Integers

Number Representation

Representing Numbers

- We, people, have many ways to represent numbers

7

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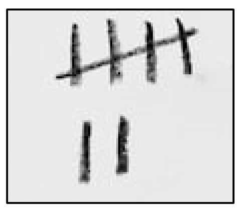
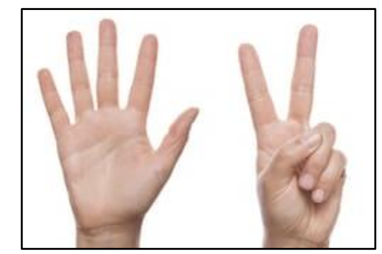




seven

sjö

- They all express the same concept
  - that some collection consists of *seven* things

Decimal Numbers

7

- The **decimal representation** is succinct and systematic
  - It uses ten symbols: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

    This comes from us having 10 fingers
    - each represents a number between 0 and 9
    - they are the **digits**
  - "ten" is the **base**
- Any number is represented as a sequence of digits
  - the **position** *i* of a digit *d* indicates its importance
    - it contributes $d \times 10^i$ to the value of the number
  - the value of the number is the sum of the contribution of each position

1 is at position 3 | **2** is at position 2 | **0** is at position 1 | **9** is at position 0

$$\mathbf{1209} = \mathbf{1}\times 10^3 + \mathbf{2}\times 10^2 + \mathbf{0}\times 10^1 + \mathbf{9}\times 10^0$$

10 is the base

Decimal Numbers

- *It uses ten symbols:*

  0, 1, 2, 3, 4, 5, 6, 7, 8, 9

  - *each represents a number between 0 and 9*

- Different languages use other symbols

	0	1	2	3	4	5	6	7	8	9
Arabic	٠	١	٢	٣	٤	٥	٦	٧	٨	٩
Bengali	০	১	২	৩	৪	৫	৬	৭	৮	৯
Chinese (simple)	〇	一	二	三	四	五	六	七	八	九
Chinese (complex)	零	壹	貳	參	肆	伍	陸	柒	捌	玖
Chinese 花碼 (huā mǎ)	〇	〡	〢	〣	〤	〥	〦	〧	〨	〩
Devanagari	०	१	२	३	४	५	६	७	८	९
Ethiopic		፩	፪	፫	፬	፭	፮	፯	፰	፱
Gujarati	૦	૧	૨	૩	૪	૫	૬	૭	૮	૯
Gurmukhi	੦	੧	੨	੩	੪	੫	੬	੭	੮	੯
Kannada	೦	೧	೨	೩	೪	೫	೬	೭	೮	೯
Khmer	០	១	២	៣	៤	៥	៦	៧	៨	៩
Lao	໐	໑	໒	໓	໔	໕	໖	໗	໘	໙
Limbu	᥆	᥇	᥈	᥉	᥊	᥋	᥌	᥍	᥎	᥏
Malayalam	൦	൧	൨	൩	൪	൫	൬	൭	൮	൯
Mongolian	᠐	᠑	᠒	᠓	᠔	᠕	᠖	᠗	᠘	᠙
Myanmar	၀	၁	၂	၃	၄	၅	၆	၇	၈	၉
Oriya	୦	୧	୨	୩	୪	୫	୬	୭	୮	୯
Tamil	௦	௧	௨	௩	௪	௫	௬	௭	௮	௯
Telugu	౦	౧	౨	౩	౪	౫	౬	౭	౮	౯
Thai	๐	๑	๒	๓	๔	๕	๖	๗	๘	๙
Tibetan	༠	༡	༢	༣	༤	༥	༦	༧	༨	༩
Urdu	۰	۱	۲	۳	۴	۵	۶	۷	۸	۹

Decimal Numbers

- Positional systems make it easy to do calculations
  - **addition** is done position by position

We used our
10 fingers for that

```
   1  1      carry
    1209
+   9517
--------
   10726
```

9+7 = 6
with a carry of 1

  - **multiplication** is done as iterated additions

```
     1209
 ×    402
---------
     2418
       0
+ 4836
---------
   486018
```

Binary Numbers

There are two voltages in computer chips: on and off
(in reality, it's more complicated)

- Computers have *one* way to represent information: **binary**
  - they use two symbols, 0 and 1

• 1 = on
• 0 = off

- In particular, they represent numbers in positional notation using base 2
  - that's the **binary representation**

That's what we call the binary digits 0 and 1

- Any number is represented as a sequence of **bits**
  - the **position** *i* of a bit *b* indicates its importance
    - it contributes $b \times 2^i$ to the value of the number
  - the value of the number is the sum of the contribution of each position

1 is at position 5

… **0** is at position 3 …

1 is at position 0

$$\mathbf{100101} = \mathbf{1}\times 2^5 + \mathbf{0}\times 2^4 + \mathbf{0}\times 2^3 + \mathbf{1}\times 2^2 + \mathbf{0}\times 2^1 + \mathbf{1}\times 2^0$$

2 is the base

Binary Numbers

- Positional systems make it easy to do calculations
  - **addition** is done position by position

```
  1111      carry
   11011
+   1110
--------
  101001
```

Here, 1+1 = 0 with a carry of 1

This works **exactly** as with decimal numbers

  - **multiplication** is done as iterated additions

```
     1010
  ×   101
  -------
     1010
       0
+  1010
---------
   110010
```

Converting Binary Numbers to Decimal

- Simply use the positional formula and carry out the calculation in decimal

$$
\begin{aligned}
100101_{[2]} &= 1\times2^5 + 0\times2^4 + 0\times2^3 + 1\times2^2 + 0\times2^1 + 1\times2^0 \\
&= 32 + 0 + 0 + 4 + 0 + 1 \\
&= 37_{[10]}
\end{aligned}
$$

Base

- Alternatively, use ***Horner's rule***:

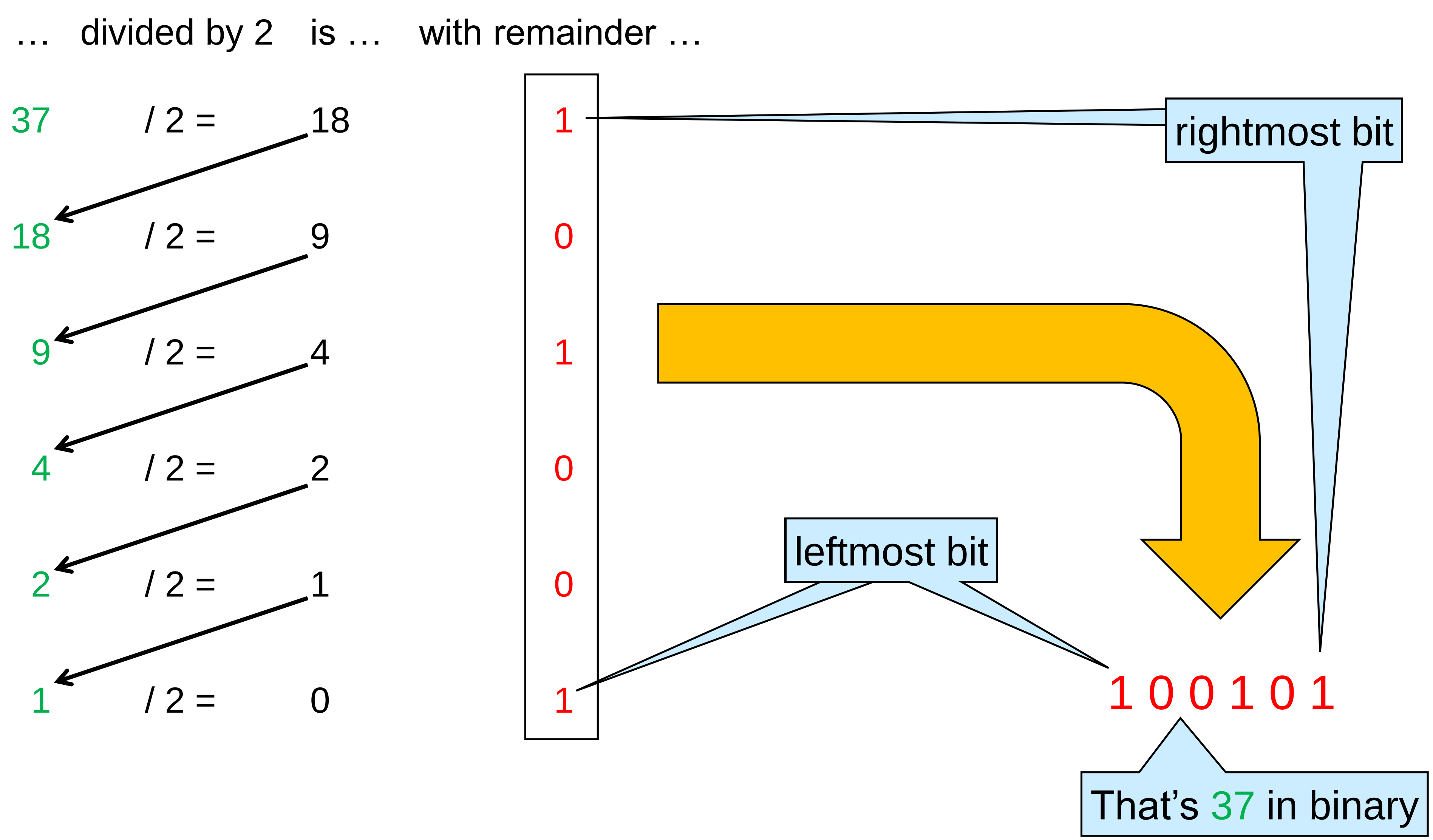
$$
\begin{aligned}
100101_{[2]} &= ((((1\times2 + 0)\times2 + 0)\times2 + 1)\times2 + 0)\times2 + 1 \\
&= (((2\times2 + 0)\times2 + 1)\times2 + 0)\times2 + 1 \\
&= ((4\times2 + 1)\times2 + 0)\times2 + 1 \\
&= (9\times2 + 0)\times2 + 1 \\
&= 18\times2 + 1 \\
&= 37_{[10]}
\end{aligned}
$$

That's because

$1\times2^5 + 0\times2^4 + 0\times2^3 + 1\times2^2 + 0\times2^1 + 1\times2^0 = ((((1\times2 + 0)\times2 + 0)\times2 + 1)\times2 + 0)\times2 + 1$

Converting Decimal Numbers to Binary

- Repeatedly divide the number by 2, harvesting the remainder, until we reach 0
  - the remainder is either 0 or 1
  - the binary representation comes out from right to left



Hexadecimal Numbers

- Binary is fine for computers, but unwieldy for people

  1100000011111111101110

  - hard to remember
  - hard to communicate

- The **hexadecimal representation** makes things simpler
  - it uses 16 symbols: the numbers 0 to 9 and the letters A to F
    - each represents a number between 0 and 15
    - they are the **hex digits**

The decimal to binary to hexadecimal conversion table (0 to 15)

$0_{[16]}$	$0000_{[2]}$	$0_{[10]}$	$8_{[16]}$	$1000_{[2]}$	$8_{[10]}$
$1_{[16]}$	$0001_{[2]}$	$1_{[10]}$	$9_{[16]}$	$1001_{[2]}$	$9_{[10]}$
$2_{[16]}$	$0010_{[2]}$	$2_{[10]}$	$A_{[16]}$	$1010_{[2]}$	$10_{[10]}$
$3_{[16]}$	$0011_{[2]}$	$3_{[10]}$	$B_{[16]}$	$1011_{[2]}$	$11_{[10]}$
$4_{[16]}$	$0100_{[2]}$	$4_{[10]}$	$C_{[16]}$	$1100_{[2]}$	$12_{[10]}$
$5_{[16]}$	$0101_{[2]}$	$5_{[10]}$	$D_{[16]}$	$1101_{[2]}$	$13_{[10]}$
$6_{[16]}$	$0110_{[2]}$	$6_{[10]}$	$E_{[16]}$	$1110_{[2]}$	$14_{[10]}$
$7_{[16]}$	$0111_{[2]}$	$7_{[10]}$	$F_{[16]}$	$1111_{[2]}$	$15_{[10]}$

