Contracts

A Mystery Function

The Story

Your first task at your new job is to debug this code written by your predecessor, who was fired for being a poor programmer.

```
int f(int x, int y) {
  int r = 1;
  while (y > 1) {
    if (y % 2 == 1) {
      r = x * r;
    }
    x = x * x;
    y = y / 2;
  }
  return r * x;
}
```

This is all you are given

How do you go about this "friendly" challenge?

The Language

- This code is written in **C0**
  - The language we will use for most of this course
- This is also valid **C** code
  - For the most part, C0 programs are valid C programs
  - We will use C0 as a gentler language to
    - learn to write complex code that is correct
    - learn to write code in C itself
- *But what does this function do?*

```
int f(int x, int y) {
  int r = 1;
  while (y > 1) {
    if (y % 2 == 1) {
      r = x * r;
    }
    x = x * x;
    y = y / 2;
  }
  return r * x;
}
```

The Programmer

- Is this good code?
  - there are no comments
  - the names are non-descript
    - the function is called f
    - the variables are called x, y, r

  No! ✘

- No wonder your predecessor was fired as a bad programmer!

- *But what does this function do?*

```
int f(int x, int y) {
  int r = 1;
  while (y > 1) {
    if (y % 2 == 1) {
      r = x * r;
    }
    x = x * x;
    y = y / 2;
  }
  return r * x;
}
```

The Function

- *But what does this function do?*
- We can run experiments
  - call f with various inputs and observe the outputs
- We do so by loading it in the **C0 interpreter** – coin

The command for the C0 interpreter

The file where we saved the function

Linux Terminal

```
# coin mystery.c0
C0 interpreter (coin) 0.3.3 'Nickel' (r590, Mon Aug 29 12:04:13 UTC 2016)
Type `#help' for help or `#quit' to exit.
-->
```

The coin prompt

Running Experiments

- *Call f with various inputs and observe the outputs*

Linux Terminal

```
# coin mystery.c0
C0 interpreter (coin) …
…
--> f(7, 12);
956385313 (int)
--> f(3, 17);
129140163 (int)
-->
```

We are calling f with inputs 7 and 12

The result is 956385313

The result has type int

- These are not very good experiments
  - they don’t help us understand what f does

Running Experiments

- *Call f with various inputs and observe the outputs*
  - we are better off calling f with small inputs
  - and vary them by just a little bit so we can spot a pattern

Linux Terminal

```
--> f(2, 3);
8 (int)
--> f(2, 4);
16 (int)
--> f(2, 5);
32 (int)
--> f(2, 6);
64 (int)
-->
```

Much better!

  - It looks like f(x, y) computes x^y
  - Let's confirm with more experiments

Confirming the Hypothesis

- *It looks like f(x, y) computes x^y*
- *Let's confirm with more experiments*

Linux Terminal

```
--> f(2, 2);
4 (int)
--> f(3, 2);
9 (int)
--> f(4, 2);
16 (int)
--> f(5, 2);
25 (int)
-->
```

Yep! That's x^y

- We find a secret memo in a hidden drawer

Power not working.
Fix by tonight or you're out

Not the friendliest of work places!

- Let's run a few more experiments to identify the problem

Discovering the Bug

- f(x, y) is *meant to* computes x^y
  - but it doesn't
- Let's find where it fails with more experiments

Linux Terminal

```
--> f(-2, 3);
-8 (int)
--> f(-2, 2);
4(int)
--> f(2, 1);
2 (int)
--> f(2, 0);
2 (int)
--> f(2, -1);
2 (int)
-->
```

It seems to work for negative values of x

That's not 2^0

That's definitely not 2^{-1}

- Now we have something to chew on

Preconditions

The Power Function

- What does it mean to be the power function x^y ?
  - $\underbrace{x \;*\; \ldots \;*\; x}_{y \text{ times}}$
    - Yes, but that's not very precise

• *y* times what? *x*? *?
• What if *y* is 0?

- Let's write a *mathematical* definition
  - $$\begin{cases} x^{\mathbf{0}} = 1 \\ x^{\mathbf{y}} = x^{\mathbf{y-1}} * x \end{cases}$$

This is a *recursive* definition

and this is its base case

The Power Function

- *What does it mean to be the power function x^y ?*

$$\begin{cases} x^{\mathbf{0}} = 1 \\ x^{\mathbf{y}} = x^{\mathbf{y-1}} * x \end{cases}$$

  - What happens if y is negative?
    - we never reach the base case …

- The power function x^y on integers is **undefined** if $y < 0$

$$\begin{cases} x^{\mathbf{0}} = 1 \\ x^{\mathbf{y}} = x^{\mathbf{y-1}} * x \quad \text{if } y > 0 \end{cases}$$

This defines x^y for $y \geq 0$ **only**

The Power Function

```
int f(int x, int y) {
  int r = 1;
  while (y > 1) {
    if (y % 2 == 1) {
      r = x * r;
    }
    x = x * x;
    y = y / 2;
  }
  return r * x;
}
```

- *What does it mean to be the power function x^y ?*

$$\begin{cases} x^{0} = 1 & \\ x^{y} = x^{y-1} * x & \text{if } y > 0 \end{cases}$$

- To implement the power function, f must disallow negative exponents
  - It can raise an error
  - It can tell the caller that the exponent should be ≥ 0

We need to test y.
This would slow f down a bit.

Better!
no need to test y

Preconditions

- Disallow negative exponents
  - by telling the caller that the exponent should be ≥ 0
- A restriction on the admissible inputs to a function is called a **precondition**
  - We need to impose a precondition on f
- In most languages, we are limited to writing a comment
  - and hope the caller reads it

This is how we would write a precondition in C

```
// y must be greater than or equal to 0
int f(int x, int y) {
 int r = 1;
  while (y > 1) {
   if (y % 2 == 1) {
     r = x * r;
   }
   x = x * x;
   y = y / 2;
  }
 return r * x;
}
```

Preconditions in C0

- *We need to impose a precondition on f*
  - *to tell the caller that y should be ≥ 0*

- In C0 we can write an **executable contract directive**

  `//@requires y >= 0;`

  C0 keyword to specify a precondition
  • written between the function header and the body
  • before the first "{"

  - We check contracts by invoking coin with the **-d** flag
    - "dynamic checking"
      - but everybody understands it as *debug mode*
  - without the **-d** flag, contracts are treated as comments

```
int f(int x, int y)
//@requires y >= 0;
{
 int r = 1;
  while (y > 1) {
    if (y % 2 == 1) {
      r = x * r;
    }
    x = x * x;
    y = y / 2;
  }
 return r * x;
}
```

Using Contract

Running with contracts disabled

Linux Terminal

```
# coin mystery.c0
C0 interpreter (coin) …
--> f(2, 3);
8 (int)
--> f(2, -1);
2 (int)
-->
```

Contracts are treated as comments

cc0, the C0 compiler, works the same way

Running with contracts enabled

Linux Terminal

```
# coin -d mystery.c0
C0 interpreter (coin) …
--> f(2, 3);
8 (int)
--> f(2, -1);
mystery.c0:2.4-2.20: @requires annotation failed

Last position: mystery.c0:2.4-2.20
           f from <stdio>:1.1-1.9
-->
```

Line number where contract failed

File where contract failed

Contracts are executed
- if **true**, execution proceeds normally
- if **false**, execution aborts

Safety

- If we call f(x,y) with a negative y
  - with **-d**, execution aborts
  - without **-d**, f can return an arbitrary result
    - there is **no** right value it could return

- Calling a function with inputs that cause a precondition to fail is **unsafe**
  - execution will never do the right thing
    - either abort
    - or compute a wrong result

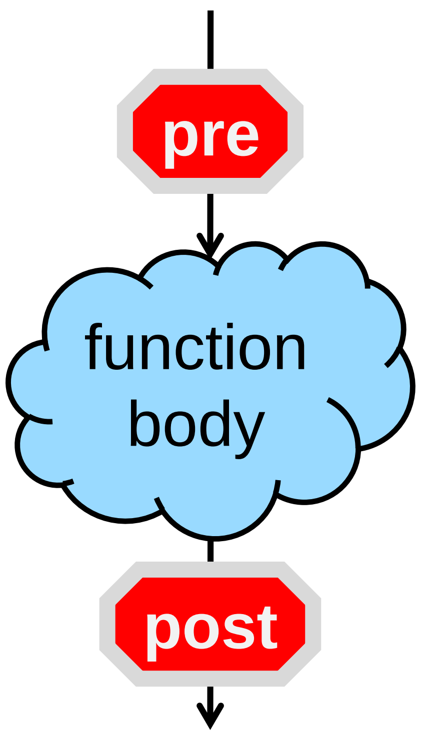
- The caller must make sure that the call is **safe**
    - that $y \geq 0$

Postconditions

Contracts about Function Outcomes

- Preconditions are checked *before* the function starts executing

- A contract that is checked *after* it is done executing could tell us if the function did the right thing
  - ➢ check that the output is what we expect
  - ○ This is a **postcondition**



Postconditions in C0

- In C0, the contract directive

  //@ensures *<some_condition>* ;

  > C0 keyword to specify a postcondition
  > - written between the function header and the body
  >   - after the preconditions (by convention)
  > - before the first "{"

  allows us to write a postcondition

  - <some_condition> can mention the contract-only variable \result
    - what the function returns
    - can only be used with //@ensures

```
int f(int x, int y)
//@requires y >= 0;
//@ensures …;
{
  int r = 1;
  while (y > 1) {
    if (y % 2 == 1) {
      r = x * r;
    }
    x = x * x;
    y = y / 2;
  }
  return r * x;
}
```

Writing a Postcondition

- The postcondition we want to write is

```
//@ensures \result == x**y;
```

That's how we write x^y in Python

  - but x**y is not defined in C0
    - C0 has no primitive power function!

