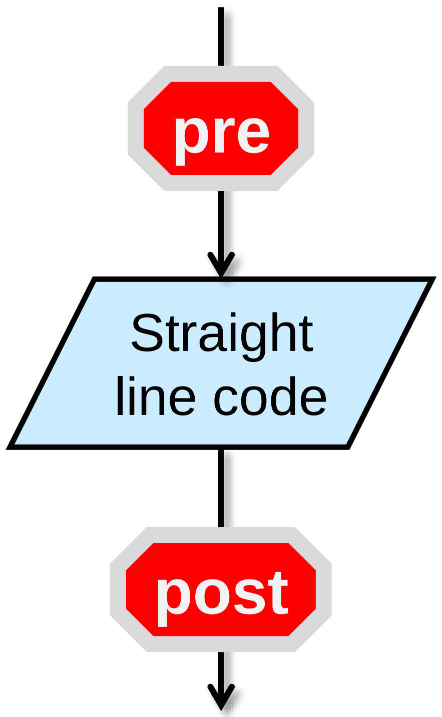
- The inputs of a function call satisfy the function's preconditions
  - we will generalize this definition in the future

We will exclusively use point-to reasoning to justify safety

Correctness

- The postconditions of a function will be true on any call that satisfies the preconditions
  - We will not need to generalize this definition

Straight Line Functions



A non-recursive function without loops

- Proving correctness amounts to combining assignments
  - **To show:** \result = x

A. b = x by line 5

B. r = 1 by line 7

C. \result = r * b by line 8

D. r * b = x by math on A, B, C

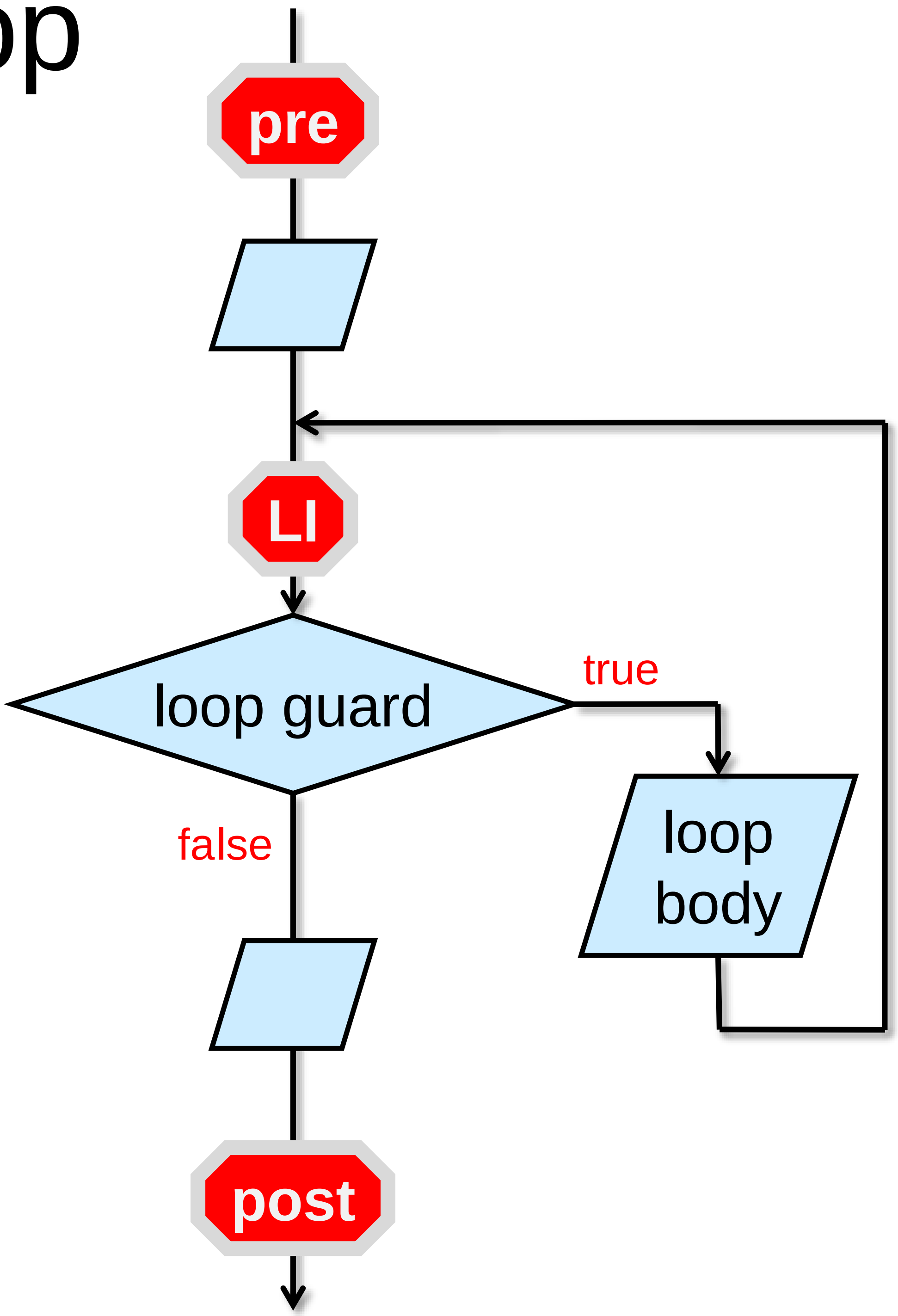
```
1. int f(int x, int y)
2. //@requires y >= 0;
3. //@ensures \result == x;
4. {
5.  int b = x;
6.  int e = y;
7.  int r = 1;
8.  return r * b;
9. }
```