Hexadecimal Numbers

$0_{[16]}$	$0000_{[2]}$	$0_{[10]}$	$8_{[16]}$	$1000_{[2]}$	$8_{[10]}$
$1_{[16]}$	$0001_{[2]}$	$1_{[10]}$	$9_{[16]}$	$1001_{[2]}$	$9_{[10]}$
$2_{[16]}$	$0010_{[2]}$	$2_{[10]}$	$A_{[16]}$	$1010_{[2]}$	$10_{[10]}$
$3_{[16]}$	$0011_{[2]}$	$3_{[10]}$	$B_{[16]}$	$1011_{[2]}$	$11_{[10]}$
$4_{[16]}$	$0100_{[2]}$	$4_{[10]}$	$C_{[16]}$	$1100_{[2]}$	$12_{[10]}$
$5_{[16]}$	$0101_{[2]}$	$5_{[10]}$	$D_{[16]}$	$1101_{[2]}$	$13_{[10]}$
$6_{[16]}$	$0110_{[2]}$	$6_{[10]}$	$E_{[16]}$	$1110_{[2]}$	$14_{[10]}$
$7_{[16]}$	$0111_{[2]}$	$7_{[10]}$	$F_{[16]}$	$1111_{[2]}$	$15_{[10]}$

- 1 hex digit corresponds to 4 bits
  - and vice versa
- This makes converting between hex and binary very simple
  - hex to binary: replace each hex digit with the corresponding 4 bits
  - binary to hex: replace each group of 4 bits with the corresponding hex digit

1100	0000	1111	1111	1110	1110
C	0	F	F	E	E

- People find it a lot simpler to remember and communicate binary information in hexadecimal
  - and not just numbers

Not all hex words are this cute, though!

Hexadecimal Numbers

- Any number has a positional representation in hex as a sequence of hex digits
  - the **position** *i* of a hex digit *h* indicates its importance
    - it contributes $h \times 16^i$ to the value of the number
  - the value of the number is the sum of the contribution of each position

$$\mathbf{C0FFEE} = \mathbf{C}\times 16^5 + \mathbf{0}\times 16^4 + \mathbf{F}\times 16^3 + \mathbf{F}\times 16^2 + \mathbf{E}\times 16^1 + \mathbf{E}\times 16^0$$

After plugging in 12 for C, etc, that's 12648430 in decimal

- We can also do arithmetic in hex
  - but hex is primarily used to represent two types of non-numerical data
    - memory addresses — next lecture
    - bit patterns — later in this lecture

Numbers in C0

- All numbers in C0 have type int
- We can enter numbers in C0
  - in decimal
  - in hexadecimal
    - by prefixing them with **0x**
- Internally, it stores them in binary
  - but there is no way to enter numbers in binary
- C0 always prints numbers back to us in decimal

Linux Terminal

```
# coin
C0 interpreter (coin) …
…
--> 0xC0FFEE;
12648430 (int)
--> 0xC0FFEE == 12648430;
true (bool)
```

When we enter C0FFEE in hex ..

… coin responds it's 12648430 in decimal

C0FFEE and 12648430 are two different ways of entering the **same number**

Numbers in C0

- *C0 always prints numbers back in decimal*
- Use the function int2hex in the <util> library to display a number in hexadecimal
  - as a string, not an int

Loads the <util> library when starting coin

Linux Terminal

```
# coin -l util
C0 interpreter (coin) …
…
--> int2hex(0xC0FFEE);
"00C0FFEE" (string)
--> int2hex(12648430);
"00C0FFEE" (string)
```

There is no int2bin

You can write your own!

Fixed-size Number Representation

Machine Words

- Computers store and manipulate binary data
  - *everything is a bit in a computer*

- Computer hardware processes batches of *k* bits in parallel
  - a batch of *k* bits is called a **machine word**
  - nowadays, a typical value of *k* is 32

- Computation is very **efficient** on whole words
  - but less so on parts of words

- Most programming languages use a word to represent an int
  - **in C0, an int is always 32 bits long**
    - internally, 37 is not represented as 100101
      but as 00000000000000000000000000100101

32 bits

Fixed-size Numbers

- A k-bit computer uses **exactly k bits** to represent an int

That's a computer whose words are k bits long

This will simplify our examples

- In our discussion, we will assume that k = 4
  - ➢ but in C0, an int is always 32 bits long

- In a 4-bit computer, 6 is not represented as 110 but as 0110
  - ➢ Numbers have a *fixed-size* in a computer

4 bits

Numbers in Math vs. in a Computer

- In math, there are infinitely many numbers
  - we visualize them as an infinite **number line**

The beginning of the number line with numbers in decimal and binary and hex

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101	1110	1111	10000	10001	10010	10011	10100	10101
0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15

- In a 4-bit computer, there are **finitely many** numbers
  - exactly $16 = 2^4$
  - the line is ***finite***

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0000	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	1110	1111
0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F

4 bits

- On a k-bit computer, we can represent only 2^k distinct numbers
  - C0 can represent only 2^{32} distinct numbers

Numbers in a Computer

- *In a 4-bit computer, we can represent only 2^4 distinct numbers*

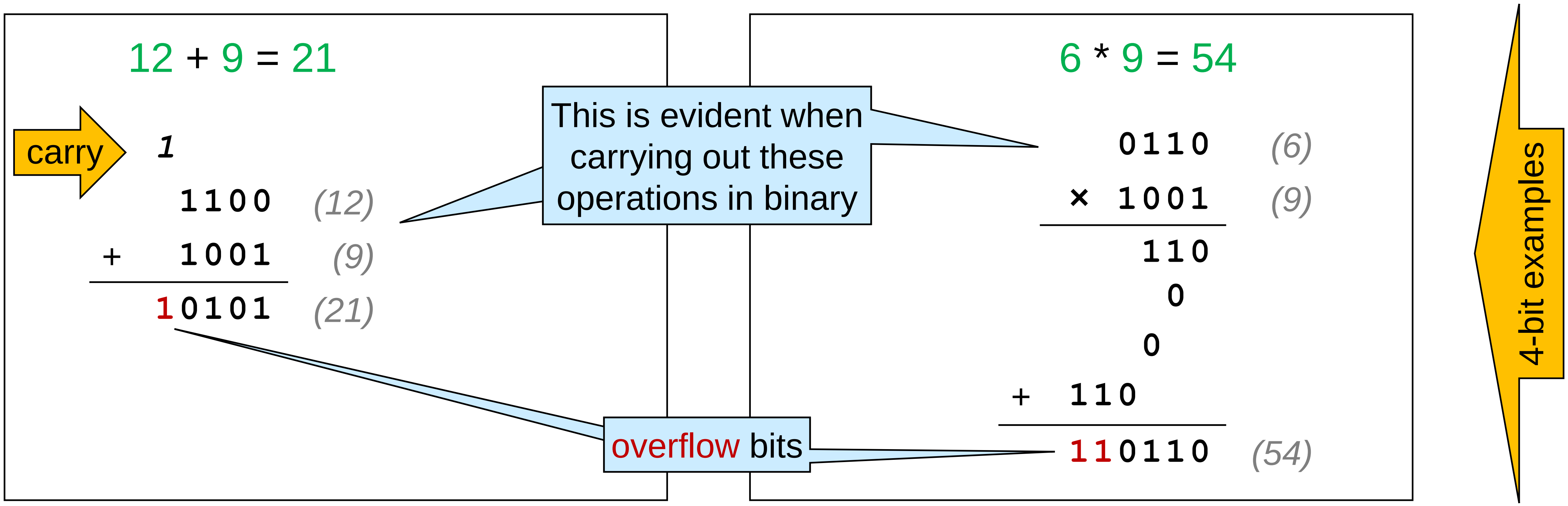
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0000	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	1110	1111
0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F

4 bits

- We cannot represent numbers larger than what fits in 4 bits
  - e.g., 21
    - in binary it’s 10101, but that requires 5 bits
- Even if we avoid writing larger numbers in a program, they may emerge during computation
  - intermediate results need to be stored in a word in memory!

Overflow

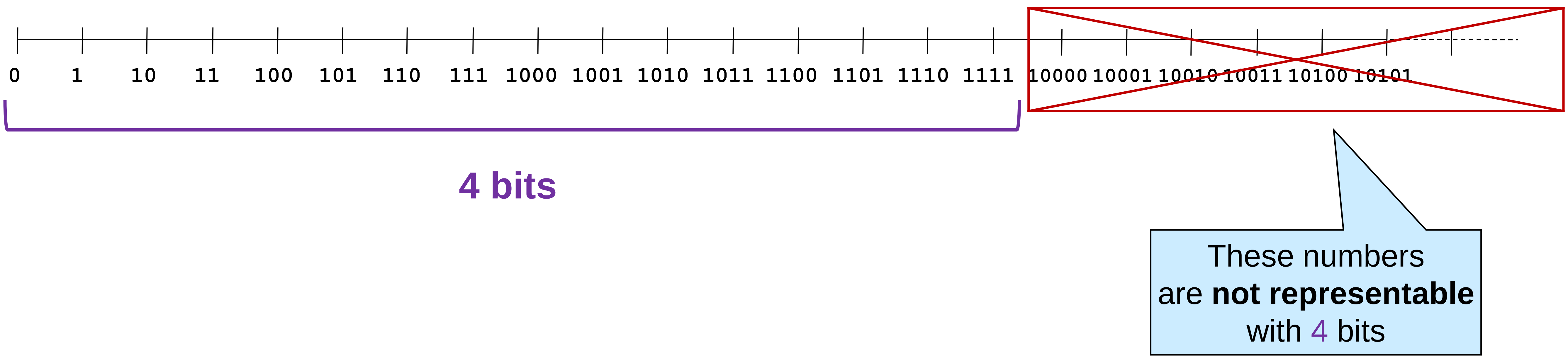
- The result of adding two int's may not fit into a k bits word
  - it may be a k+1 bit number!
  - the result may be even longer when multiplying two int's



- We have an **overflow** when the result of an operation doesn't fit in a machine word
  - *k* bit operands, but the result has more than *k* bits

How to Deal with Overflow?