- What do we do?
  - transcribe the mathematical definition into a C0 function

$$\begin{cases} x^{\mathbf{0}} = 1 & \\ x^{\mathbf{y}} = x^{\mathbf{y-1}} * x & \text{if } y > 0 \end{cases}$$

```
int POW(int x, int y)
//@requires y >= 0;
{
  if (y == 0) return 1;
  return POW(x, y-1) * x;
}
```

Writing a Postcondition

- Then our postcondition is

  //@ensures \result == POW(x, y);

  right? … almost

```
Linux Terminal
# coin -d mystery.c0
mystery.c0:18.5-18.6:error:cannot assign to
   variable 'x' used in @ensures annotation
    x = x * x;
    ~
Unable to load files, exiting...
```

  - The function modifies x (and y)
    - Which values of x and y should C0 evaluate the postcondition with?
      - We want the initial values, but it is checked when returning …
  - To avoid confusion, C0 disallows modified variables in postconditions

```
int POW(int x, int y)
//@requires y >= 0;
{
  if (y == 0) return 1;
  return POW(x, y-1) * x;
}

int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
 int r = 1;
  while (y > 1) {
    if (y % 2 == 1) {
      r = x * r;
    }
    x = x * x;
    y = y / 2;
  }
 return r * y;
}
```

Writing a Postcondition

- *C0 disallows modified variables in postconditions*
  - Make copies x and y and modify those

Linux Terminal

```
# coin -d mystery.c0
C0 interpreter (coin) …
--> f(2, 3);
8 (int)
--> f(2, 0);
mystery.c0:11.4-11.33: @ensures annotation failed
Last position: mystery.c0:11.4-11.33
               f from <stdio>:1.1-1.8
```

Line number where contract failed

  - We're good

```
int POW(int x, int y)
//@requires y >= 0;
{
  if (y == 0) return 1;
  return POW(x, y-1) * x;
}

int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
 int b = x;
 int e = y;
 int r = 1;
 while (e > 1) {
   if (e % 2 == 1) {
     r = b * r;
   }
   b = b * b;
   e = e / 2;
 }
 return r * b;
}
```

Recall Safety

- In the postcondition of f, we are making a call to POW
  - **Is it safe?**
    - We need to show that y >= 0
    - ➢ The precondition tells us that y >= 0 ✓
- The body of POW makes a call to POW
  - **Is it safe?**
    - We need to show that y-1 >= 0
    - ➢ The precondition tells us that y >= 0
    - ➢ Since we don't return on the if, y > 0
    - ➢ So y-1 >= 0 by math ✓
- These are examples of **point-to reasoning**
  - We justify something by pointing to lines of code that supports it

This should always be on our mind

```
int POW(int x, int y)
//@requires y >= 0;
{
  if (y == 0) return 1;
  return POW(x, y-1) * x;
}

int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

The Power Function

- But wait!
  - f was meant to implement the power function
  - … but POW **is** the power function!
- Let's use it!
  - There may be benefits to fixing f instead
    - it may be more efficient than POW
  - Keep reading …

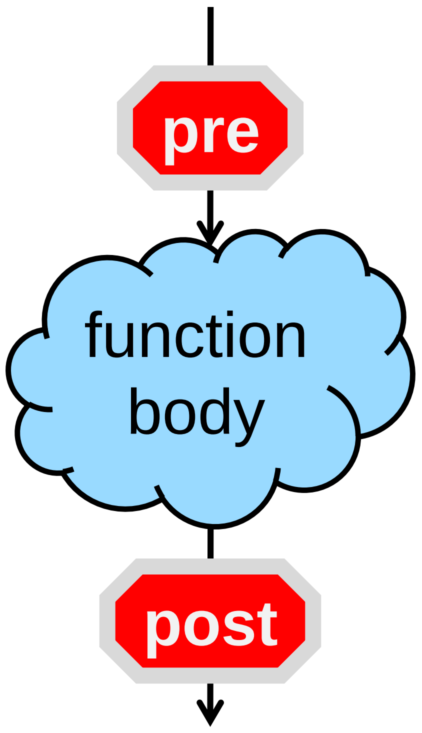
```
int POW(int x, int y)
//@requires y >= 0;
{
  if (y == 0) return 1;
  return POW(x, y-1) * x;
}

int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

Correctness

- If a call violates a function's postconditions
  (assuming its preconditions were met so it actually ran)

  the function is doing something wrong
  - the function has a **bug**



- The function is **incorrect**
  - Our mystery function f is incorrect

- When writing a function, we must make sure that it is **correct**
  - i.e., that its postconditions will be satisfied for any safe input

Blame

- If a function preconditions fail, it's the caller's fault
    - the caller passed invalid inputs
  - the call is **unsafe**

- If its postconditions fail, it's the implementation's fault
    - the function code does the wrong thing
  - the function is **incorrect**

We will develop methods to make sure that the code we write is **safe** and **correct**

How to Use Contracts

- Contract-checking helps us write code that works as expected
  - Use **-d** while writing our code
  - At this stage, this is **development code**
    - bugs are likely
- Once we are confident our code works, compile it without **-d**
  - The code can be used in its intended application
  - At this stage, this is **production code**
    - there should be no bugs
- Why not use **-d** always?
  - it slows down execution

Specification Functions

- POW is used only in contracts
  - It is not executed when contract-checking is disabled
    - without **-d**
- Functions used only in contracts are called **specification functions**
  - They help us state what the code should do
  - They are critical to writing good code

```
int POW(int x, int y)
//@requires y >= 0;
{
  if (y == 0) return 1;
  return POW(x, y-1) * x;
}

int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

Function Contracts

Where are we?

- We have learned a lot about f
  - the preconditions describe what valid inputs are
  - the postconditions describe what it is supposed to do
    - on valid inputs
- We have a fully documented function
- We have not looked at all at its body
    - but we know there is a bug in there
    - it is incorrect

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

The Caller's Perspective

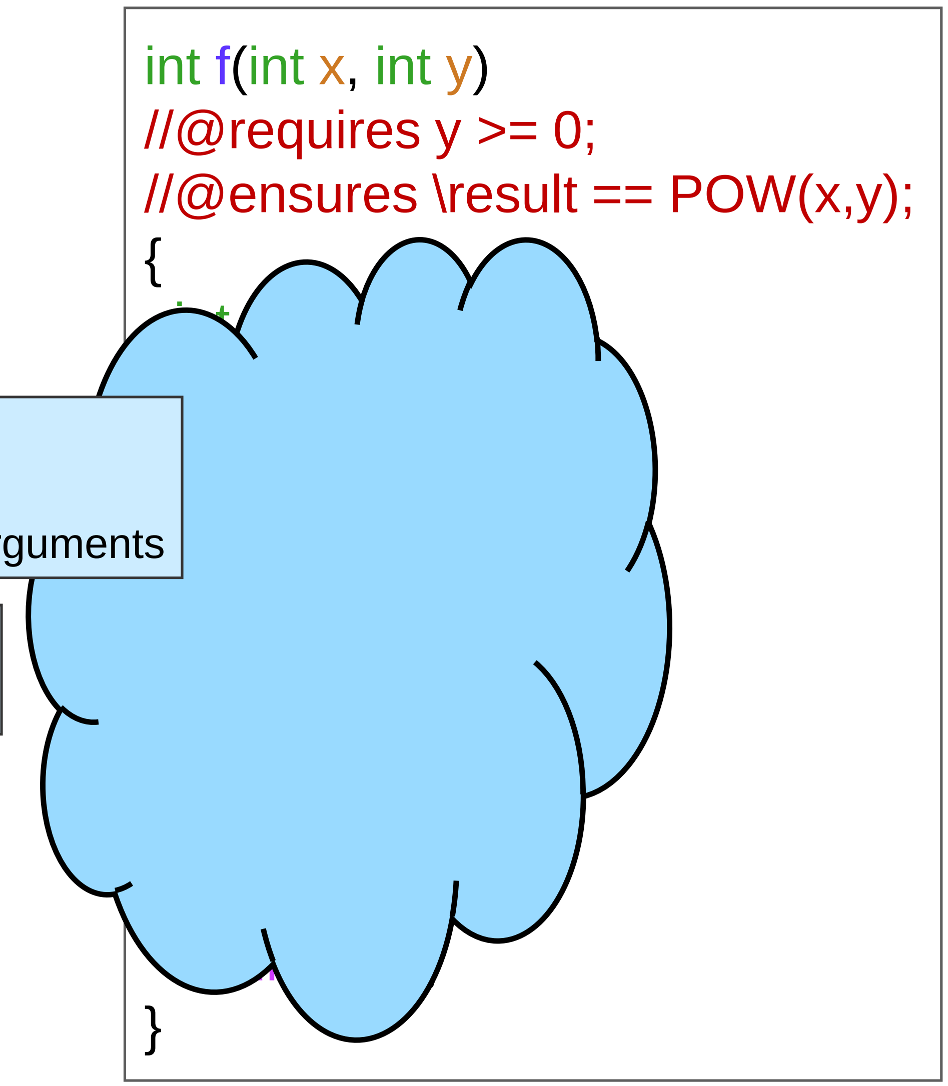
Preconditions describe valid inputs
Postconditions describe what it does

- That's what the **caller** needs to know to use the function

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
```

Header:
- function name
- number and type of its arguments

Contracts:
- pre- and post-conditions



- The caller should be able to use it without knowing anything about how it is implemented
  - The implementation details are **abstracted away**

Abstraction

- Split a complex system into **small** chunks that can be understood **independently**

Bother with as few details as possible at any time

- Computer science is all about abstraction

The Function's Perspective

Preconditions describe valid inputs
Postconditions describe what it does

- That's what the implementation is to do
  - guidelines to write the body of the function
- How to write good code
  - **First write the contracts**
  - and then the body
    - in this way, you always know what you are aiming for

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

Now, we need to look at the body of f to find the bug

Loop Invariants

Diving In

- We need to look at the body of f
  - The complicated part is the **loop**
    - the values of the variables change at each iteration
    - it's unclear how many iterations there are
  - If we understand the loop, we understand the function

- How to go about that?

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

Abstraction

- *If we understand the loop,*
  we understand the function

- How to go about that?
  - Contracts summarize what a function does so we don't need to bother with the details of its implementation
    - ➢ An abstraction over functions
  - Come up with a summary of the loop so we don't need to bother with the details of its implementation
    - ➢ **An abstraction over loops!**