Functions with One Loop

- Proving correctness involves 3 steps
  - Show that the loop invariants are ***valid***
    - **INIT:** the LI are true initially
    - **PRES:** the LI are preserved by an arbitrary iteration of the loop
  - **EXIT:** the LI and the negation of the loop guard imply the postcondition
  - **TERM:** the loop terminates

That's exactly what we did for our mystery function

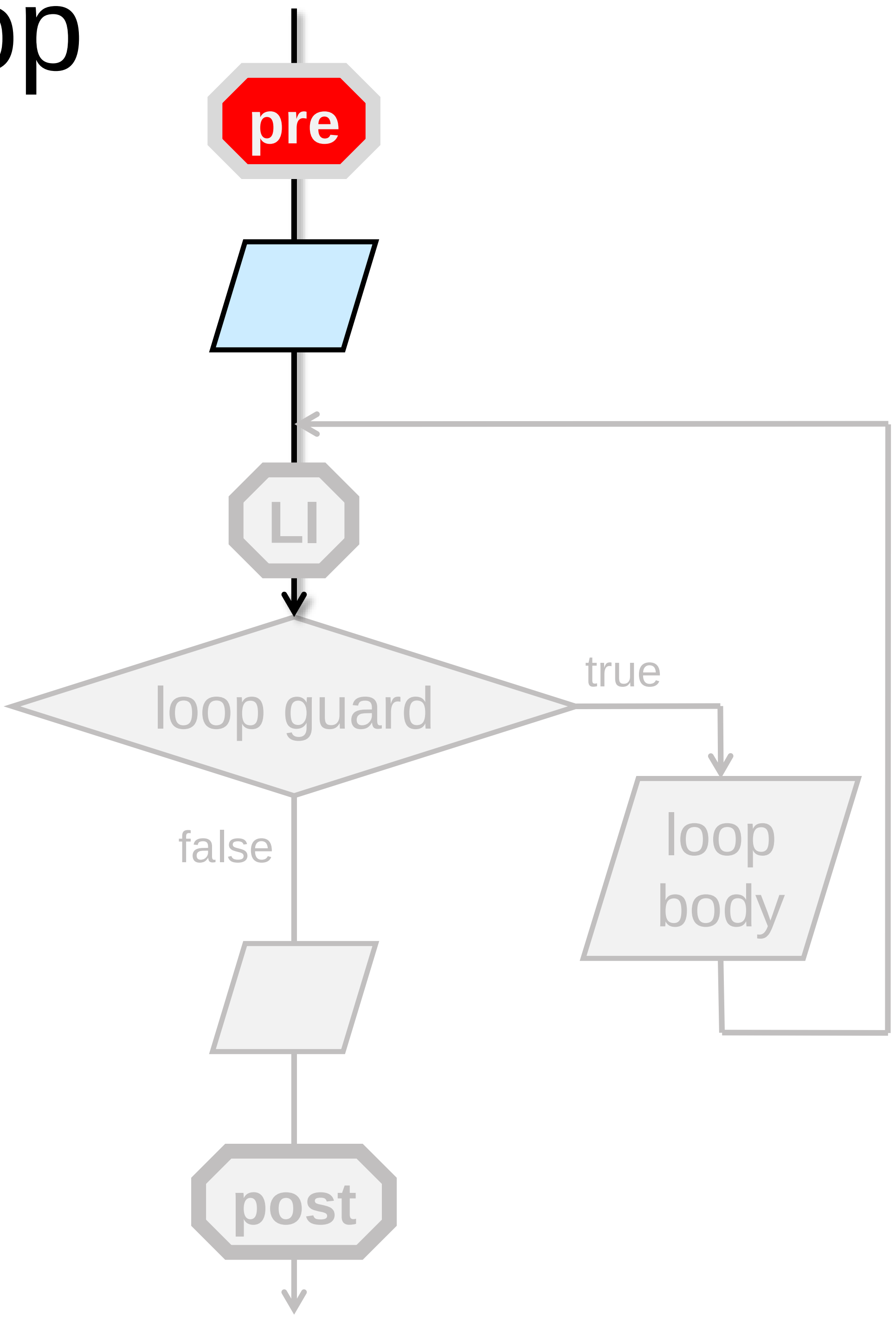
These steps can be proved in any order



Functions with One Loop

INIT: the loop invariant is true initially

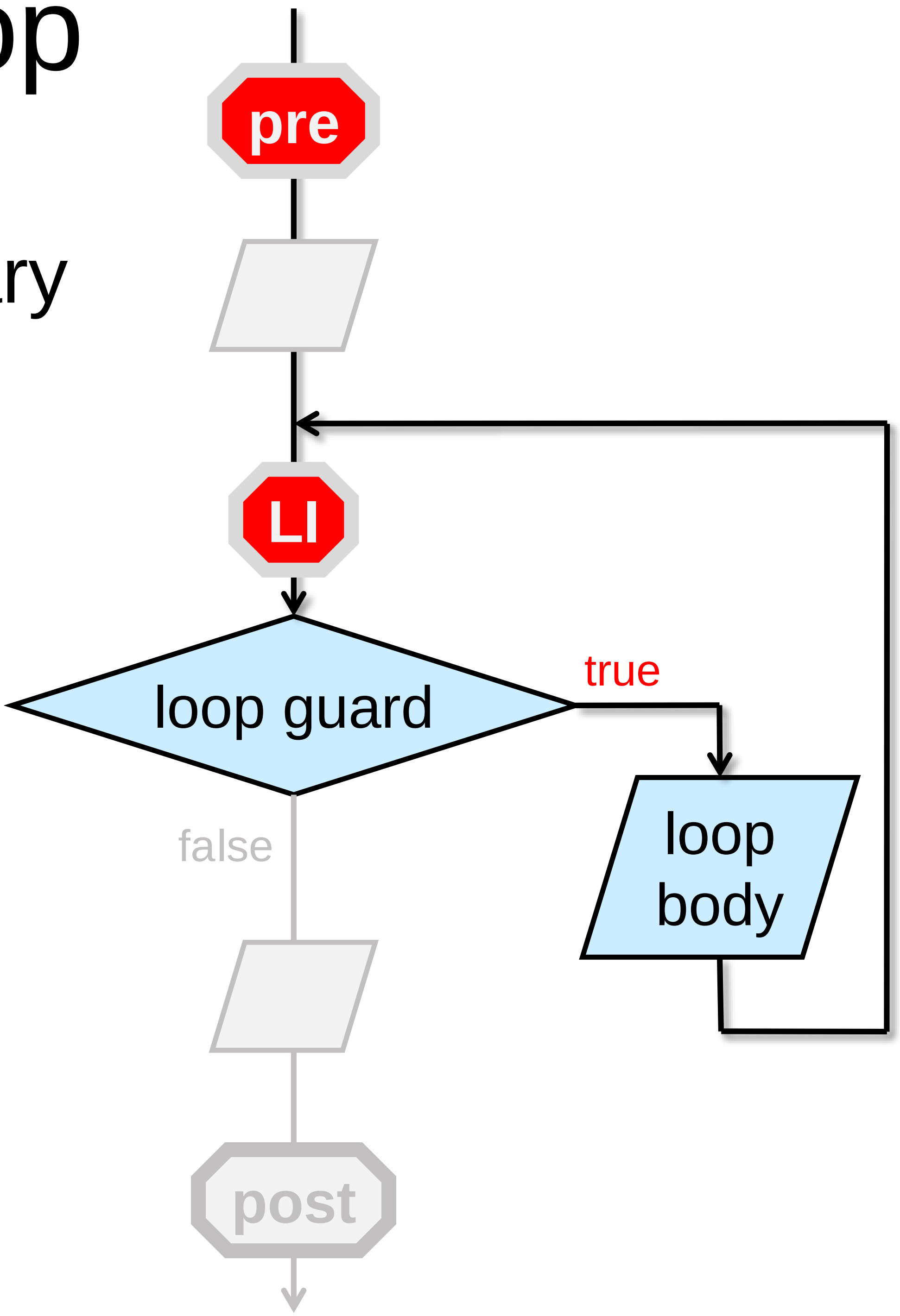
- proved by <u>point-to reasoning</u> typically using
  - the preconditions
  - simple assignments before the loop