- The result of an operation does not fit into a k-bit word



- Two common approaches to handling overflow
  1. Raise an error or an exception
     - an *error* aborts the program
     - an *exception* is an error that can be handled to continue computation
  2. Continue execution in some meaningful way

Handling Overflow as Error

- Signaling an error is not always the right thing to do
  - The **Ariane 5** rocket exploded on its first launch because an unexpected overflow raised an unhandled exception

```
L_M_BV_32 := TBD.T_ENTIER_32S ((1.0/C_M_LSB_BV) *
if L_M_BV_32 > 32767 then
   P_M_DERIVE(T_ALG.E_BV) := 16#7FFF#;
elsif L_M_BV_32 < -32768 then
   P_M_DERIVE(T_ALG.E_BV) := 16#8000#;
else
   P_M_DERIVE(T_ALG.E_BV) := UC_16S_EN_16NS(TDB
end if;
P_M_DERIVE(T_ALG.E_BH) :=
    UC_16S_EN_16NS (TDB.T_ENTIER_16S ((1.0/C_M_LS
```

Handling Overflow as Error

- Treating overflows as errors makes it **hard to write correct code** involving ints
  - hard to debug
  - hard to reason about
- Example
  - $n + (n - n)$ and $(n + n) - n$ are equal **in math**
  - but with fixed size numbers, they may yield different outcomes
    - $n + (n - n)$ is **always** equal to n
    - $(n + n) - n$ may overflow

Writing one or the other is the same

Writing one or the other is **not** the same; although it feels like it is

- People instinctively use math when writing code
  - we want the laws of arithmetic to hold
    - whenever possible

Modular Arithmetic

Continuing Computation on Overflow

- Instead of aborting execution, just **ignore the overflow bits**

12 + 9 = …

carry *1*

1100 *(12)*

\+ 1001 *(9)*

~~1~~0101 *(~~21~~ **5**)*

6 * 9 = …

0110 *(6)*

× 1001 *(9)*

110

0

0

\+ 110

~~11~~0110 *(~~54~~ **6**)*

overflow bits

4-bit examples

- The result of the operation is what fits in the word

… = 5

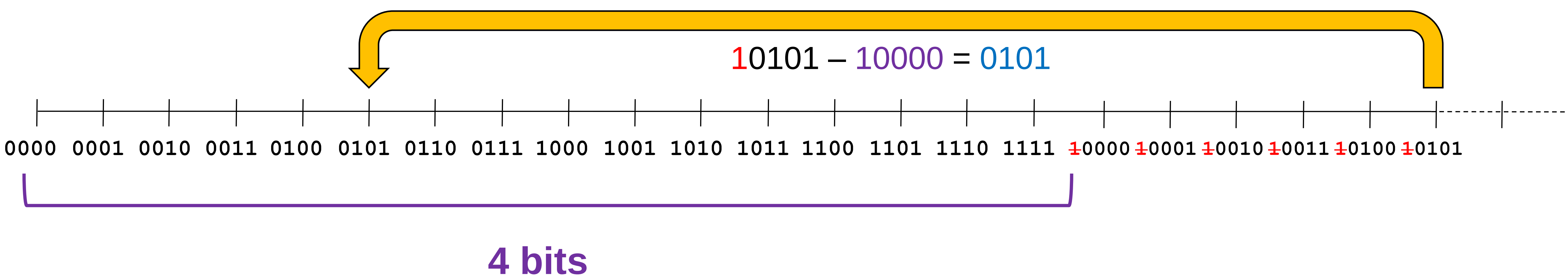
… = 6

  - This is **not** the correct mathematical value
    - *but does it relate to it in any way?*

Ignoring the Overflow Bits

```
      1
     1100      (12)
+    1001       (9)
 ---------
    ~~1~~0101  (~~21~~ 5)
```

- Throwing out the overflow bit amounts to subtracting **10000** from the result
  - that's 16 in decimal

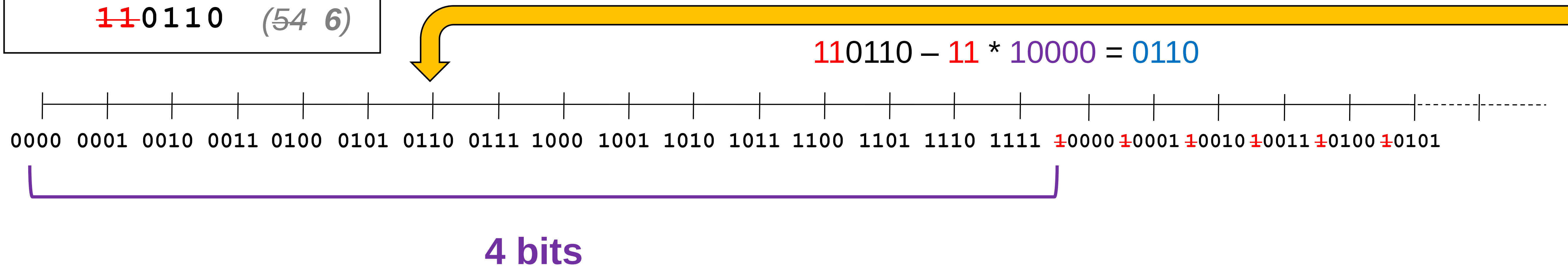


- Note that 16 is 2^4
  - 4 is how many bits our words have

Ignoring the Overflow Bits

```
      0110     (6)
  ×   1001     (9)
  ----------
       110
        0
       0
 +   110
 ------------
   110110     (54 6)
```

- Throwing out the overflow bits amounts to subtracting a *multiple* of **10000** from the result
  - that's 16 in decimal



- In general, we subtract as many multiples of 16 (= 2^4) as necessary so that the result fits in 4 bits
- Ignoring the overflow bits computes the result **modulo 16**

Computing Modulo *n*

$n > 1$

- Evaluate an expression normally but return the remainder of dividing it by *n*
  - a number between 0 and *n-1*
  - $12 + 9 =_{\text{mod } 16} 5$
  - $9 * 6 =_{\text{mod } 16} 6$
- This is called **modular arithmetic**
- Modular arithmetic works just like traditional arithmetic

Modular Arithmetic

- Modular arithmetic obeys the **same laws** as traditional arithmetic
  - *for expressions involving + and * so far*

$x + y =_{\text{mod } n} y + x$	Commutativity of addition
$(x + y) + z =_{\text{mod } n} x + (y + z)$	Associativity of addition
$x + 0 =_{\text{mod } n} x$	Additive unit
$x * y =_{\text{mod } n} y * x$	Commutativity of multiplication
$(x * y) * z =_{\text{mod } n} x * (y * z)$	Associativity of multiplication
$x * 1 =_{\text{mod } n} x$	Multiplicative unit
$x * (y + z) =_{\text{mod } n} x * y + x * z$	Distributivity
$x * 0 =_{\text{mod } n} 0$	Annihilation

- We use these laws ***implicitly*** every time we do arithmetic
  - in particular when writing programs

Handling Overflow in C0

- C0 discards overflow bits
  - **C0 handles overflow using modular arithmetic**
  - numerical expressions are computed modulo 2^{32}
    - because C0 assumes 32-bit words

- This makes it easy to reason about programs
  - modular arithmetic works like traditional arithmetic
    - we apply it innately
  - there is no need to consider special cases for overflow
    - for expressions using + and * so far

Overflow does <u>not</u> abort computation in C0

Reasoning about int Code

- This function always returns "Good"

```
string foo(int x) {
  int z = 1+x;
  if (x+1 == z)
    return "Good";
  else
    return "Bad";
}
```

This is equivalent to
x+1 == 1+x
by substitution

x+1 == 1+x
is always **true**
by commutativity of addition

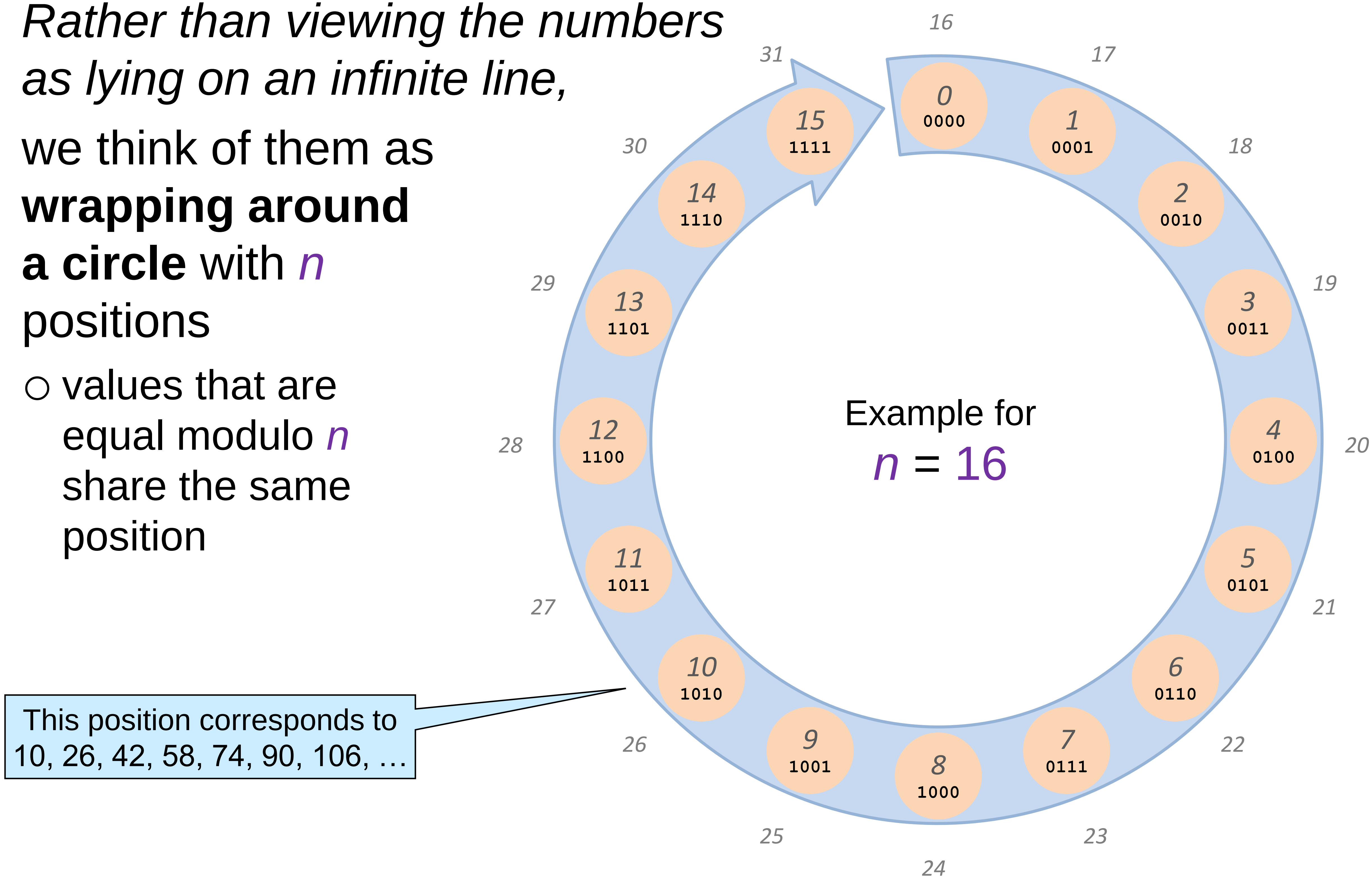
(modulo 2^{32})

- We don't need to worry about 1+x or x+1 overflowing
  - they may, but that doesn't matter
    - overflow doesn't abort computation
    - the laws of (modular) arithmetic tell us they always evaluate to the same value

What does Computing Modulo *n* Mean?