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
  }
  return r * b;
}
```

Loop Invariants

The values of the variables change at each iteration

- One valuable abstraction is what does **not** change
  - This is called a **loop invariant**
    - a quantity that remains constant at each iteration of the loop
      - a quantity may be an expression, not just a variable

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

We will see what makes some loop invariants **really valuable** shortly

Tracing Code

- How to find a **loop invariant**?
  - *a quantity that remains constant at each iteration of the loop*

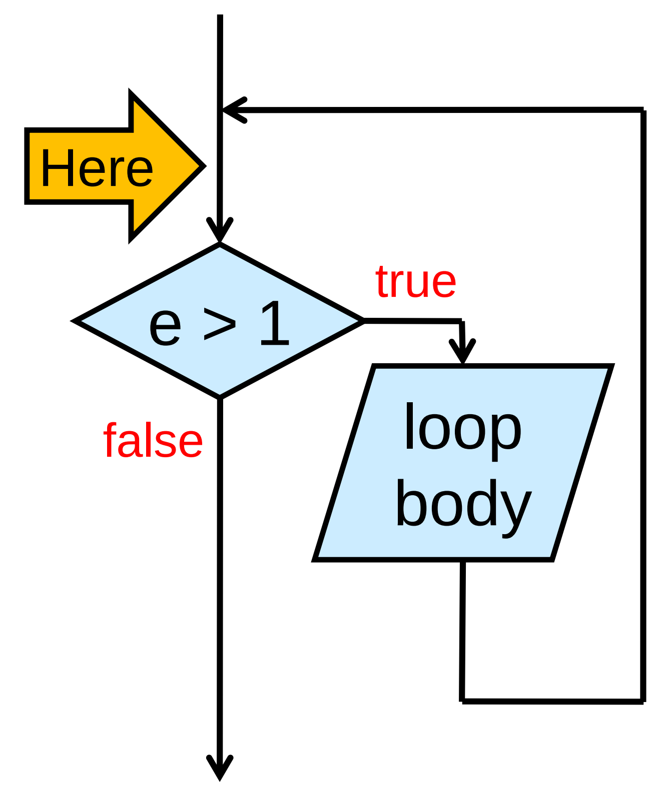
- Run the function on sample inputs
- Track the value of the variables
  - b, e, r
    - no need to bother with x and y since they don't change
  - just before the loop guard is tested
    - That's e > 1

This is called **tracing** an execution

- Look for patterns

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

Loop guard



Tracing Code

- Run the function on sample inputs and track the value of the variables
  - Let's try with f(2,8)

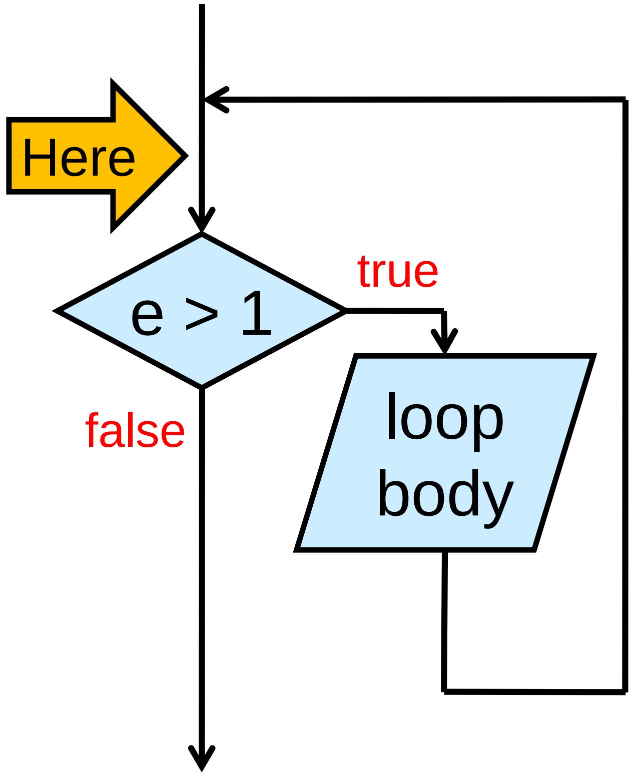
b	e	r
2	8	1
4	4	1
16	2	1
256	1	1

At this point we exit the loop

  - Can we spot a quantity that doesn't change?

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

This checks if e is odd



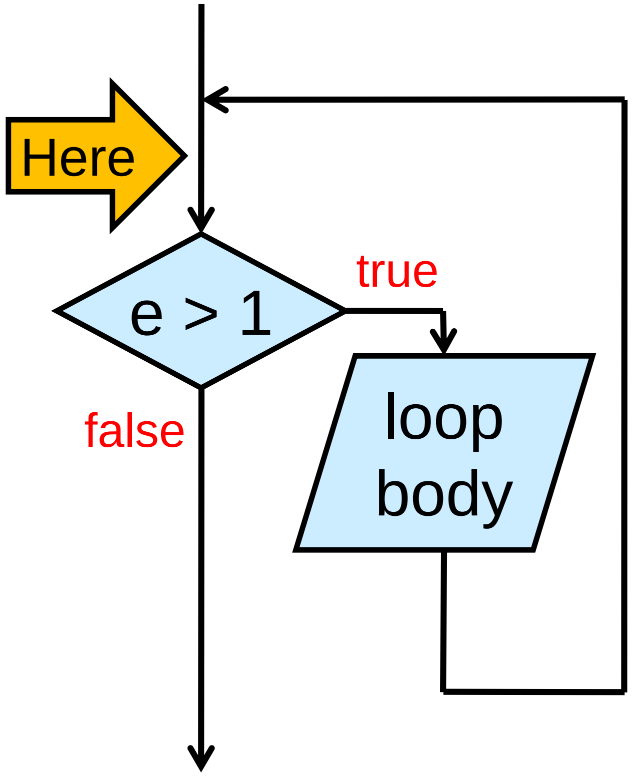
Tracing Code

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

- Trying with f(2,8)
  - *Can we spot a quantity that doesn't change?*
  - $\mathbf{b^e}$ is always 256

b	e	r	$\mathbf{b^e}$
2	8	1	**256**
4	4	1	**256**
16	2	1	**256**
256	1	1	**256**

  - This is a **candidate loop invariant**
    - $\mathbf{b^e}$ is constant on one set of inputs
    - a loop invariant must stay constant on all inputs



Tracing Code

- **b^e** is a *candidate* loop invariant

- Let's try with f(2,7)

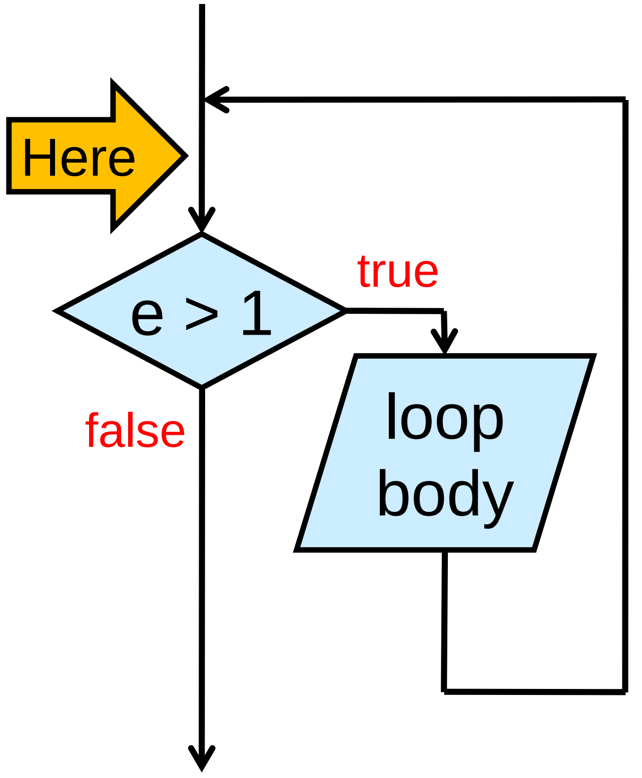
b	e	r	**b^e**
2	7	1	**128**
4	3	2	**64**
16	1	8	**16**

Not constant on these inputs

  - **b^e** is **not** invariant on these inputs!
    - It was a candidate that didn't pan out

- Can we spot another quantity that doesn't change?

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```



Tracing Code

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

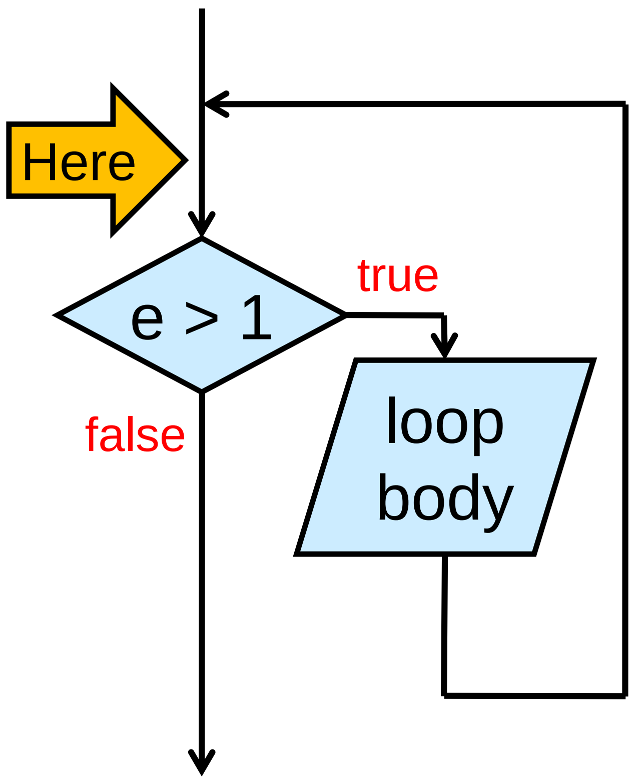
- *Trying with f(2,7)*
  - *Can we spot a quantity that doesn't change?*
  - **b^e * r** is always 128

b	e	r	b^e	**b^e * r**
2	7	1	128	**128**
4	3	2	64	**128**
16	1	8	16	**128**

- This is another candidate loop invariant
  - Let's test it on f(3,5)

b	e	r	**b^e * r**
3	5	1	**243**
9	2	3	**243**
81	1	3	**243**

  - This seems to work



A Candidate Loop Invariant

- **b^e * r** is a promising candidate loop invariant
  - It works on *three* inputs!
- How do we know it works in general?
  - We can't test it on all inputs
  - We need to provide a **proof**
- *But first, let's add it to our code*

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1) {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

Loop Invariants in C0

- In C0, we use the directive

  //@loop_invariant

  to specify a loop invariant

C0 keyword to specify a loop invariant
• written between the loop guard and the loop body

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1)
  //@loop_invariant  … ;
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

- Then, simply write

  //@loop_invariant POW(b, e) * r;

  - … this won't work
    - C0 would need to keep track of the values of this expression across all iterations of the loop
    - also, what if the loop runs 0 times?

- In C0, loop invariants must be **boolean expressions**
  - **true** means it was satisfied in the current iteration
  - **false** means it wasn't

Loop Invariants in C0

- They are boolean expressions
  - **true** means satisfied
- What can we use?

b	e	r	$b^e * r$
2	7	1	**128**
4	3	2	**128**
16	1	8	**128**

  - As we enter the loop, b is x and e is y
    - so x^y is 128 too
    - thus, $b^e * r = x^y$
- Then, we can write

  //@loop_invariant POW(b, e) * r == POW(x, y);

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1)
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

Execution will abort when ran with **-d** if LI is ever **false**

Safety

We have two new calls to POW

- **Are they safe?**

- POW(x, y)
  - **To show:** y >= 0
  - y >= 0 by line 2 (precondition of f)

- POW(b, e)
  - **To show:** e >= 0
  - "e is *initially* equal to y which is >= 0 and it is halved at *each* iteration of the loop so e is *always* >= 0"
  - This is an example of **operational reasoning**
    - The justification relies on what is happening in all the iterations of the loop
      - This is error-prone
    - We will disallow safety proofs based on operational reasoning on loops

```
1.  int f(int x, int y)
2.  //@requires y >= 0;
3.  //@ensures \result == POW(x,y);
4.  {
5.    int b = x;
6.    int e = y;
7.    int r = 1;
8.    while (e > 1)
9.    //@loop_invariant POW(b,e) * r == POW(x,y);
10.   {
11.     if (e % 2 == 1) {
12.       r = b * r;
13.     }
14.     b = b * b;
15.     e = e / 2;
16.   }
17.   return r * b;
18. }
```

Safety