Functions with One Loop

PRES: the LI are preserved by an arbitrary iteration of the loop

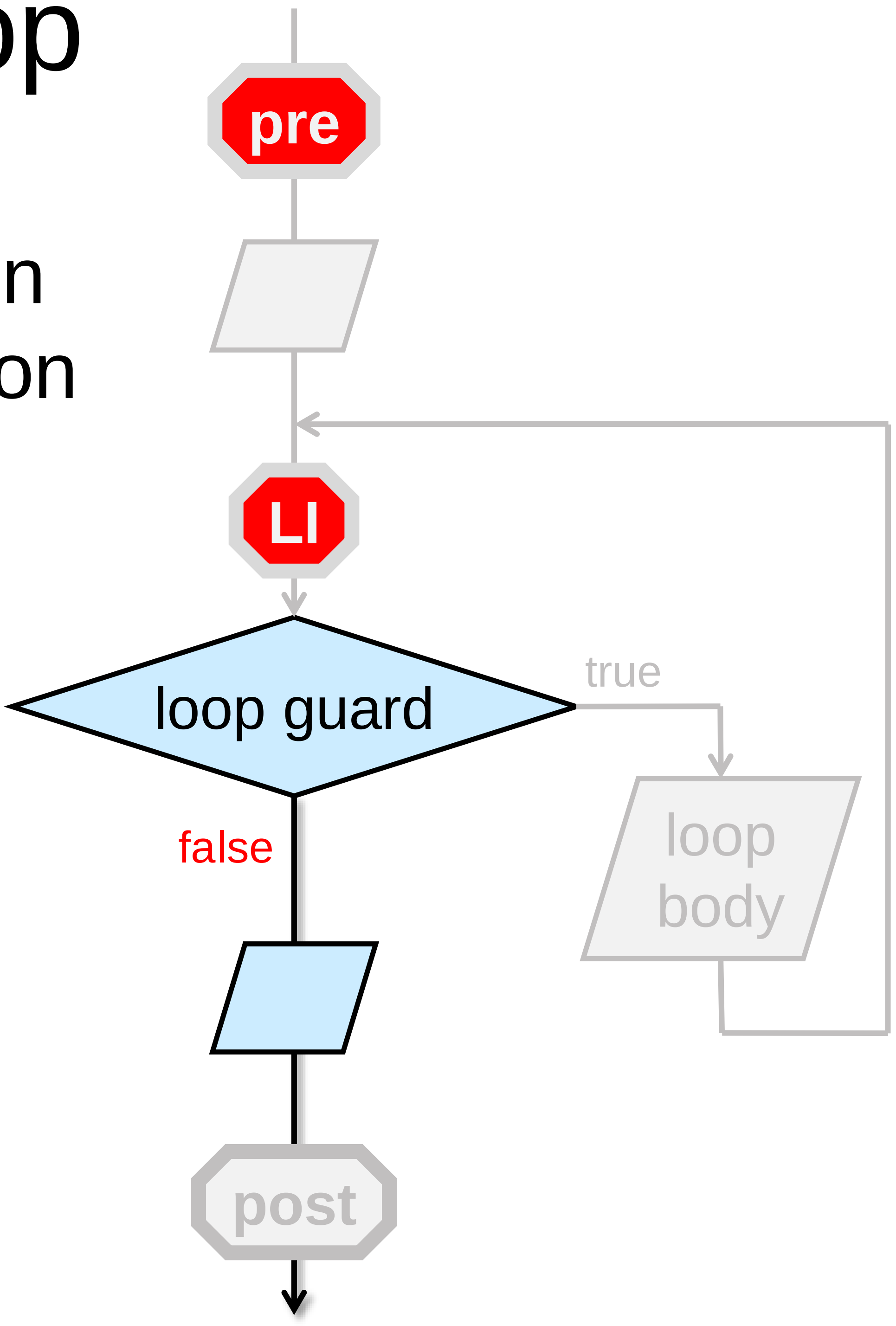
- proved by <u>point-to reasoning</u> typically using
  - the assumption that the LI is true at the beginning of the iteration
  - the loop guard being true
    - we are running an iteration
  - simple assignments and conditionals in the loop body
  - the preconditions (sometimes)