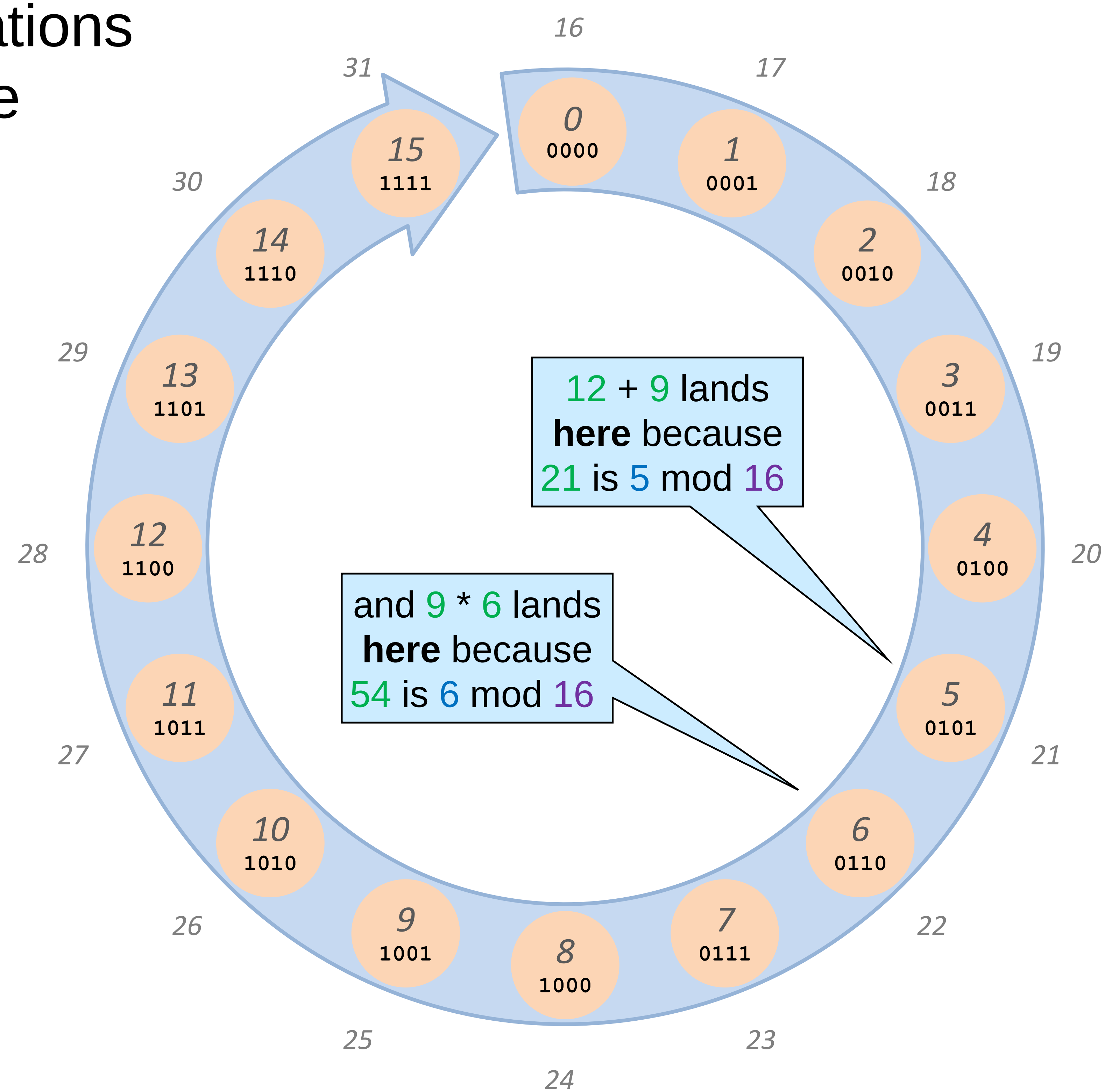
- *Rather than viewing the numbers as lying on an infinite line,*

  we think of them as **wrapping around a circle** with *n* positions
  - values that are equal modulo *n* share the same position



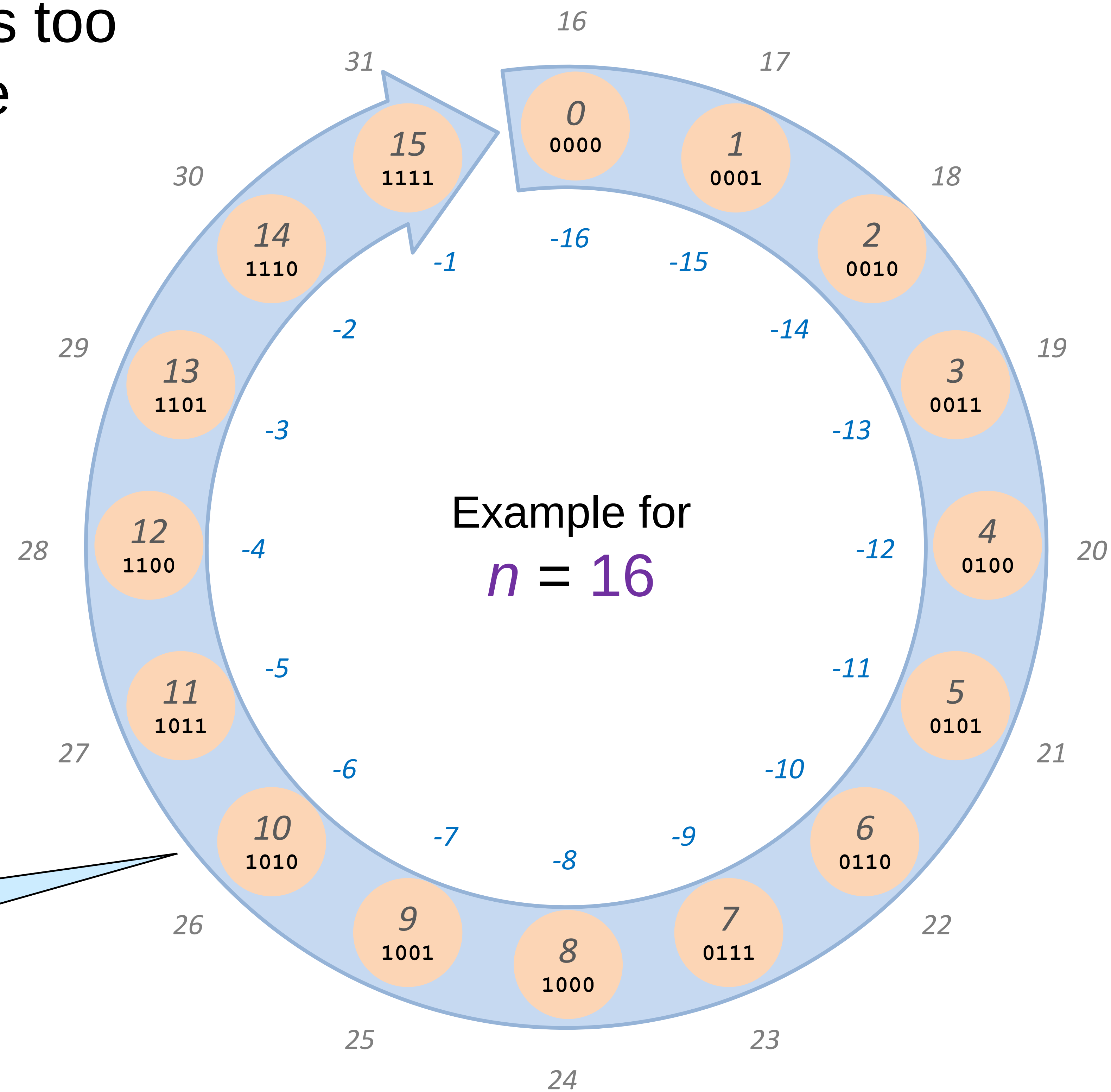
What does Computing Modulo n Mean?

- We carry out computations normally but return the position of the result on the circle
  - $12 + 9 =_{\text{mod } 16} 5$
  - $9 * 6 =_{\text{mod } 16} 6$
- Then, addition corresponds to *moving clockwise* around the circle
  - to compute 12 + 9 start from 12 and step 9 times clockwise



What about the Negatives?

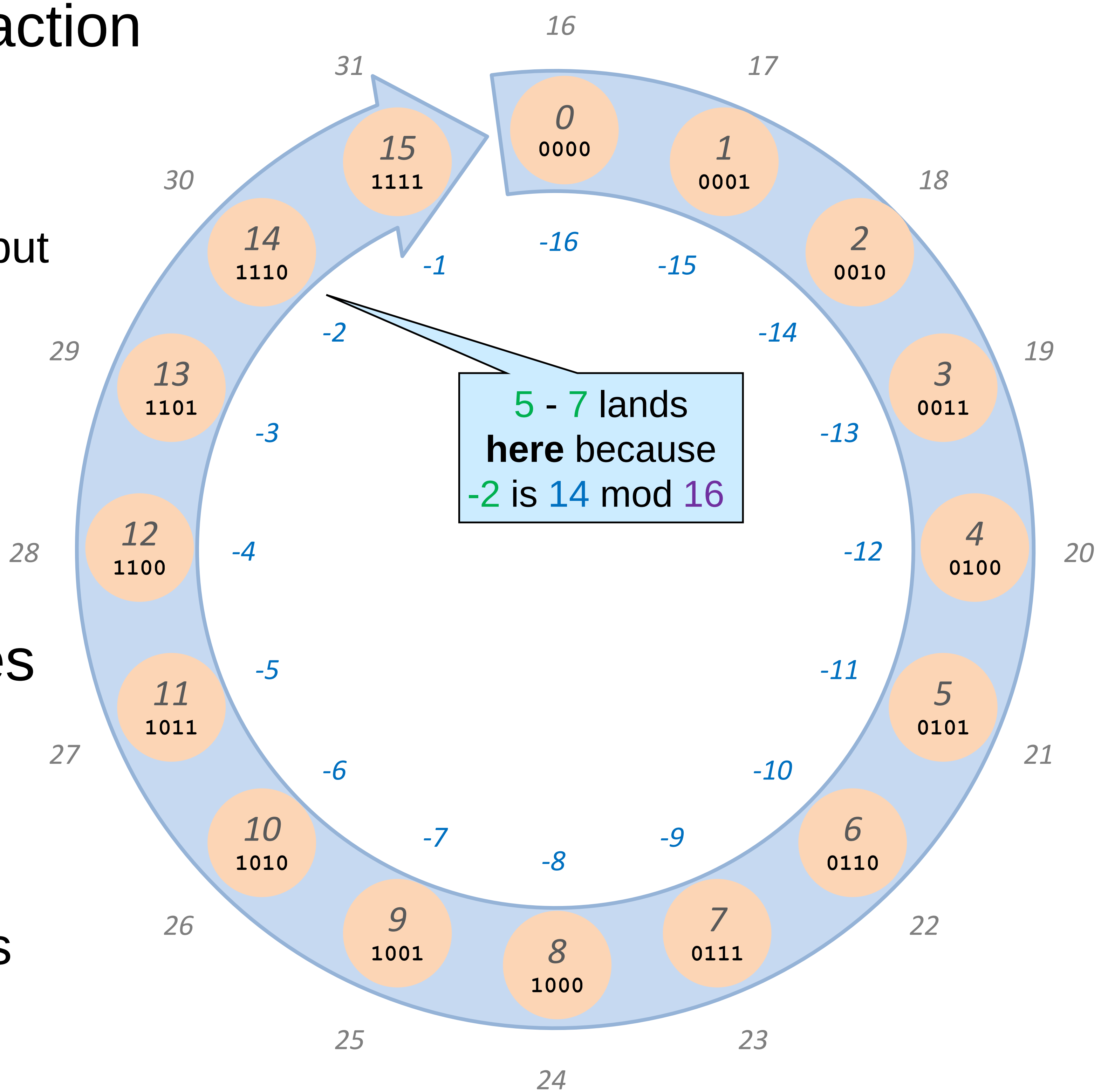
- The negative numbers too wrap around the circle
  - $-1 =_{\text{mod } 16} 15$
  - $-6 =_{\text{mod } 16} 10$



This position corresponds to
…, -86, -70, -54, -38, -22, -6,
10, 26, 42, 58, 74, 90, 106, …