```
1.  int f(int x, int y)
2.  //@requires y >= 0;
3.  //@ensures \result == POW(x,y);
4.  {
5.    int b = x;
6.    int e = y;
7.    int r = 1;
8.    while (e > 1)
9.    //@loop_invariant e >= 0;
10.   //@loop_invariant POW(b,e) * r == POW(x,y);
11.   {
12.     if (e % 2 == 1) {
13.       r = b * r;
14.     }
15.     b = b * b;
16.     e = e / 2;
17.   }
18.   return r * b;
19. }
```

POW(b, e)

- ➢ **To show:** e >= 0
- ○ We can sort of do it with operational reasoning
  - ➢ error prone!
- ○ but we really want to prove it using point-to reasoning

- We do believe that e >= 0 at every iteration of the loop
  - ○ Turn it into a candidate loop invariant!

    //@loop_invariant e >= 0;

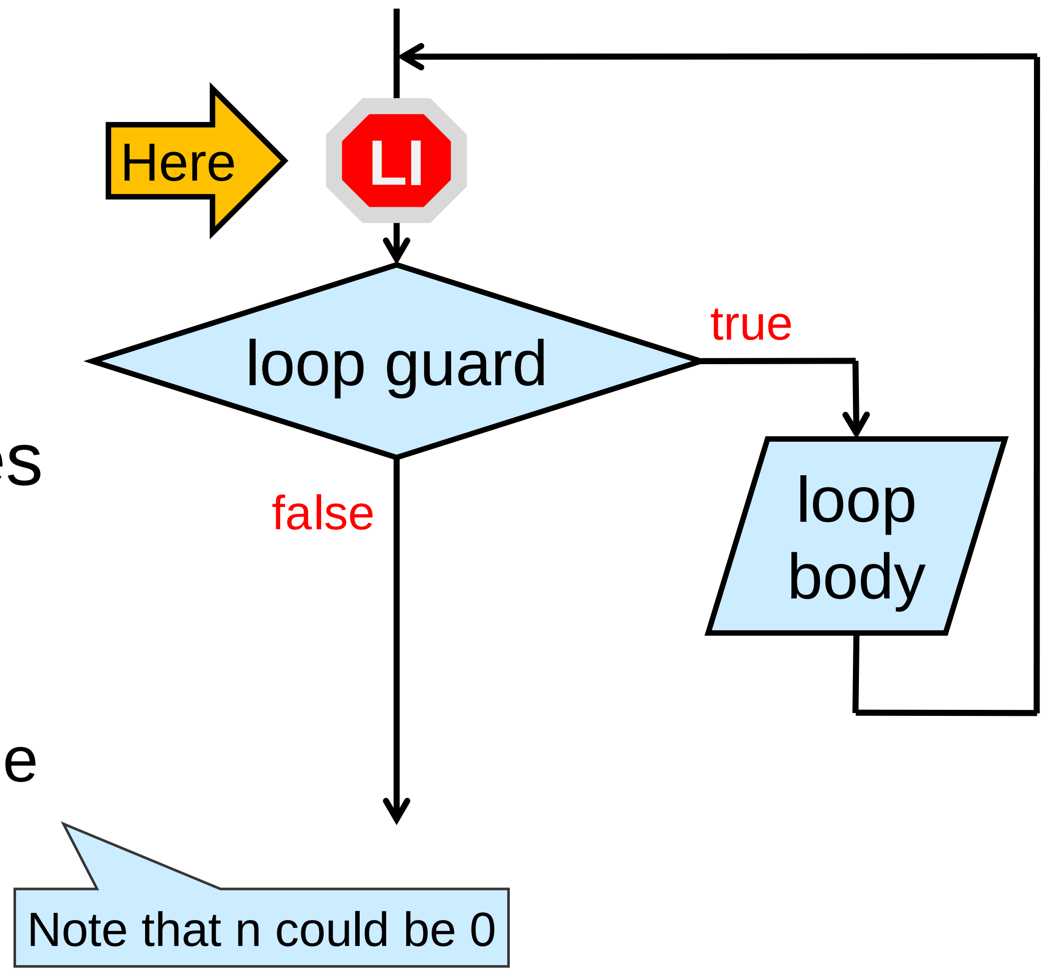
    - ➢ We will need to prove later that it is valid
  - ○ Then we prove that POW(b, e) is safe by pointing to line 9

An operational hunch is often a good candidate loop invariant

How Loop Invariants Work

- Loop invariants are checked **just before** the loop guard is tested
- If the loop body runs n times,
  - the loop invariant is checked n+1 times
    - must be **true** all n+1 times
  - the loop guard is tested n+1 times too
    - **true** the first n times and **false** the last time
- When we exit the loop
  - the loop invariant is **true**
  - the loop guard **false**

Important!



Validating Loop Invariants

Where are we?

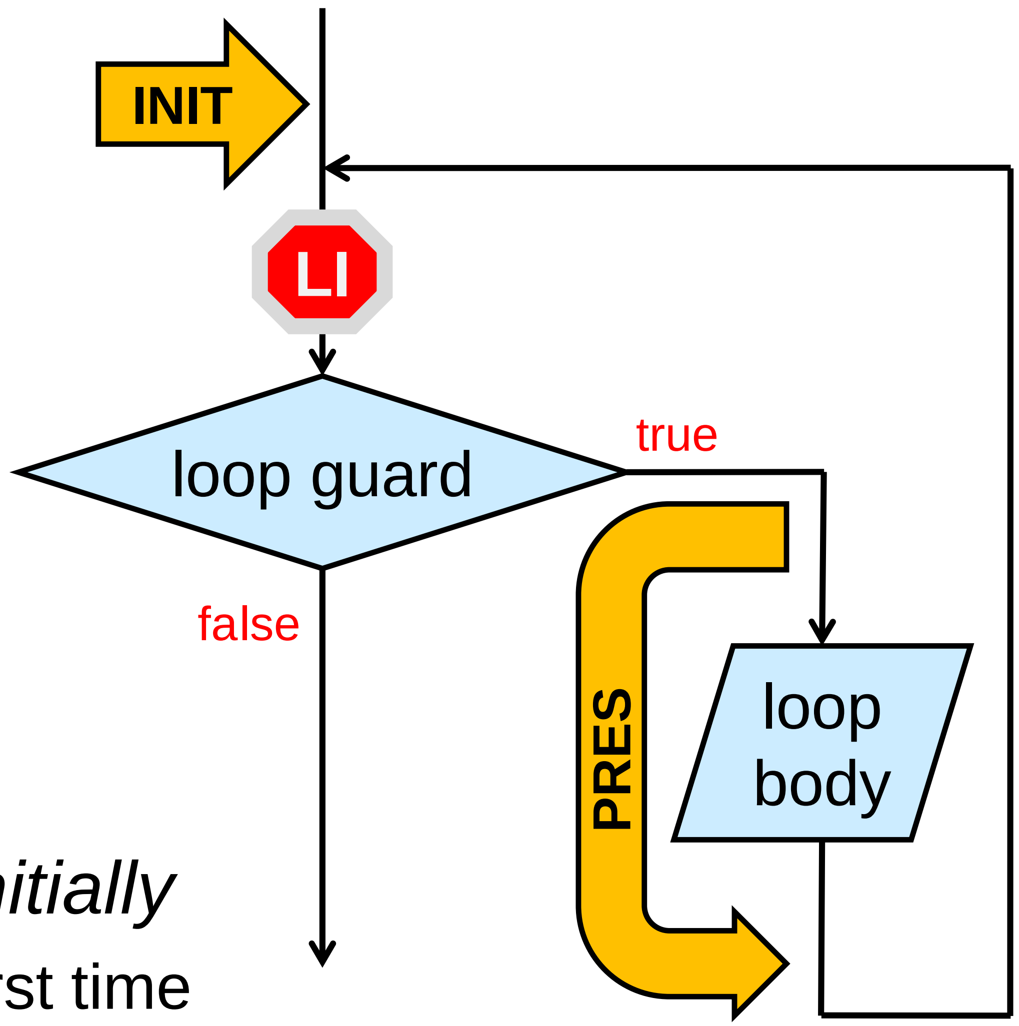
- We have learned even more about f
  - The contracts tell us what it is meant to do
  - The loop invariants give us useful information about how the loop works
    - but these are **candidate** loop invariants
    - we need to prove that they are valid

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

- We have started learning about proving things about code
    - just safety so far
  - point-to reasoning: good
  - operational reasoning: error prone

Proving a Loop Invariant Valid

- We cannot show a loop invariant is valid by running it on all possible inputs
  - We need to supply a proof
    - using point-to reasoning



- Two steps

INIT: show that the loop invariant is true *initially*

- just before we test the loop guard the very first time

PRES: show that the loop invariant is **preserved** by the loop

- if it is true at the beginning of an **arbitrary iteration** of the loop,
- then it is also true at the end of this iteration

But it may become false temporarily in the middle of the loop body

We use math notation for brevity

Validity of e ≥ 0

```
1.  int f(int x, int y)
2.  //@requires y >= 0;
3.  //@ensures \result == POW(x,y);
4.  {
5.    int b = x;
6.    int e = y;
7.    int r = 1;
8.    while (e > 1)
9.    //@loop_invariant e >= 0;
10.   //@loop_invariant POW(b,e) * r == POW(x,y);
11.   {
12.     if (e % 2 == 1) {
13.       r = b * r;
14.     }
15.     b = b * b;
16.     e = e / 2;
17.   }
18.   return r * b;
19. }
```

INIT:

- **To show:** $e \geq 0$ initially

A. $y \geq 0$ by line 2

B. $e = y$ by line 6

C. $e \geq 0$ by math on A and B ✓

This is a typical proof format in this course

PRES:

LI at **start** of current iteration

LI at **end** of current iteration

- **To show:** if $e \geq 0$, then $e \geq 0$

But isn't this trivially true?

- The value of e changes in the body of the loop
- We need a way to distinguish the value at the start and end of the current iteration
  - e ⇐ value of e at the **start** of the current iteration
  - e' ⇐ value of e at the **end** of the current iteration

Validity of e ≥ 0

INIT: e ≥ 0 initially

LI at **start** of current iteration

LI at **end** of current iteration

PRES:

- **To show:** if e ≥ 0, then e’ ≥ 0

A. e’ = e/2 by line 16

B. e ≥ 0 by assumption

C. e/2 ≥ 0 by math on B

D. e’ ≥ 0 by A and C

Both INIT and PRES were proved by point-to reasoning

```
1.  int f(int x, int y)
2.  //@requires y >= 0;
3.  //@ensures \result == POW(x,y);
4.  {
5.    int b = x;
6.    int e = y;
7.    int r = 1;
8.    while (e > 1)
9.    //@loop_invariant e >= 0;
10.   //@loop_invariant POW(b,e) * r == POW(x,y);
11.   {
12.     if (e % 2 == 1) {
13.       r = b * r;
14.     }
15.     b = b * b;
16.     e = e / 2;
17.   }
18.   return r * b;
19. }
```

Validity of $b^e r = x^y$

```
1.  int f(int x, int y)
2.  //@requires y >= 0;
3.  //@ensures \result == POW(x,y);
4.  {
5.    int b = x;
6.    int e = y;
7.    int r = 1;
8.    while (e > 1)
9.    //@loop_invariant e >= 0;
10.   //@loop_invariant POW(b,e) * r == POW(x,y);
11.   {
12.     if (e % 2 == 1) {
13.       r = b * r;
14.     }
15.     b = b * b;
16.     e = e / 2;
17.   }
18.   return r * b;
19. }
```

INIT:

- **To show:** $b^e r = x^y$ initially

A. $b = x$ by line 5

B. $e = y$ by line 6

C. $r = 1$ by line 7

D. $b^e r = x^y$ by math on A, B, C ✓

PRES:

LI at **start** of current iteration

LI at **end** of current iteration

x and y don't change in the loop

- **To show:** if $b^e r = x^y$, then $b'^{e'} r' = x^y$

○ We need to distinguish 2 cases based on the test e % 2 == 1

- **e % 2 == 1** is **true** — *e is odd*
- **e % 2 == 1** is **false** — *e is even*

Validity of $b^e\ r = x^y$

```
1. int f(int x, int y)
2. //@requires y >= 0;
3. //@ensures \result == POW(x,y);
4. {
5.   int b = x;
6.   int e = y;
7.   int r = 1;
8.   while (e > 1)
9.   //@loop_invariant e >= 0;
10.  //@loop_invariant POW(b,e) * r == POW(x,y);
11.  {
12.    if (e % 2 == 1) {
13.      r = b * r;
14.    }
15.    b = b * b;
16.    e = e / 2;
17.  }
18.  return r * b;
19. }
```

PRES:

- **To show:** if $b^e\ r = x^y$, then $b'^{e'}\ r' = x^y$
- Case **e is odd** (e % 2 == 1)
  - Then e = 2n+1 for some n

A.	$b' = b*b$	by line 15
B.	$e' = e/2$	by line 16
C.	$= n$	by case assumption and math
D.	$r' = b * r$	by line 13
E.	$b'^{e'}\ r' = (b*b)^n\ b*r$	by A, B, C, D
F.	$= b(b^2)^n\ r$	by math
G.	$= b^{2n+1}\ r$	by math
H.	$= b^e\ r$	by case assumption
I.	$= x^y$	by assumption

This is one of the most complex proofs in this course

- This proves the first case

Validity of $b^e\ r = x^y$