Functions with One Loop

EXIT: the loop invariants and the negation of the loop guard imply the postcondition

- proved by <u>point-to reasoning</u> typically using
  - the loop invariant
  - the negation of the loop guard
  - simple assignments and conditionals after the loop

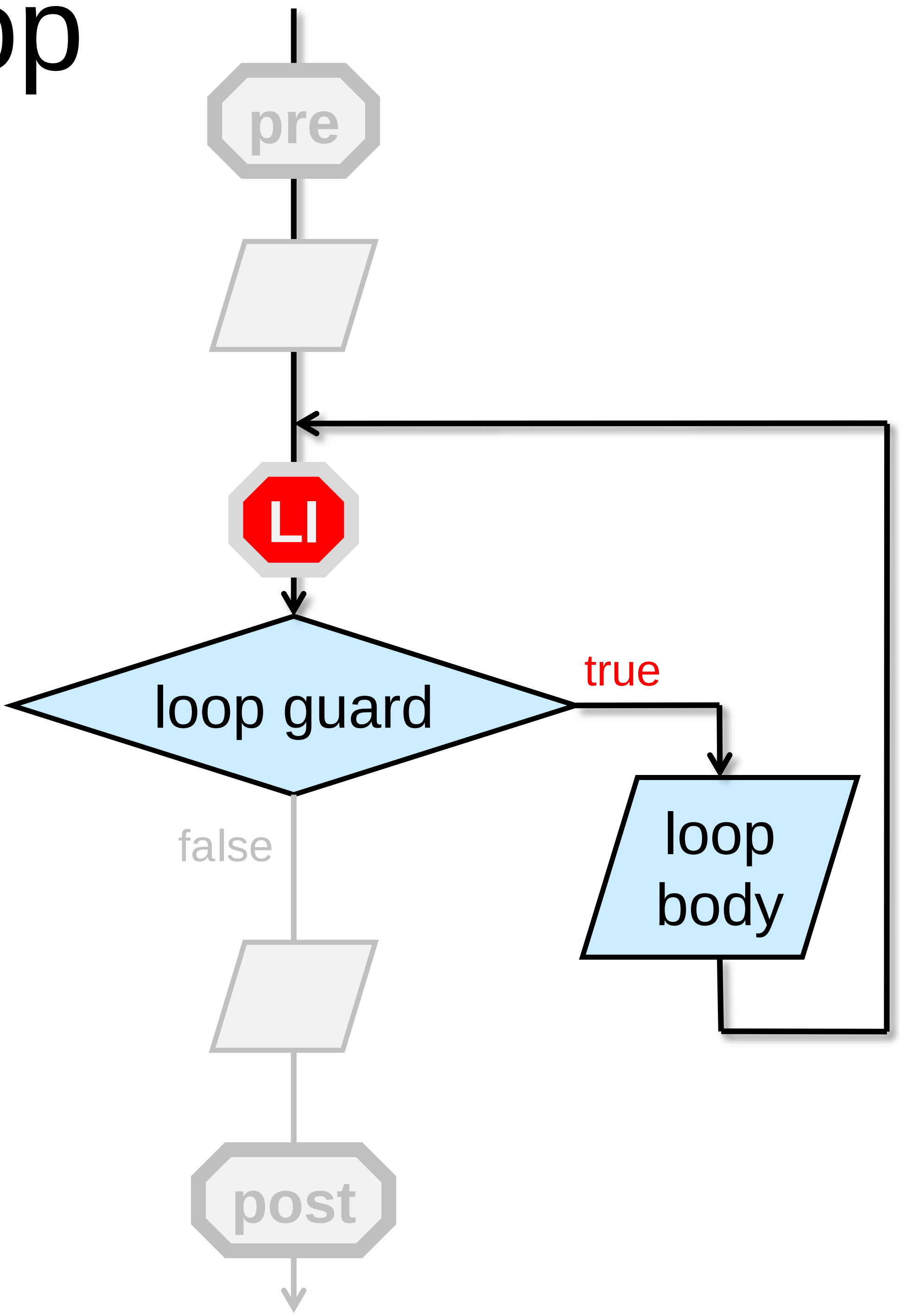


Functions with One Loop

TERM: the loop terminates

- proved by operational reasoning typically using
  - the assumption that the LI is true at the beginning of the iteration
  - the loop guard
  - simple assignments and conditionals in the loop body