Subtraction modulo *n*

- We can then do subtraction modulo *n*
  - $5 - 7 =_{\text{mod } 16} 14$
    - We evaluate it normally but return the remainder of dividing it by *n*
    - Equivalently, return the position of the result on the circle
- x - y is stepping y times *counter-clockwise* from x
  - to compute 5 - 7 start from 5 and step 7 times counter-clockwise



Subtraction modulo *n*

- With subtraction, we can define the *additive inverse -x* of any number x
  - ➢ the number that added to x yields 0

  $-x =_{\text{mod } n} 0 - x$

- Then, more laws of traditional arithmetic are valid in modular arithmetic

$x + (-x) =_{\text{mod } n} 0$	Additive inverse
$-(-x) =_{\text{mod } n} x$	Cancelation

  - ○ More programs behave as if we were using normal arithmetic
    - ➢ even in the presence of overflows

Reasoning about int Code

```
string foo(int x) {
  int z = x + x - x;
  if (z == x)
    return "Good";
  else
    return "Bad";
}
```

- This function always returns "Good"
  - x + x - x = x in normal arithmetic
  - so x + x - x == x in C0
- If the compiler understands x + x - x
  - as x + (x - x), then
    - x + (x - x) = x + 0 by additive inverse
      = x by additive unit
  - as (x + x) - x, then
    - (x + x) - x = x + (x - x) by associativity of +
      = x as above

x + x may overflow
but it doesn't matter

Two’s Complement

Printing Numbers

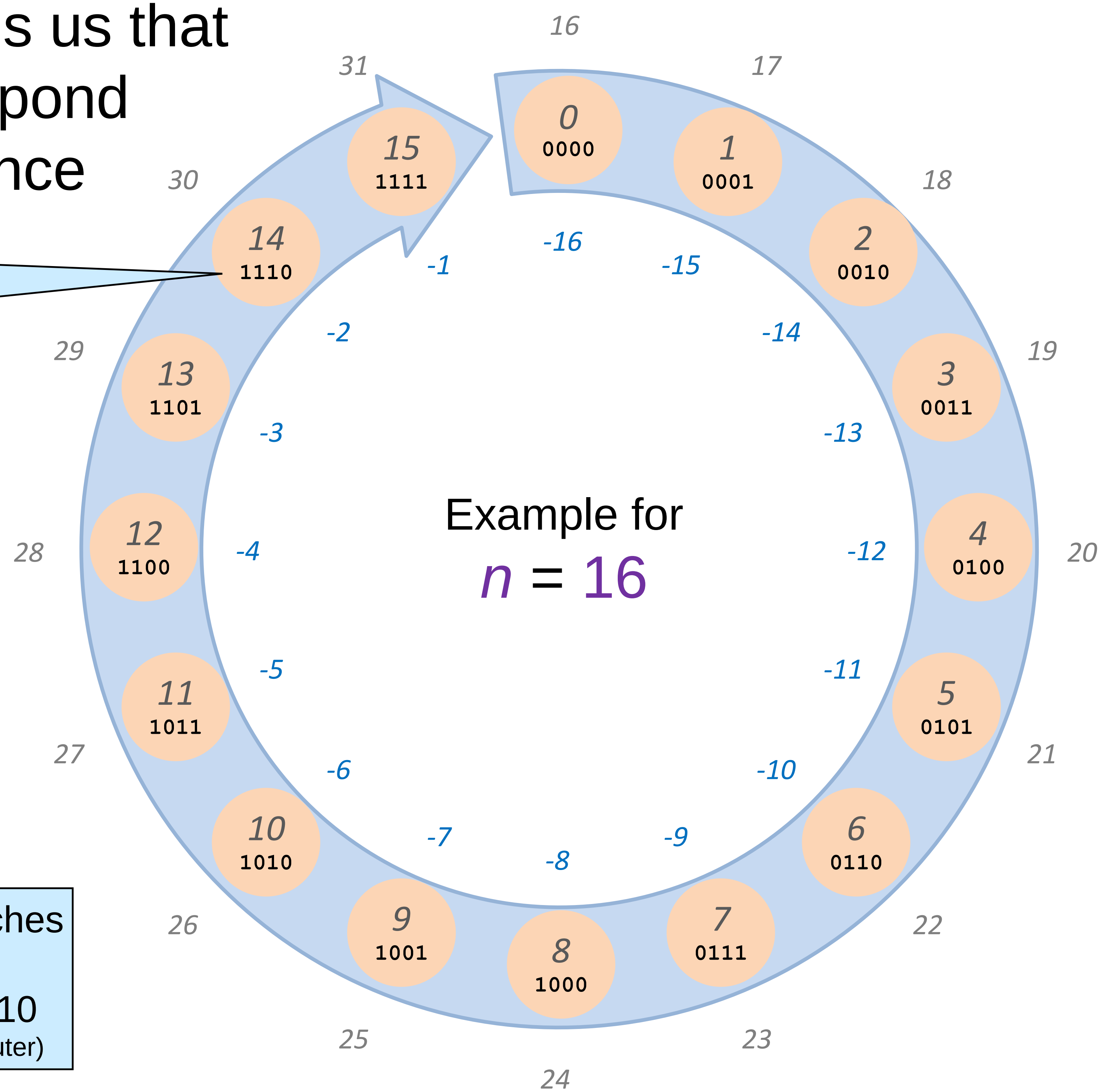
- Modular arithmetic tells us that many numbers correspond to the same bit sequence

int x

1110 could stand for
…, -82, -66, -50, -34, -18, -2,
14, 30, 46, 62, 78, 94, 110, …

- But what number should the computer print **1110** as?
  - 14?
  - -2?
  - 78?
  - …

Say our program reaches
printint(x);
where x contains 1110
(on a hypothetical 4-bit computer)



Comparing Numbers

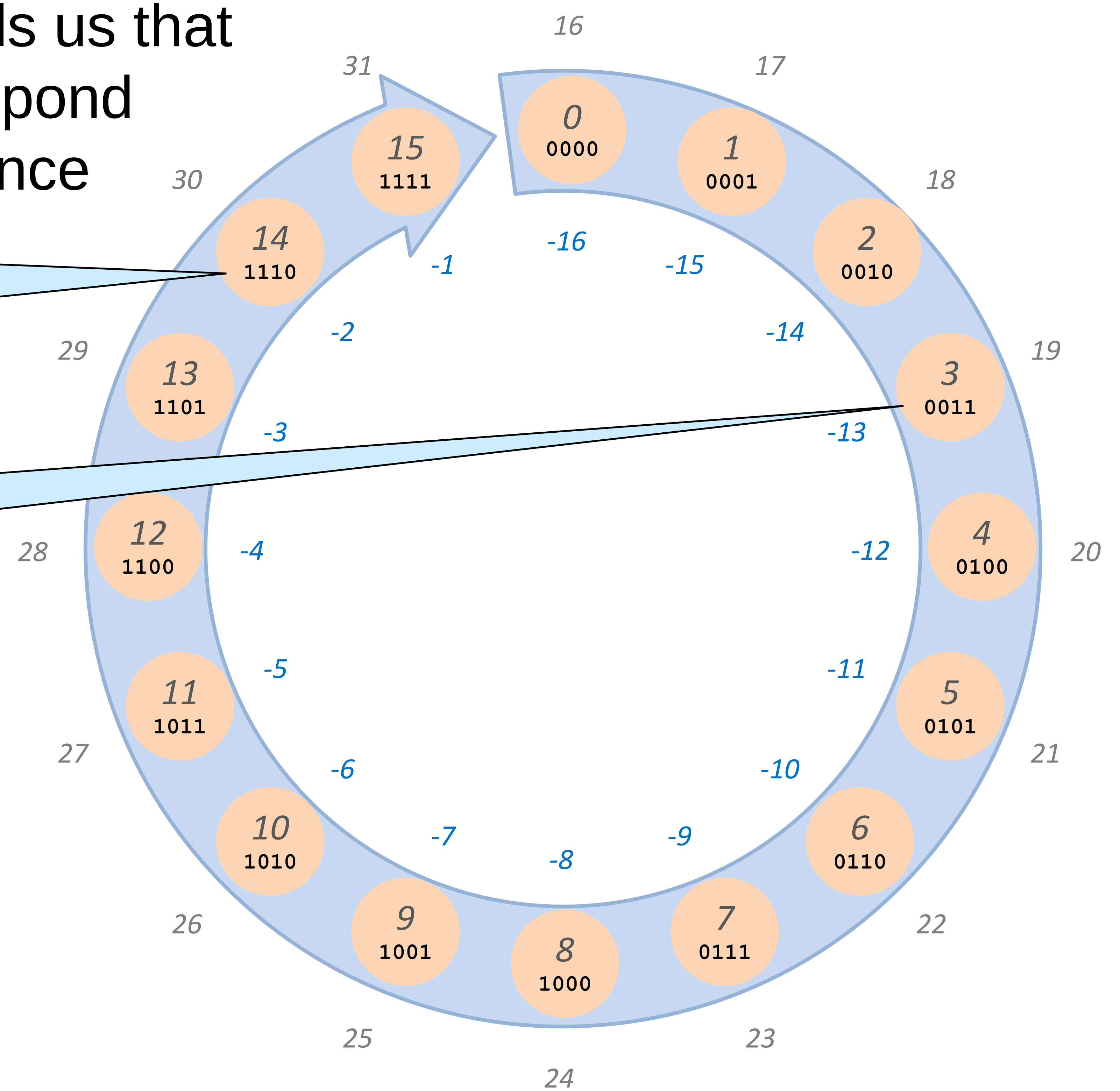
- Modular arithmetic tells us that many numbers correspond to the same bit sequence

int x

1110 could stand for
…, -82, -66, -50, -34, -18, -2,
14, 30, 46, 62, 78, 94, 110, …

int y

0011 could stand for
…, -77, -61, -45, -29, -13, 3,
19, 35, 51, 67, 83, 99, 115, …