```
1. int f(int x, int y)
2. //@requires y >= 0;
3. //@ensures \result == POW(x,y);
4. {
5.   int b = x;
6.   int e = y;
7.   int r = 1;
8.   while (e > 1)
9.   //@loop_invariant e >= 0;
10.  //@loop_invariant POW(b,e) * r == POW(x,y);
11.  {
12.    if (e % 2 == 1) {
13.      r = b * r;
14.    }
15.    b = b * b;
16.    e = e / 2;
17.  }
18.  return r * b;
19. }
```

PRES:

- **To show:** if $b^e\ r = x^y$, then $b'^{e'}\ r' = x^y$
- Case **e is even** (e % 2 == 0)
  - Then e = 2n for some n

A.	$b' = b*b$	by line 15
B.	$e' \quad = e/2$	by line 16
C.	$= n$	by case assumption and math
D.	$r' = r$	since r is unchanged
E.	$b'^{e'}\ r' = (b*b)^n\ r$	by A, B, C, D
F.	$= (b^2)^n\ \ r$	by math
G.	$= b^{2n}\ r$	by math
H.	$= b^e\ r$	by case assumption
I.	$= x^y$	by assumption

- This proves the second case too

PRES holds for $b^e\ r = x^y$ ✓

Loop Invariants

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

- $e \geq 0$ is valid ✓
  - it holds **INIT**ially
  - it is **PRES**erved by an arbitrary iteration of the loop
    - if $e \geq 0$, then $e' \geq 0$
- $b^e\ r = x^y$ is valid ✓
  - it holds **INIT**ially
  - it is **PRES**erved by an arbitrary iteration of the loop
    - if $b^e\ r = x^y$, then $b'^{e'}\ r' = x^y$
- This shows that both are **genuine loop invariants**
  - not just candidates
  - we can forget about the body of the loop when reasoning about this function

Proof-directed Debugging

Where are we?

- The contracts tell us what the function is *meant to* do
  - but we know there is a bug in there
- The loop invariants abstract away the details of the loop

But what to do with them is still a bit mysterious

- *Let's find the bug!*

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
  }
  return r * b;
}
```

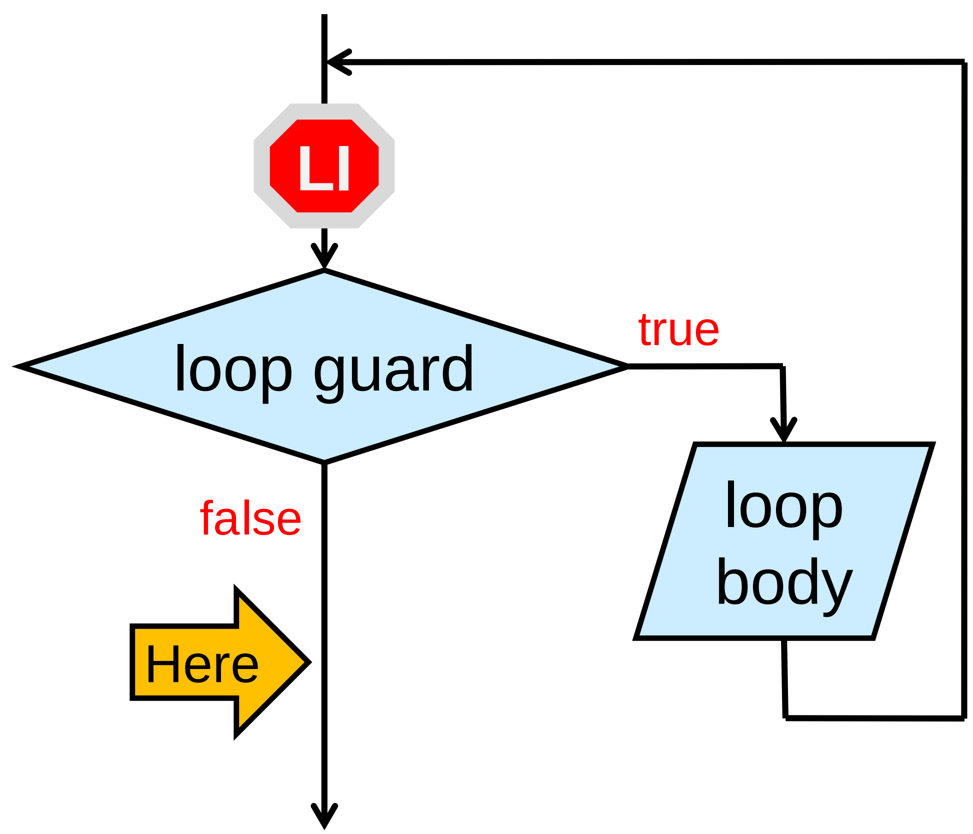
After the Loop

- What do we know when execution exits the loop?
  - the loop guard is **false**
    - $e \leq 1$
  - the loop invariants are **true**
    - $e \geq 0$
    - $b^e\ r = x^y$
- Knowing this will
  - enable us to prove correctness
  - or expose a bug

Since f is incorrect, this should happen

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
  }
  return r * b;
}
```

Here



After the Loop

- What do we know when execution exits the loop?
  - the loop guard is **false**
    - $e \leq 1$
  - the loop invariants are **true**
    - $e \geq 0$
    - $b^e\ r = x^y$
- From $e \leq 1$ and $e \geq 0$, we have that
  - either $e = 0$
  - or $e = 1$

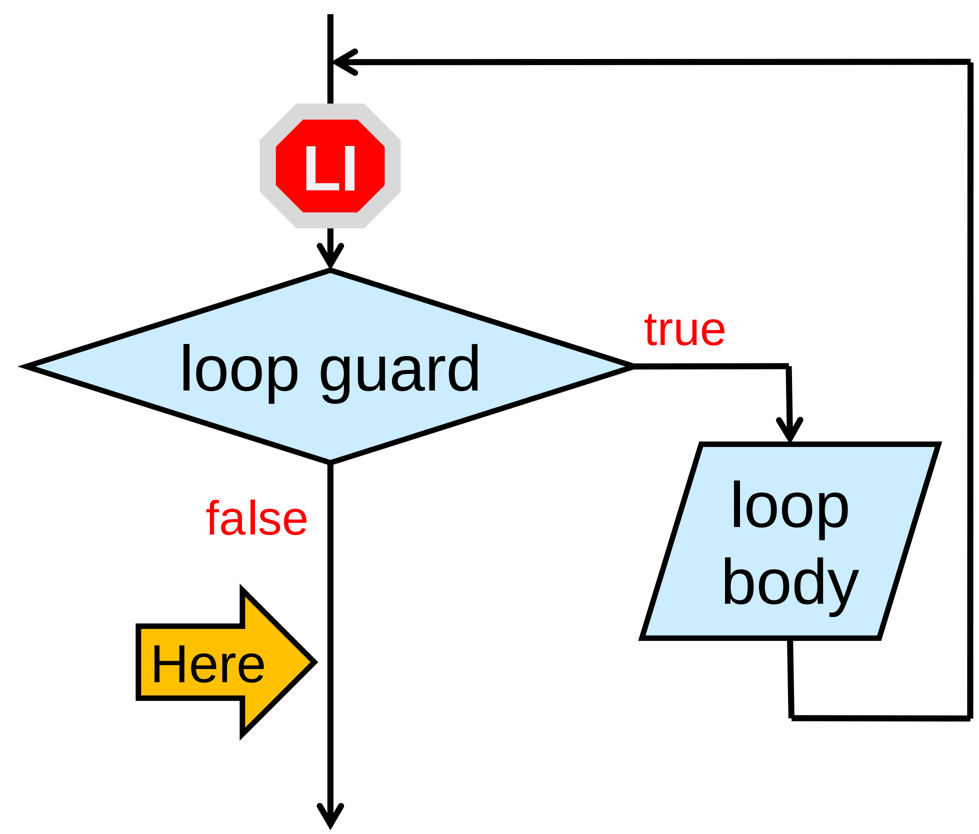
  as we exit the loop

Recall that e has type int

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {

  }
  return r * b;
}
```

Here



After the Loop

- Either $e = 0$ or $e = 1$
  - Let's plug these values in the other loop invariant, $b^e\ r = x^y$

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
  }
  return r * b;
}
```

Here

➔ If $e = 1$, then $x^y = b^e\ r = b^1\ r = r\ b$
  - Thus, $x^y = r\ b$ in this case

This is exactly what f returns.

➔ if $e = 0$, then $x^y = b^e\ r = b^0\ r = r$
  - Thus, $x^y = r$ in this case
    - $x^y \neq r\ b$

This is **not** what f returns.

This is the bug!

Tracking the Bug

- The bug is when e = 0 as we exit the loop
- This can happen **only** if f is called with 0 as y
  - if e = 1, the loop doesn't run and e stays 1
  - if e > 1 at the start of an iteration, then e' ≥ 1 as we end it

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

Here

Fixing the Bug

Idea #1: return 1 if y = 0

- This works but it introduces a **special case** in the code
- Special cases leads to contrived, unmaintainable code
  - sometimes unavoidable
  - but let's see if we can do better

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  if (y == 0) return 1;
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

Fixing the Bug

Idea #2: change the precondition to y > 0

- This forces the caller to have special cases in their code!
  - calls to f need to be **guarded**

```
int c = f(a, b)
```

⇒

```
int c = 1;
if (b > 0) c = f(a, b);
```

```
int f(int x, int y)
//@requires y > 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

- This also means that f is not the power function any more
  - undefined when exponent is 0

- Not a great solution ✘

Fixing the Bug

Idea #3: forget about f and use POW instead

- Recall the trace of f(2,8)
  - the loop ran 4 times

b	e	r
2	8	1
4	4	1
16	2	1
256	1	1

- Trace POW(2, 8)
  - 9 recursive calls

x	y
2	8
2	7
2	6
2	5
2	4
2	3
2	2
2	1
2	0

- f is much more efficient

✘

```
int POW(int x, int y)
//@requires y >= 0;
{
  if (y == 0) return 1;
  return POW(x, y-1) * x;
}

int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 1)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  return r * b;
}
```

Fixing the Bug

Observations: with this body,

- if e == 1, then
  - e/2 == 0
  - r becomes b*r by line 13

Idea #4: make f return only when e = 0

- change the loop guard to e > 0
  - the loop always end with e = 0
- return r instead of r * b
  - that's what we had to return when e = 0

```
1.  int f(int x, int y)
2.  //@requires y >= 0;
3.  //@ensures \result == POW(x,y);
4.  {
5.    int b = x;
6.    int e = y;
7.    int r = 1;
8.    while (e > 0)
9.    //@loop_invariant e >= 0;
10.   //@loop_invariant POW(b,e) * r == POW(x,y);
11.   {
12.     if (e % 2 == 1) {
13.       r = b * r;
14.     }
15.     b = b * b;
16.     e = e / 2;
17.   }
18.   return r;
19. }
```

No special cases!

✓

Rather than getting rid of the bad case (e = 0), we make it the good case and do away with the other case (e = 1)

How's this for a movie plot?

Correctness

Did we Really Fix the Bug?