But it can also be proved by **point-to reasoning**



Functions with One Loop

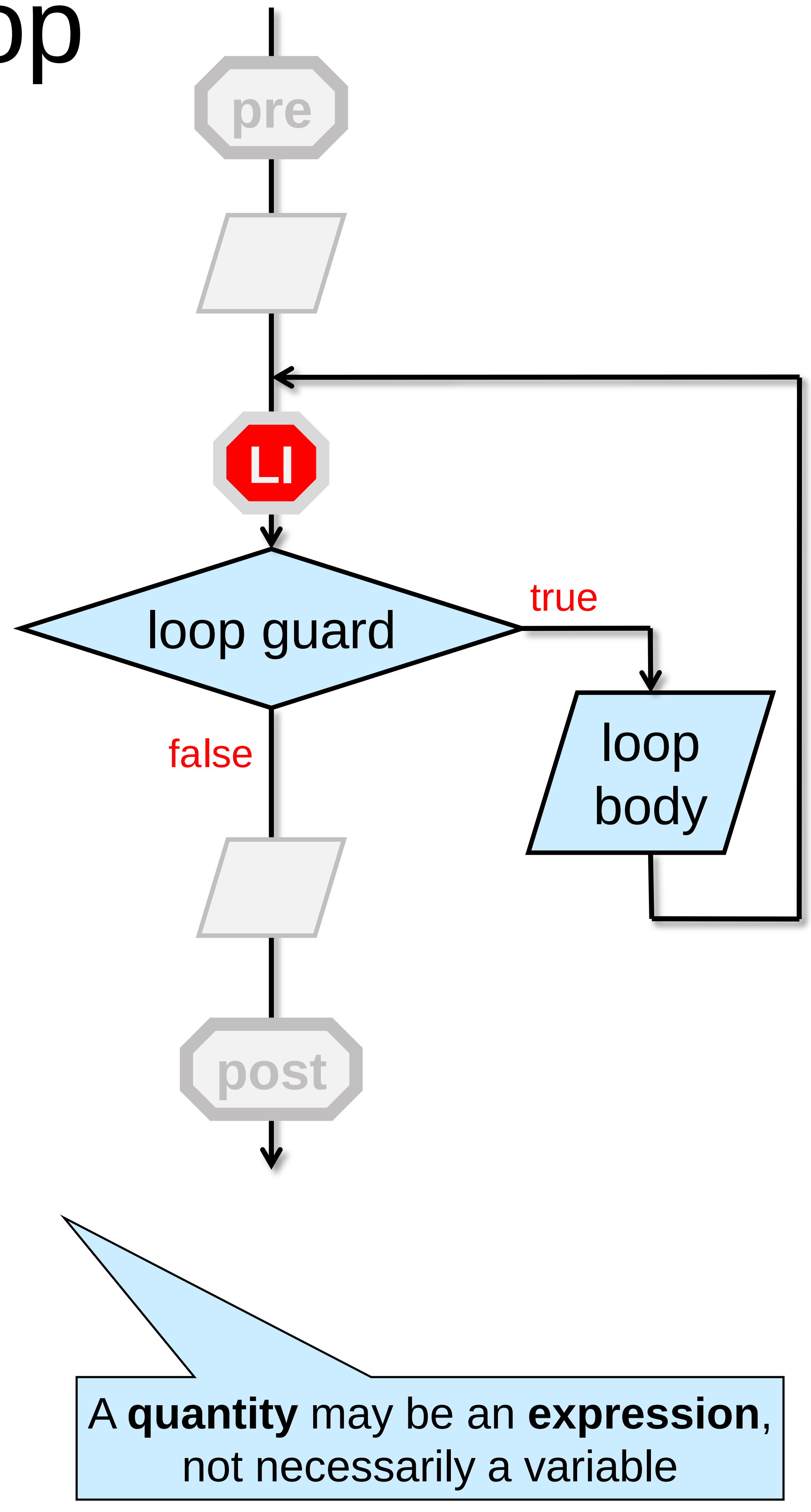
TERM: the loop terminates

- Format of a termination proof using <u>operational reasoning</u>

  “*on an arbitrary iteration of the loop,*
  the quantity ______ gets strictly smaller
  but it can’t ever get smaller than ______”
  on which the loop guard is false

  or

  “*on an arbitrary iteration of the loop,*
  the quantity ______ gets strictly bigger
  but it can’t ever get bigger than ______”
  on which the loop guard is false



More Complex Functions

- These techniques can be extended
  - but we will rarely deal with functions with more than one loop
- We can also factor out nested loops and the like into helper functions
  - and then use the technique we just saw

Seriously??

- *All these proofs and complicated reasoning seem overkill!*
  - *the mystery function wasn't all that hard after all*
  - *we could just spot what was going on*

- Yes, but it won't be that easy for more complex functions
  - the technique we saw is **systematic** and **scalable**
  - reasoning about code will pay off

- Point-to reasoning is what we do in our head all the time when programming
  - writing it down as loop invariants and contracts makes it easier not to get confused
  - and the **-d** flag will catch lingering issues at run time

Epilogue

Where are we?

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 0)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
 //@assert e == 0;
return r;
}
```

- We fully documented f
  - function contracts
  - loop invariants
  - key assertions
- We fixed the bug
- We gave mathematical proofs that
  - all the calls it makes are safe
  - it is correct
- Let’s enjoy the fruit of our labor with some more testing!

Sanity Checks

- Let's do a last round of testing

Linux Terminal

```
# coin -d mystery.c0
C0 interpreter (coin) …
--> f(2, 0);
1 (int)
--> f(2, 1);
2 (int)
--> f(2, 7);
128 (int)
--> f(2, 8);
256 (int)
--> f(2, 19);
524288 (int)
--> f(2, 31);
-2147483648 (int)
--> f(2, 32);
0 (int)
-->
```

Bug fixed!

Looking good

Plausible

What?

What?

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 0)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
 //@assert e == 0;
return r;
}
```

The story continues …