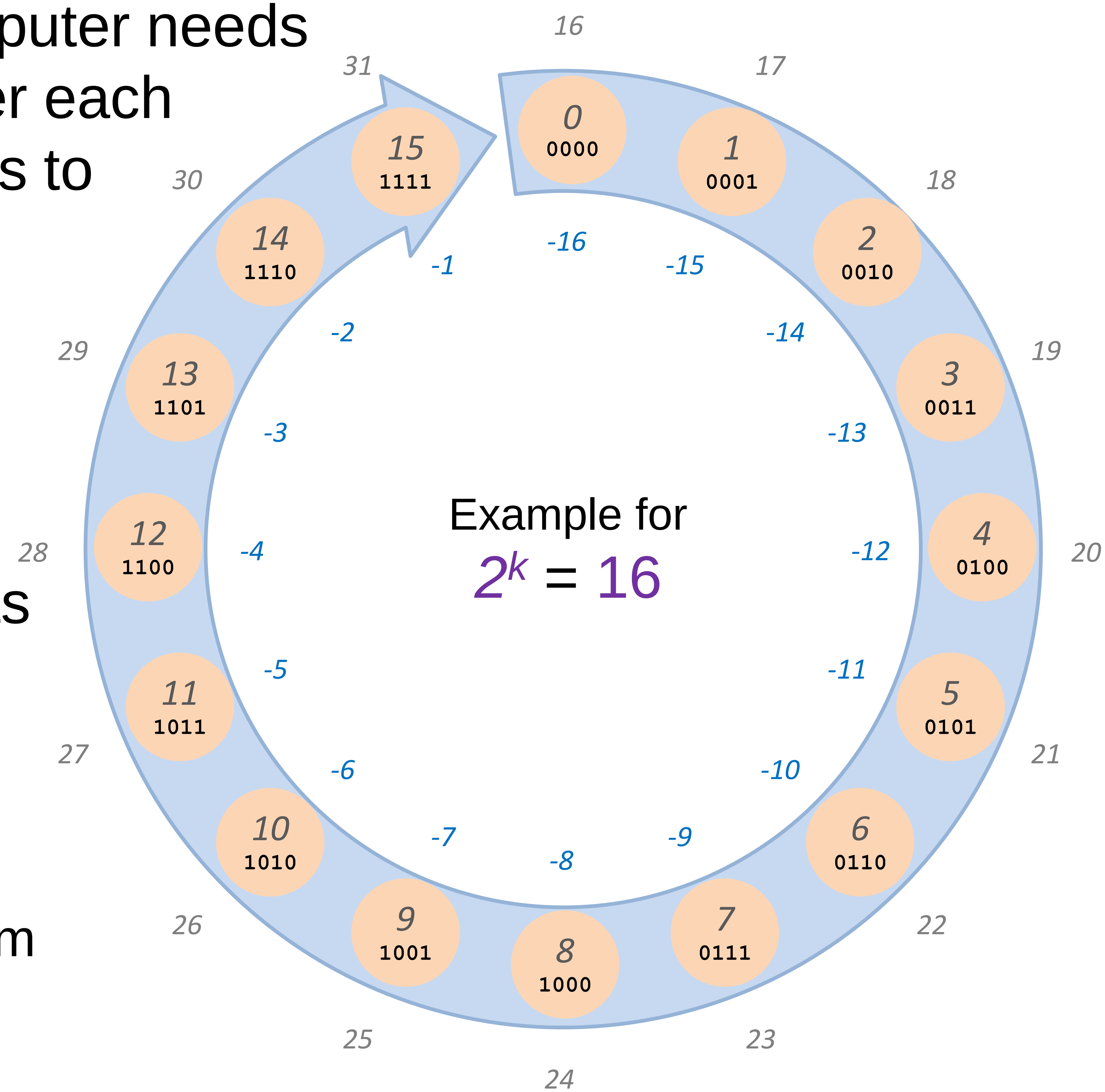
- But what should x > y evaluate to?
  - true?
  - false?

The Range of int's

- In both case, the computer needs to decide what number each k-bit word corresponds to

This is the opposite of the earlier problem: *what k-bit word does each number correspond do*

- Common requirements
  - successive bit values should correspond to successive numbers
    - 16, 1, -14, … won't do
  - 0 should be one of them

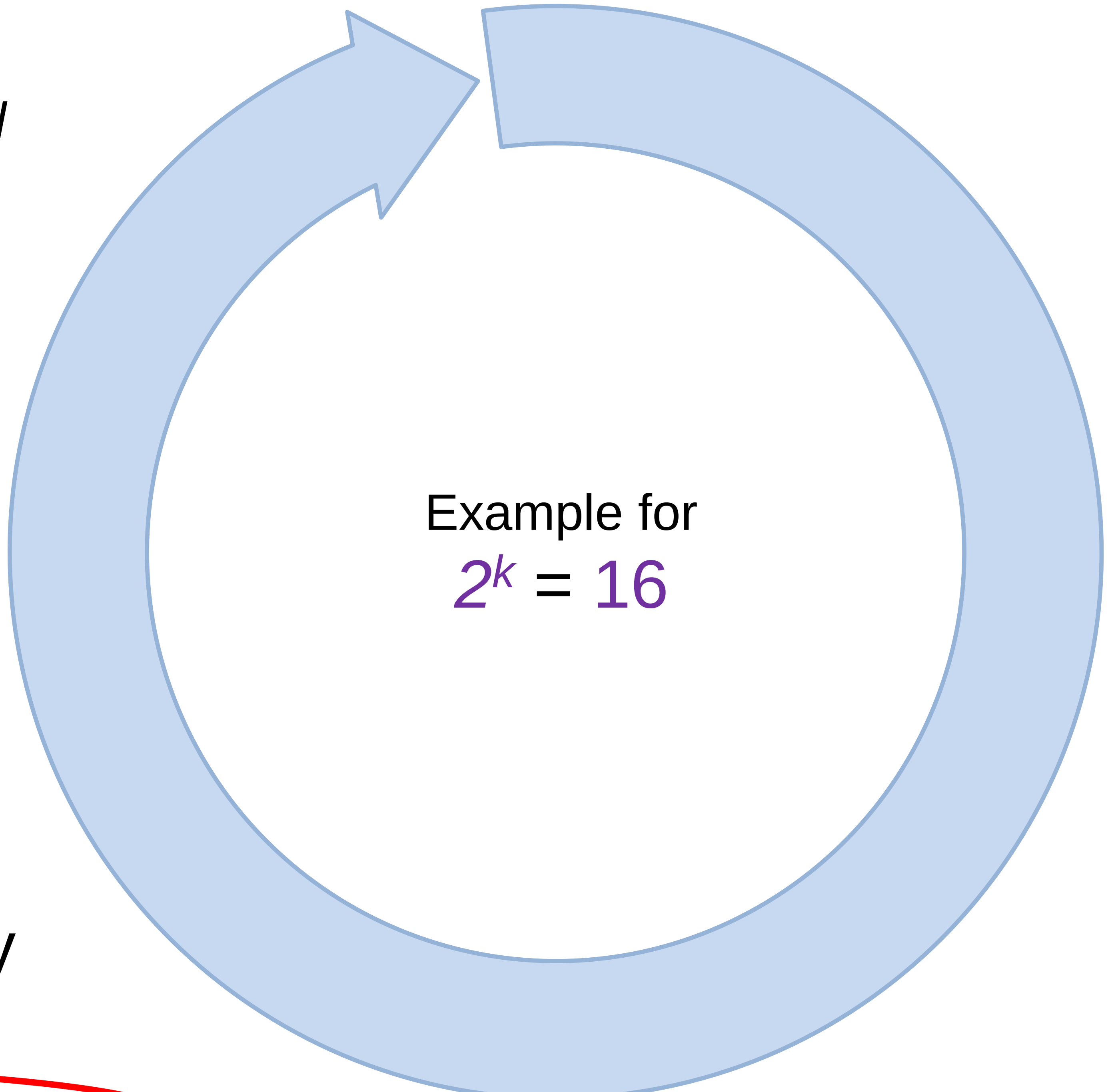


The Range of int's

- *What number does each k-bit word correspond to?*
  - *successive bit values should correspond to successive numbers*
  - *0 should be one of them*

- Pick the first 2^k integers starting at 0
  - here 0, 1, … 15
    - **1110** is printed as 14
    - **1110** > **0011** returns true

- int's that behave this way are called **unsigned**
  - This is **not** how C0's int's work

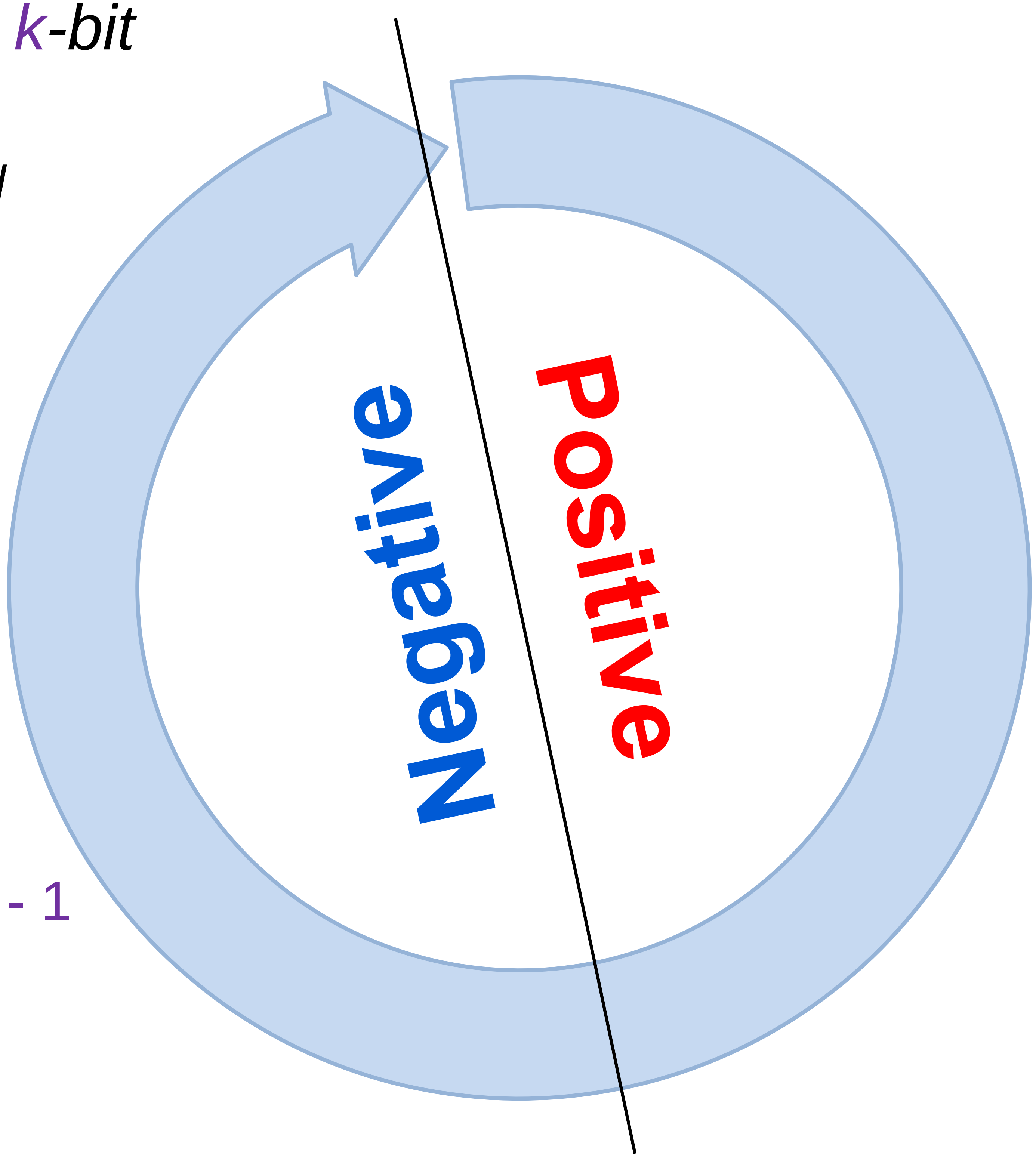


The Range of int's

- *What number does each k-bit word correspond to?*
  - *successive bit values should correspond to successive numbers*
  - *0 should be one of them*

- We also want some negative numbers
  - about half

- One common option
  - Pick the range -2^{k-1} to $2^{k-1} - 1$
  - This choice is called **two's complement**