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 0)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {

  }
  return r;
}
```

- The loop invariants are still valid
  - we didn't change the body of the loop
  - we changed the loop guard
    - but we didn't use it in the validity proof

Go back and check

- Right after the loop, we know that
  - the loop guard is **false**: $e \leq 0$
  - the 1st loop invariant is **true**: $e \geq 0$

  so $e = 0$

  - the 2nd loop invariant is **true**: $b^e\ r = x^y$
    - so $x^y = b^e\ r = b^0\ r = r$

This is what f returns now

✓

Assertions

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 0)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  //@assert e == 0;
  return r;
}
```

//@assert can appear anywhere a statement is expected

Right after the loop, we know that e = 0

- We can note this with the directive
  //@assert e == 0;
  - checked only when running with **-d**
  - aborts execution if the test is **false**
- //@assert is a great way to note
  - intermediate steps of reasoning
  - expectations about execution
- These are all the run-time directives of C0
  //@requires, //@ensures, //@loop_invariant, //@assert
  There are no others

Is the Function Correct?

Correctness: for any safe input, the postconditions are true

- We just proved that, as we exit the loop, $r = x^y$
  - ➢ just before return r;
- This tells us that f will **never return the wrong result**

… but will it ***always return the right result***?

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 0)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
  //@assert e == 0;
  return r;
}
```

Is the Function Correct?

Correctness: *for any safe input, the postconditions are true*

- Can a function **never return the wrong result** and yet not necessarily ***always return the right result*** ?
  - Let's empty out the loop body in our example

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 0)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  { }
  return r;
}
```

This is legal C0 code

The loop invariants are valid
- **INIT** is unchanged
- **PRES** holds trivially

If execution were to reach return r,
- e == 0 would have to be **true**
- r would have to contain x^y

But it never reaches return r!
So the postcondition will never be **true**

This code is **not** correct.

- … only if it never returns
  - if the loop runs for ever

Termination

- We need to have a reason to believe the loop terminates
  - ➢ it doesn't run for ever

- Here's a proof of termination
  - ○ *as the loop runs,*
    - ➢ *e gets strictly smaller at each iteration*
    - ➢ *it can never become smaller than 0*
    - ➢ *the loop guard is false when e = 0*
  - ○ so the loop must terminate

✓

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 0)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
 //@assert e == 0;
 return r;
}
```

This is an **operational** proof:
we are not pointing to anything

Termination

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 0)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
 //@assert e == 0;
 return r;
}
```

- Operational proof
  - *as the loop runs, e gets strictly smaller, it can never become smaller than 0, and the loop guard is false when e = 0*
  - so the loop must terminate
- Can we prove it using point-to reasoning?
  - Yes! Here's what we need to show
  - in an arbitrary iteration of the loop,
    - if $e \geq 0$,
    - then $e' < e$
    - and $e' \geq 0$

0 is a lower bound for e

e is strictly decreasing

0 stays a lower bound for e

if e starts >= 0,
it gets strictly smaller and
can never becomes smaller than 0

- the loop guard is false when e = 0
  - $0 > 0$ is false

Termination

- Point-to-proof
  - **To show:** if e ≥ 0, then e’ < e and e’ ≥ 0

A. e > 0 by line 8 (loop guard)

B. e’ = e/2 by line 16

C. e’ < e by math

D. e’ ≥ 0 by math

However,
for termination proofs,
we will generally be Ok with an operational argument

```
1.  int f(int x, int y)
2.  //@requires y >= 0;
3.  //@ensures \result == POW(x,y);
4.  {
5.    int b = x;
6.    int e = y;
7.    int r = 1;
8.    while (e > 0)
9.    //@loop_invariant e >= 0;
10.   //@loop_invariant POW(b,e) * r == POW(x,y);
11.   {
12.     if (e % 2 == 1) {
13.       r = b * r;
14.     }
15.     b = b * b;
16.     e = e / 2;
17.   }
18.  //@assert e == 0;
19.  return r;
20. }
```

Reasoning about Code

Reasoning about C0

- C0 programs have a precise behavior
  - we can reason about them mathematically

- We used two types of reasoning
  - **Operational reasoning:** drawing conclusions about how things <u>change</u> when certain lines of code are executed
  - **Point-to reasoning:** drawing conclusions about what we <u>know to be true</u> by pointing to specific lines of code that justify them
    - boolean expressions
    - basic mathematical properties
    - variable assignments

This is operational reasoning, but really simple

Operational Reasoning

- Examples
  - Value of variables right after an assignment ✓
  - Things happening in the body of a loop from outside this loop ✗
  - Things happening in the body of a function being called ✗
  - Previously true statement after variables in it have changed ✗
- Operational reasoning is hard to do right consistently
  - ➢ very error prone!
  - We want to stay away from anything beyond simple assignments
    - ➢ except in termination proofs

If a proof about loops uses words like "always", "never", "each", you are doing operational reasoning

But operational intuitions are a good way to form conjectures that we can then prove using point-to reasoning

Point-to Reasoning

- Examples
  - Boolean conditions ✓
    - condition of an if statement in the “then” branch ✓
    - negation of the condition of an if statement in the “else” branch ✓
    - loop guard inside the body of a loop ✓
    - negation of the loop guard after the loop ✓
  - Contract annotations ✓
    - preconditions of the current function ✓
    - postconditions of a function just called ✓
    - loop invariant inside the loop body ✓
    - loop invariant after the loop ✓
    - earlier fully justified assertions ✓
  - Math ✓
    - laws of logic ✓
    - some laws of arithmetic ✓
  - Value of variables right after an assignment ✓

Point-to Reasoning: Tips and Tricks

- When reasoning about an earlier loop, **pretend the body of the loop is not there**
  - Only rely on the **loop guard** and **loop invariants**

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 0)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {

  }
  //@assert r = POW(x,y);
  return r;
}
```

When reasoning about
an earlier loop,
pretend its body is not there

Point-to Reasoning: Tips and Tricks

- When reasoning about a function being called, **pretend the body of the function is not there**
  - ➢ unless it's a specification function
  - ○ Only rely on its **contracts**

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{

}

int main() {
  int k = f(2,5);
  //@assert k >= 0;
  return 0;
}
```

When reasoning about the outcome of a function call, pretend its body is not there

Safety

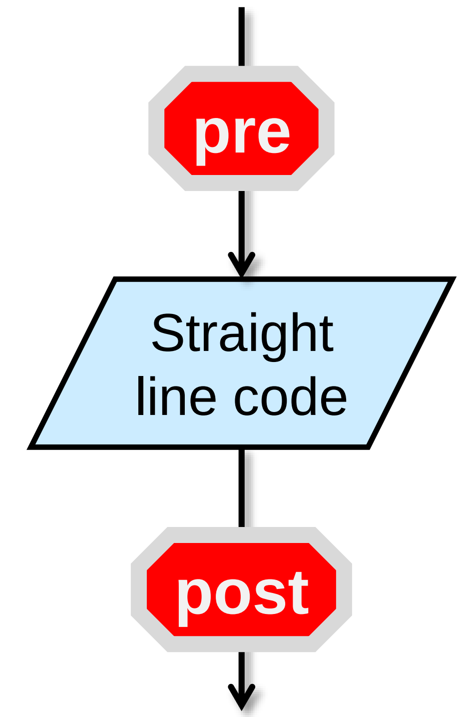
- The inputs of a function call satisfy the function’s preconditions
  - we will generalize this definition in the future

We will exclusively use point-to reasoning to justify safety

Correctness

- The postconditions of a function will be true on any call that satisfies the preconditions
  - We will not need to generalize this definition

Straight Line Functions



A non-recursive function without loops

- Proving correctness amounts to combining assignments
  - **To show:** \result = x

A. b = x — by line 5

B. r = 1 — by line 7

C. \result = r * b — by line 8

D. r * b = x — by math on A, B, C

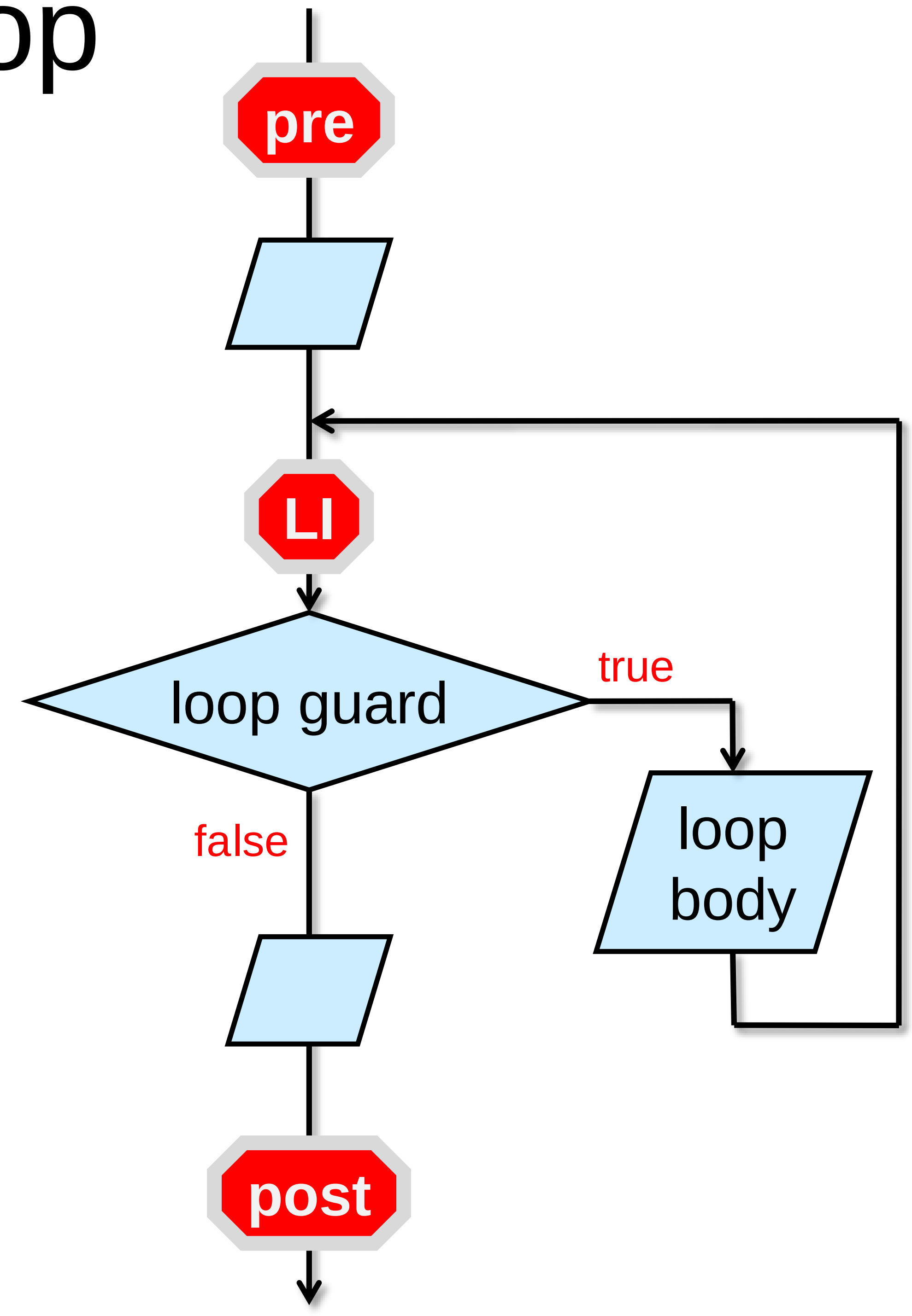
```
1. int f(int x, int y)
2. //@requires y >= 0;
3. //@ensures \result == x;
4. {
5.   int b = x;
6.   int e = y;
7.   int r = 1;
8.   return r * b;
9. }
```

Functions with One Loop

- Proving correctness involves 3 steps
  - Show that the loop invariants are ***valid***
    - **INIT:** the LI are true initially
    - **PRES:** the LI are preserved by an arbitrary iteration of the loop
  - **EXIT:** the LI and the negation of the loop guard imply the postcondition
  - **TERM:** the loop terminates

That's exactly what we did for our mystery function

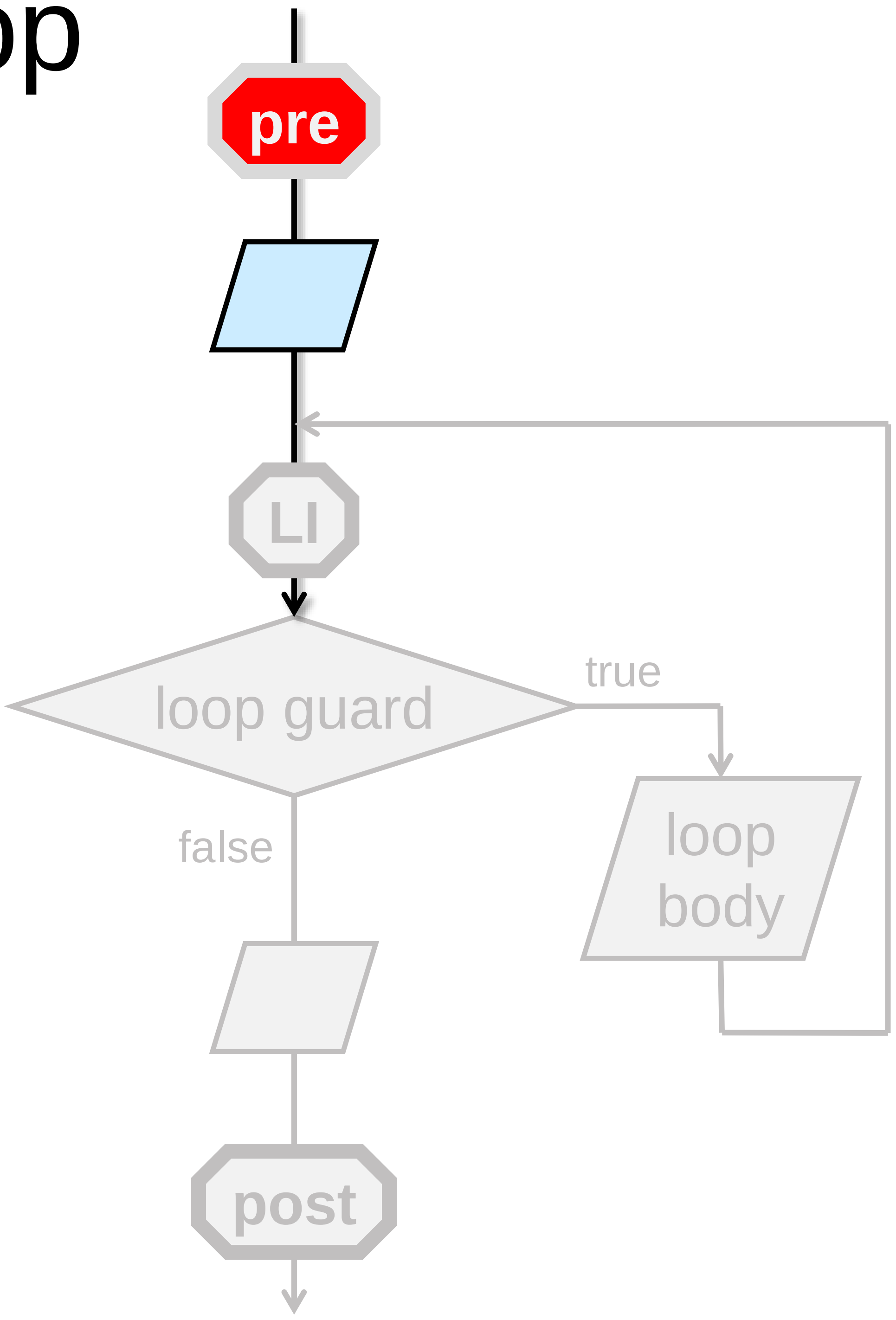
These steps can be proved in any order



Functions with One Loop

INIT: the loop invariant is true initially

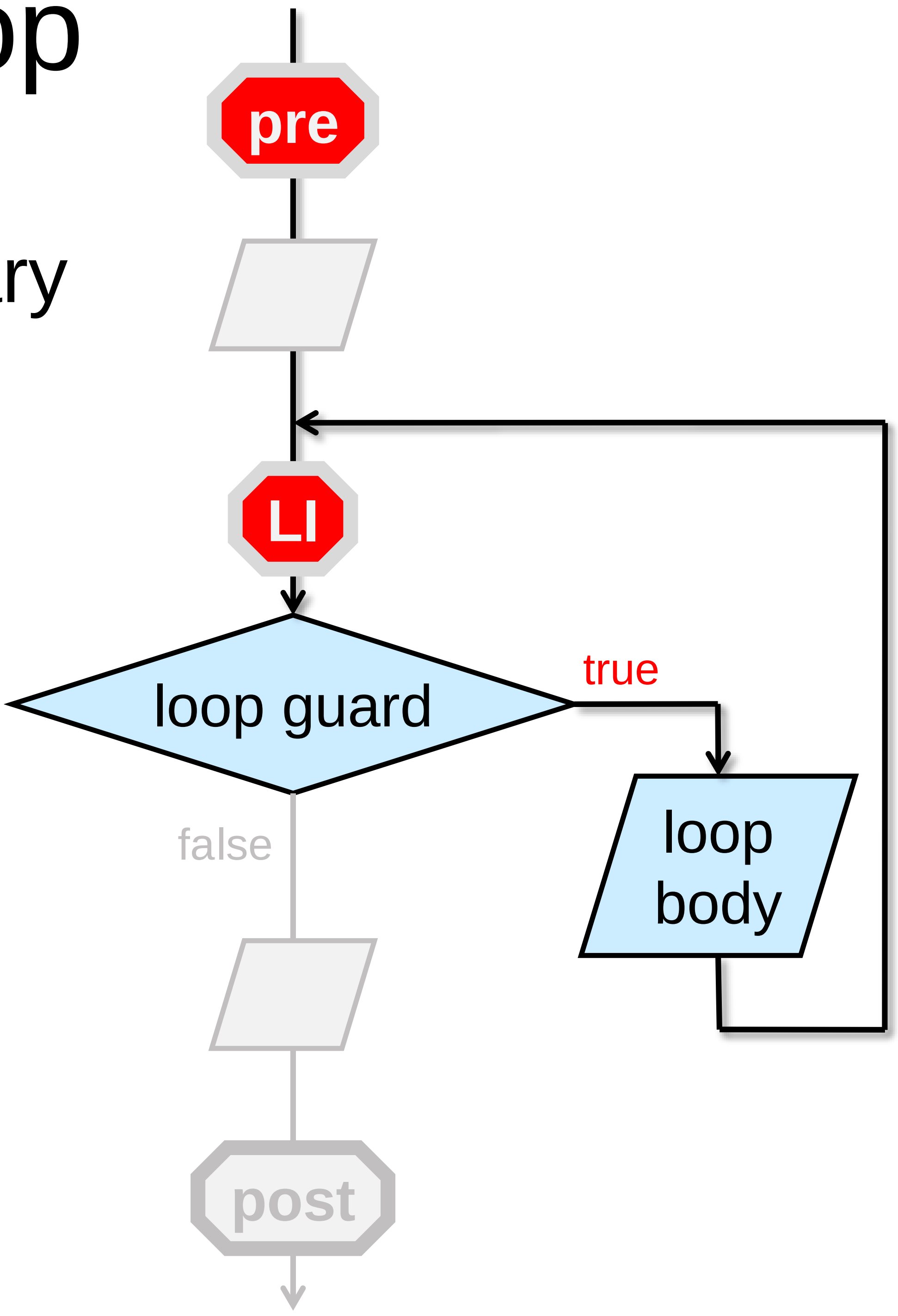
- proved by point-to reasoning typically using
  - the preconditions
  - simple assignments before the loop



Functions with One Loop

PRES: the LI are preserved by an arbitrary iteration of the loop

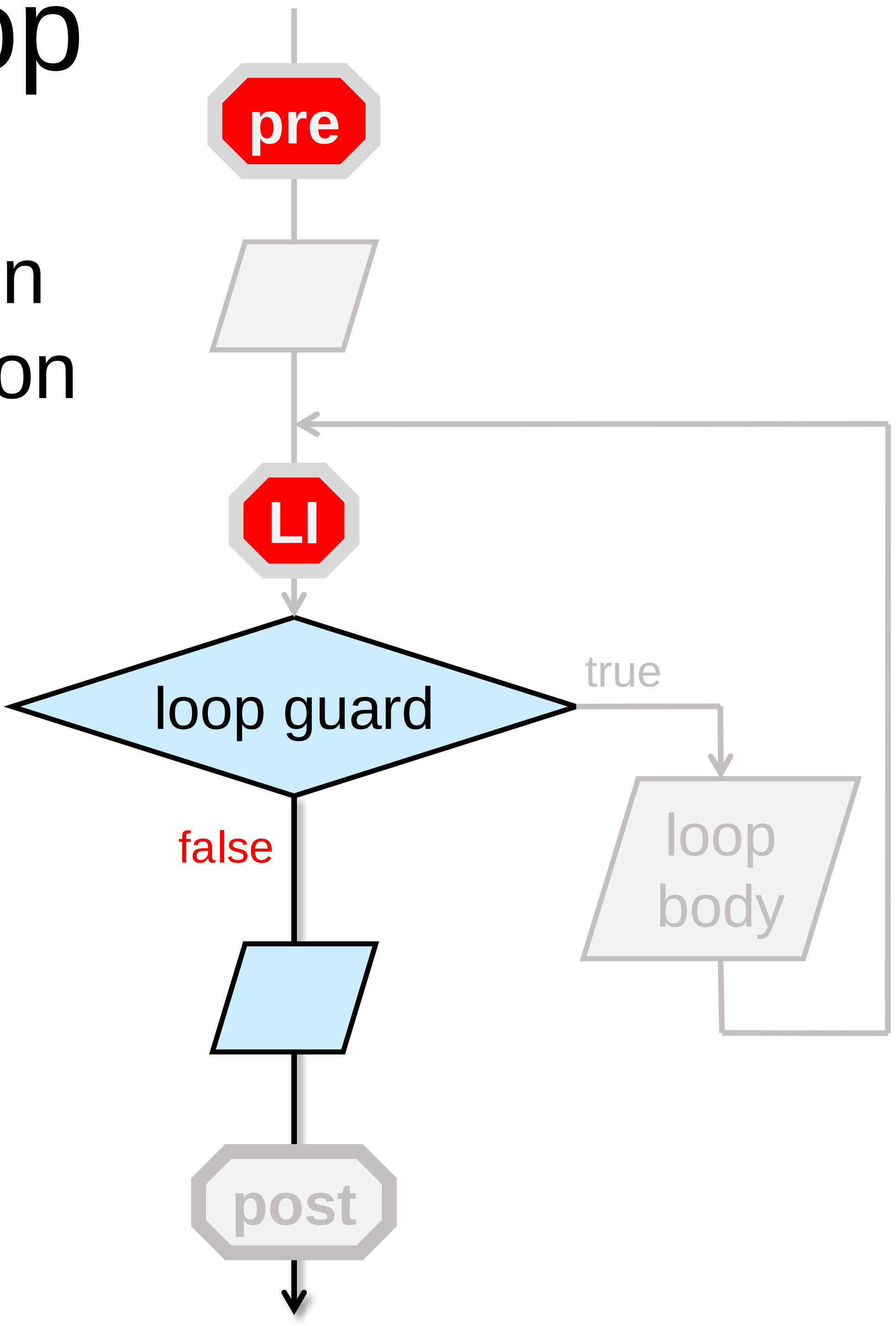
- proved by <u>point-to reasoning</u> typically using
  - the assumption that the LI is true at the beginning of the iteration
  - the loop guard being true
    - we are running an iteration
  - simple assignments and conditionals in the loop body
  - the preconditions (sometimes)