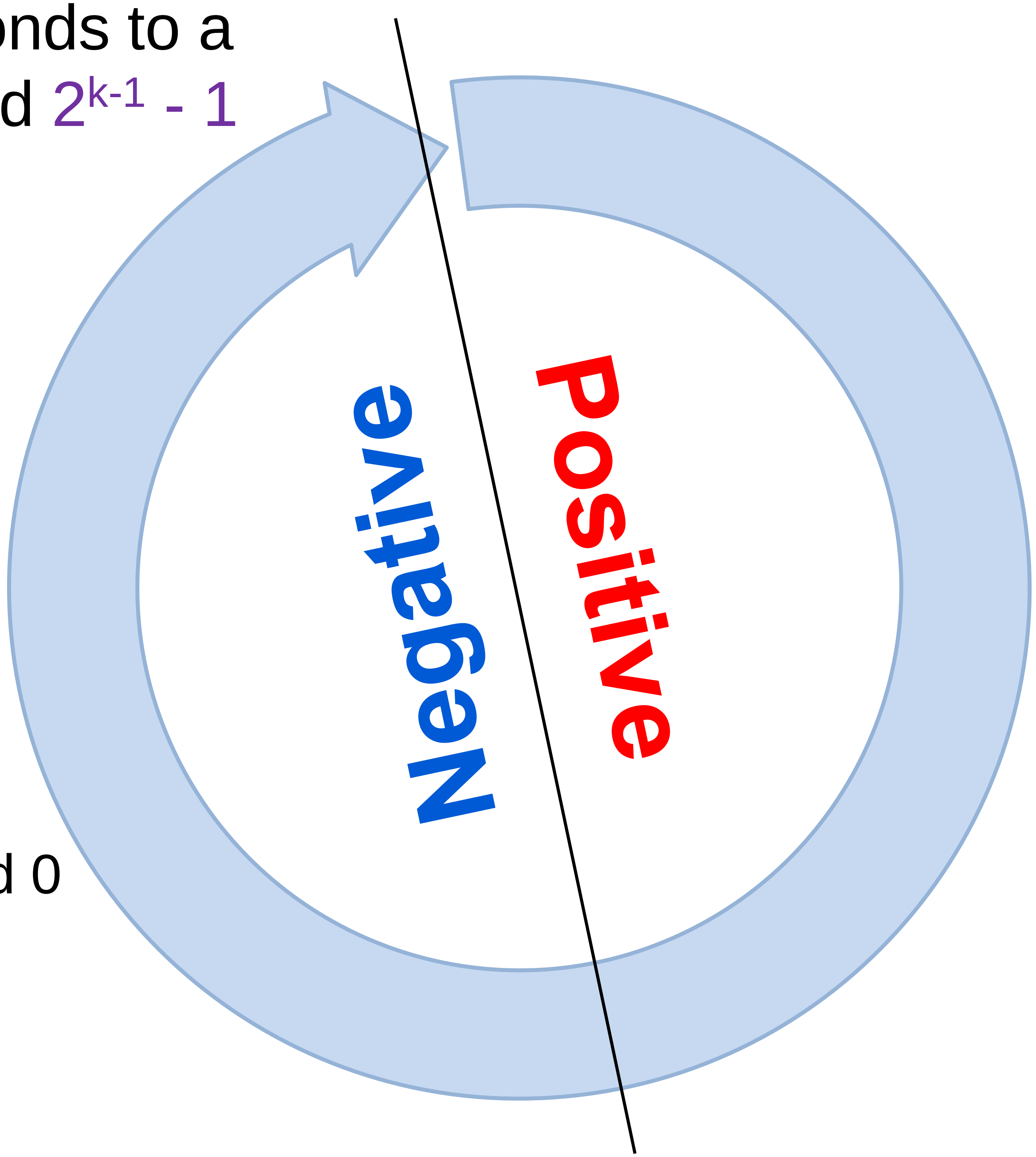


Two's Complement

- Each k-bit word corresponds to a number between -2^{k-1} and $2^{k-1} - 1$
  - the negative numbers go from -1 to -2^{k-1}
  - the positive numbers go from 1 to $2^{k-1} - 1$
  - and there is 0
- The leftmost bit tells the sign
  - 1 for negative numbers
  - 0 for positive numbers and 0

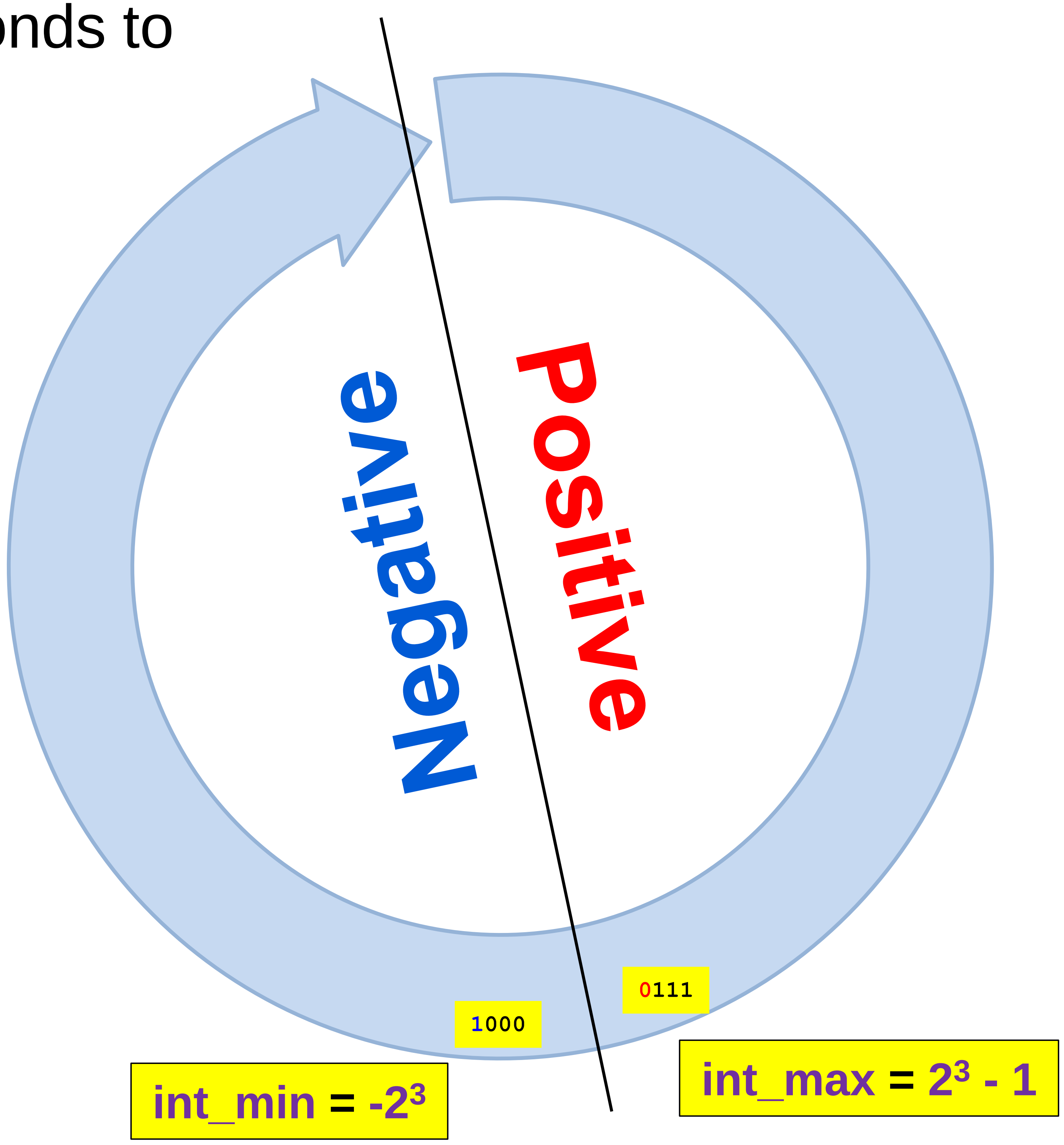
  It is called the **sign bit**

Efficient way to determine the sign of a number



Two's Complement

- Each k-bit word corresponds to a number in the range -2^{k-1} to $2^{k-1} - 1$
  - The ***smallest number*** is called **int_min**
    - -2^{k-1}
    - 100…000 in binary
  - The ***largest number*** is called **int_max**
    - $2^{k-1} - 1$
    - 011…111 in binary
  - Other notable numbers:
    - 0 is 000…000
    - -1 is 111…111

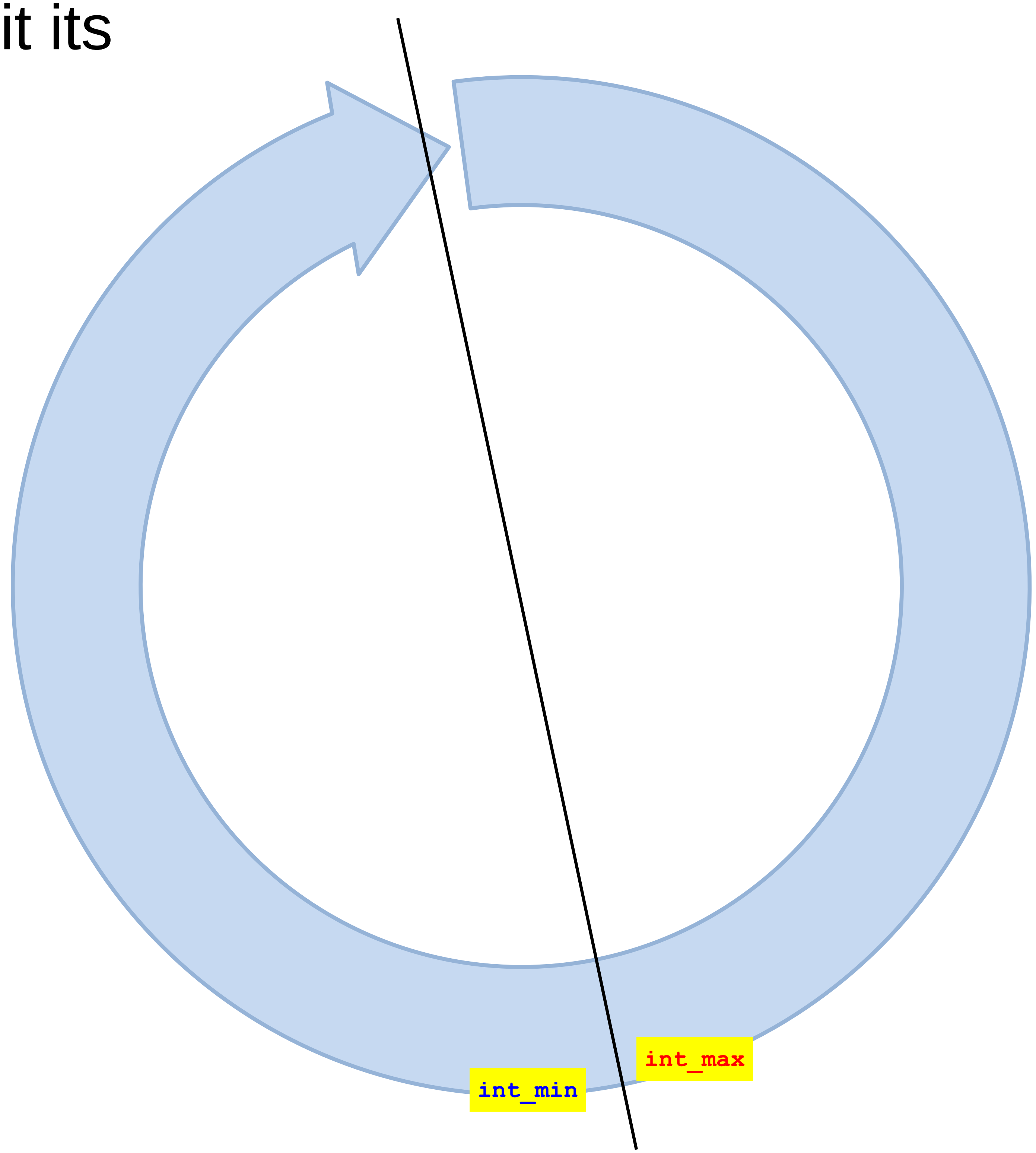


Two's Complement Overflow

- An operation **overflows** it its mathematical result is outside the range -2^{k-1} to $2^{k-1} - 1$

If it is < -2^{k-1} , this is sometimes called **underflow**

- E.g.,
  - int_max + 1
  - int_min - 3
  - 2 * int_max
  - 17 * int_min



int's in C0

- C0 represents integers as 32-bit words
- It handles overflow using modular arithmetic
- The range of int's is based on two's complement
  - **int_max** = $2^{31} - 1$ = 2147483647
  - **int_min** = -2^{31} = -2147483648
  - Their values are defined as the functions int_max() and int_min() in the <util> system library

Linux Terminal

```
# coin -l util
C0 interpreter (coin) …
…
--> int_max();
2147483647 (int)
--> int_min();
-2147483648 (int)
-->
```

Reasoning about int Code

- Comparing int values in C0 does **not** work like comparing numbers in normal arithmetic

```
string bar(int x) {
  if (x+1 > x)
    return "Good";
  else
    return "Strange";
}
```

- This function does not always return "Good"
  - if x is **int_max**, it returns "Strange"!
    - but in math x+1 > x for ***any*** x!