Functions with One Loop

EXIT: the loop invariants and the negation of the loop guard imply the postcondition

- proved by <u>point-to reasoning</u> typically using
  - the loop invariant
  - the negation of the loop guard
  - simple assignments and conditionals after the loop

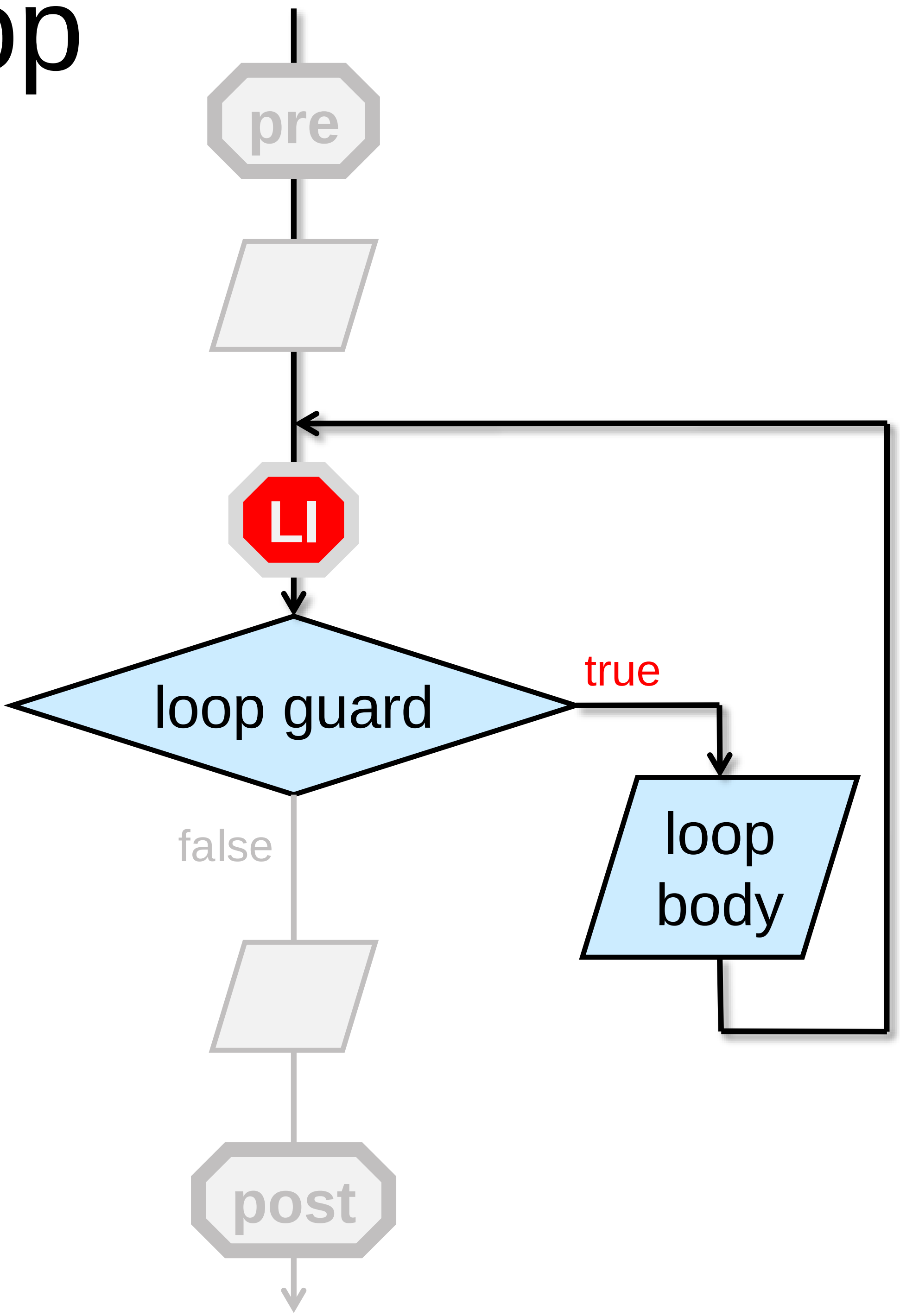


Functions with One Loop

TERM: the loop terminates

- proved by <u>operational reasoning</u> typically using
  - the assumption that the LI is true at the beginning of the iteration
  - the loop guard
  - simple assignments and conditionals in the loop body

But it can also be proved by **point-to reasoning**



Functions with One Loop

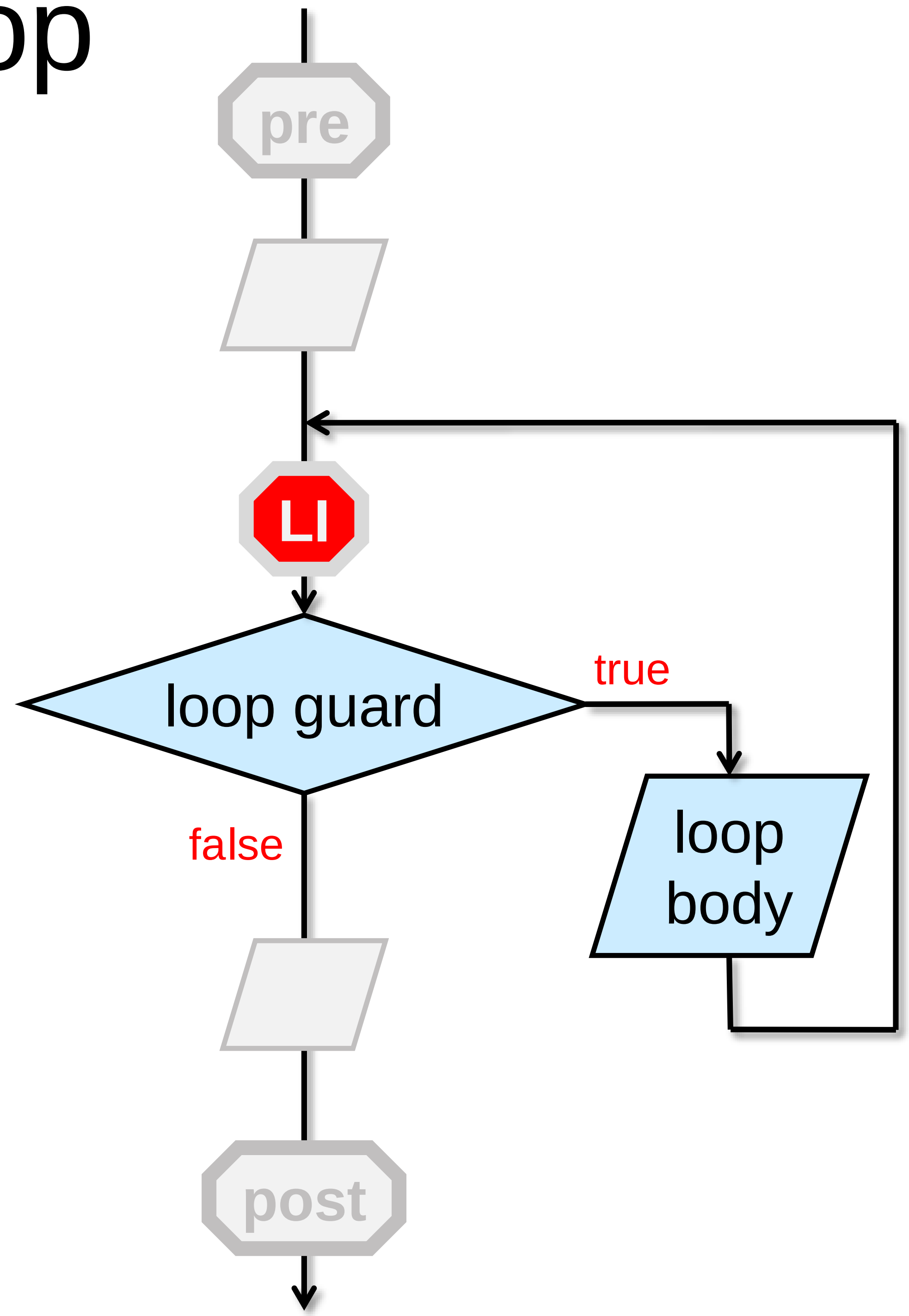
TERM: the loop terminates

- Format of a termination proof using <u>operational reasoning</u>

  "*on an arbitrary iteration of the loop, the quantity ______ gets strictly smaller but it can't ever get smaller than ______" on which the loop guard is false*

  or

  "*on an arbitrary iteration of the loop, the quantity ______ gets strictly bigger but it can't ever get bigger than ______" on which the loop guard is false*



A **quantity** may be an **expression**, not necessarily a variable

More Complex Functions

- These techniques can be extended
  - but we will rarely deal with functions with more than one loop

- We can also factor out nested loops and the like into helper functions
  - and then use the technique we just saw

Seriously??

- *All these proofs and complicated reasoning seem overkill!*
  - *the mystery function wasn't all that hard after all*
  - *we could just spot what was going on*
- Yes, but it won't be that easy for more complex functions
  - the technique we saw is **systematic** and **scalable**
  - reasoning about code will pay off
- Point-to reasoning is what we do in our head all the time when programming
  - writing it down as loop invariants and contracts makes it easier not to get confused
  - and the **-d** flag will catch lingering issues at run time

Epilogue

Where are we?

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 0)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
 //@assert e == 0;
return r;
}
```

- We fully documented f
  - function contracts
  - loop invariants
  - key assertions
- We fixed the bug
- We gave mathematical proofs that
  - all the calls it makes are safe
  - it is correct
- Let’s enjoy the fruit of our labor with some more testing!

Sanity Checks

- Let's do a last round of testing

Linux Terminal

```
# coin -d mystery.c0
C0 interpreter (coin) …
--> f(2, 0);
1 (int)
--> f(2, 1);
2 (int)
--> f(2, 7);
128 (int)
--> f(2, 8);
256 (int)
--> f(2, 19);
524288 (int)
--> f(2, 31);
-2147483648 (int)
--> f(2, 32);
0 (int)
-->
```

Bug fixed!

Looking good

Plausible

What?

What?

```
int f(int x, int y)
//@requires y >= 0;
//@ensures \result == POW(x,y);
{
  int b = x;
  int e = y;
  int r = 1;
  while (e > 0)
  //@loop_invariant e >= 0;
  //@loop_invariant POW(b,e) * r == POW(x,y);
  {
    if (e % 2 == 1) {
      r = b * r;
    }
    b = b * b;
    e = e / 2;
  }
 //@assert e == 0;
return r;
}
```

The story continues …