- When reasoning about code that uses >, >=, < and <=, we often need to account for overflow
    - by considering special cases
  - Code that only uses +, * and - doesn’t need a special treatment

Also operators dealing with sign

Division and Modulus

Operations on int's

- So far, we learned how C0 handles
  - +, -, *: using modular arithmetic
  - >, >=, <, <=: using two's complement
  - *Division is missing!*

== and != too

- We are used to division on *real numbers*:
  - *x/y* is the number *z* such that *z***y* = *x*
    - if $y \neq 0$
- But this definition doesn't work with integers
  - there is no ***integer*** *z* such that 2**z* = 3

Integer Division

- With ***integers***, there is not always z such that $z * y = x$
  - z is x/y in calculus
- We introduce a new operation, the **modulus**, to pick up the slack
  - We want to define the operations x/y and $x\%y$ so that

integer division of x by y

modulus of x by y

$$(x/y) * y + (x\%y) = x$$

- That's not enough!
  - defining x/y to always return 0 and $x\%y$ to return x would work
    - we don't want that!

Integer Division and Modulus

$$(x/y) * y + (x\%y) = x$$

- We also want the modulus to be between 0 and $y\text{-}1$
  - Also require

$$0 \le |x \% y| < |y|$$

We take the absolute value in case y is negative

- This is still not enough!
  - defining 9/4 to be 3 and 9%4 to be -3 would work
    - (9/4) * 4 + (9%4) = 3*4 - 3 = 9   and   0 ≤ |-3| < 4
    - We don't want that!
- We want division to "round down"
  - in a calculator, 9/4 = 2.25
  - so with integer division, we want 9/4 = 2
    - and therefore 9%4 = 1

Integer Division and Modulus

$$(x/y) * y + (x\%y) = x$$

$$0 \le |x \% y| < |y|$$

Division should "round down"

- But what does "rounding down" mean for negative numbers?
  - does -2.25 rounds down to -2? "down" towards 0
  - or does -2.25 round down to -3? "down" towards -∞
- In C0, integer division rounds toward 0
  - so -9/4 == -2 in C0
  - In other languages, it rounds towards -∞ Python, for example

Division by Zero

- In math, division by zero is **undefined**
- In a program, division by zero is an **error**
  - C0 will abort execution

Linux Terminal

```
# coin
C0 interpreter (coin) …
--> 5/0;
Error: division by zero.
Last position: <stdio>:1.1-1.4
-->
```

- Any time we have x/y in a program, we must have a reason to believe that y != 0
  - 0 is not a *valid* value for the denominator of a division
- In C0, we flag invalid values using **preconditions**
  - *some primitive operations come with preconditions*
    - not just user-defined functions

Safety Requirements

- Integer division, x/y, has the precondition

```
//@requires y != 0;
```

- There is another *invalid* input:
  int_min()/-1 also **aborts** the program
  - this is because computer chips raise errors on these values

- Integer division has a second precondition:

```
//@requires !(x == int_min() && y == -1);
```

x%y
has the same
preconditions

- Code that uses / or % must be **safe**
  - We must prove that these preconditions are satisfied

Operations on int's – Summary

- +, -, *: handled using **modular arithmetic**

== and != too

- >, >=, <, <=: handled using **two's complement**

- x/y rounds towards 0 – always
- x/y and x%y have preconditions

```
//@requires y != 0;
//@requires !(x == int_min() && y == -1);
```

Bit Patterns

Using int Beyond Numbers

- So far, we used the type int to represent integers
  - numbers!
- But in C0, an int is always 32 bits
- We can use an int to represent any data we can fit in 32 bits
  - pixels, network packets, …

  Then, an int does not represent a number but a **bit pattern**
- C0 has a special set of operations to manipulate bit patterns
  - they are the **bitwise operations** and the **shifts**
  - +, -, *, / and % are called the **arithmetic operations**

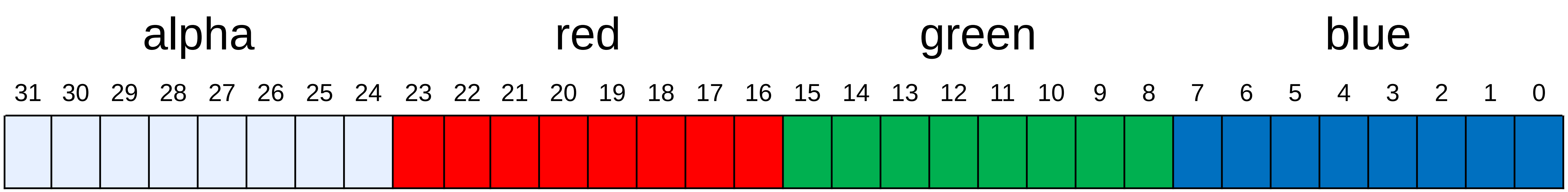
We *could* use the arithmetic operations to manipulate bit patterns but that's inefficient and error prone

Pixels as 32-bit int's

- A **pixel** is a dot of color in an image
  - The color of a pixel can be described by specifying
    - how much **red**, **green** and **blue** it contains
    - how opaque it is – this part is called the **alpha** component

This is called the ARGB representation

- Pixels are efficiently represented as bit patterns



  - bits 0-7 give the intensity of blue
  - bits 8-15 give the intensity of green
  - bits 16-23 give the intensity of red
  - bits 24-31 specify the opacity

• A value of 0 means there is no blue
• A value of 255 means maximally blue

Similar

Similar

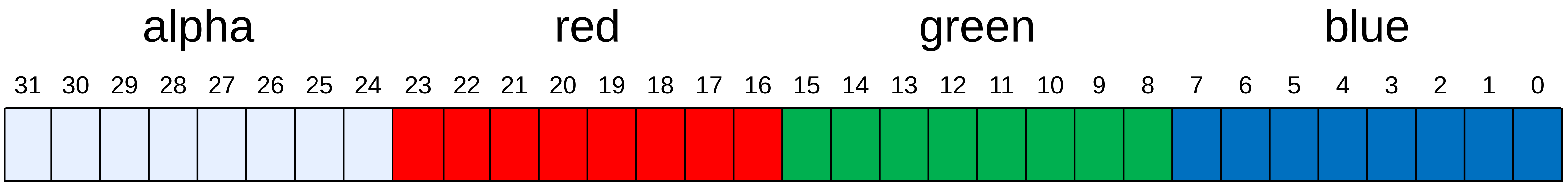
• 0 means fully transparent
• 255 means fully opaque

Pixels as Bit Patterns

- To describe a pixel, we need to give all its 32 bits
  - E.g., 10110011011100110101101011111001
  - We are better off using hexadecimal
    - 0xB3735AF9

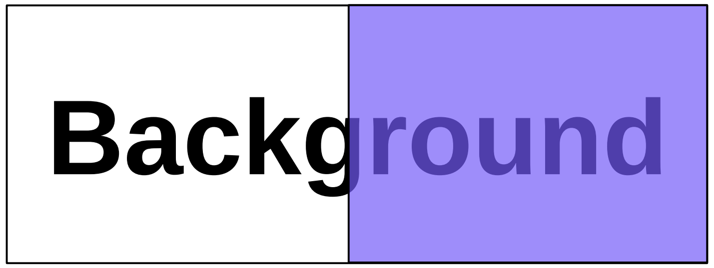
This is mind numbing!

We always use hex with bit patterns



1011 0011 0111 0011 0101 1010 1111 1001

B 3 7 3 5 A F 9



Here's the color of this pixel

Bitwise Operations

Bitwise Operations

- The **bitwise operations** manipulate the bits of a bit pattern independently of the other bits nearby

- They are

  ~ – pronounced “not”

  Takes a single argument

  & – pronounced “and”

  | – pronounced “or”

  ^ – pronounced “xor”

  Take two arguments

- Let’s see how they work on an individual bit

Bitwise Operations on One Bit

- Here are the tables that give the output for each input

This says that:
- 0 & 0 is 0
- 0 & 1 is 0
- 1 & 0 is 0
- 1 & 1 is 1

and

&	**0**	**1**
0	0	0
1	0	1

or

|	**0**	**1**
0	0	1
1	1	1

xor

^	**0**	**1**
0	0	1
1	1	0

not

~	**0**	**1**
	1	0

Bitwise Operations

- C0's bitwise operations take int's as input and return an int
  - there is no type for individual bits in C0
- They apply the tables on each bit of their inputs, **position by position**
  - so, if int's were 6 bits,

But we know they are 32 bit

```
   000111      000111      000111
 & 010101    | 010101    ^ 010101    ~ 010101
 --------    --------    --------    --------
   000101      010111      010010      101010
```

6-bit examples

- **&** and **|** are related to **&&** and **||** but
  - **&** and **|** take two int's and return an int
  - **&&** and **||** take two bool's and return a bool

Bitwise And – &

Let's see how to use the bitwise operations to manipulate bit patterns

- If we "and" any bit **b** with
  - **0**, we always get **0**
    - b & 0 = 0
  - **1**, we always get **b** back
    - b & 1 = b

b

&	0	1
0	0	0
1	0	1

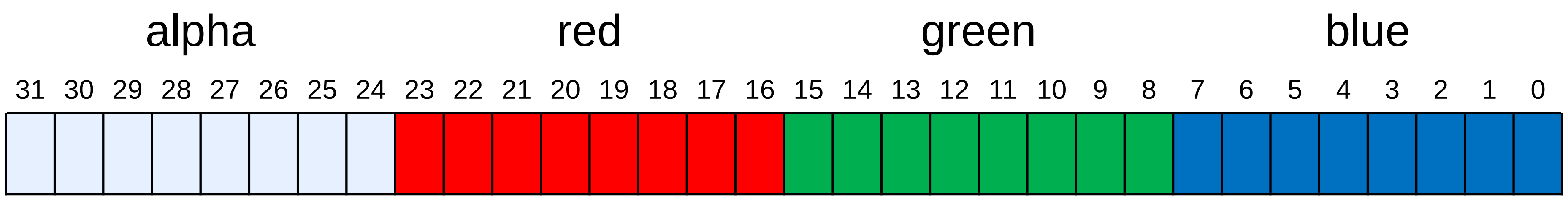
mask

b & 0 = 0

b & 1 = b

- If the int x is a bit pattern, then **x & m** is an int that
  - has the same bits as x where m is 1
  - and has a zero where m is 0
- The int m is called a **mask**
  - it allows us to retain specific bits of interest in x

&: Clearing Bits



b

&	0	1
0	0	0
1	0	1

mask

b & 0 = 0

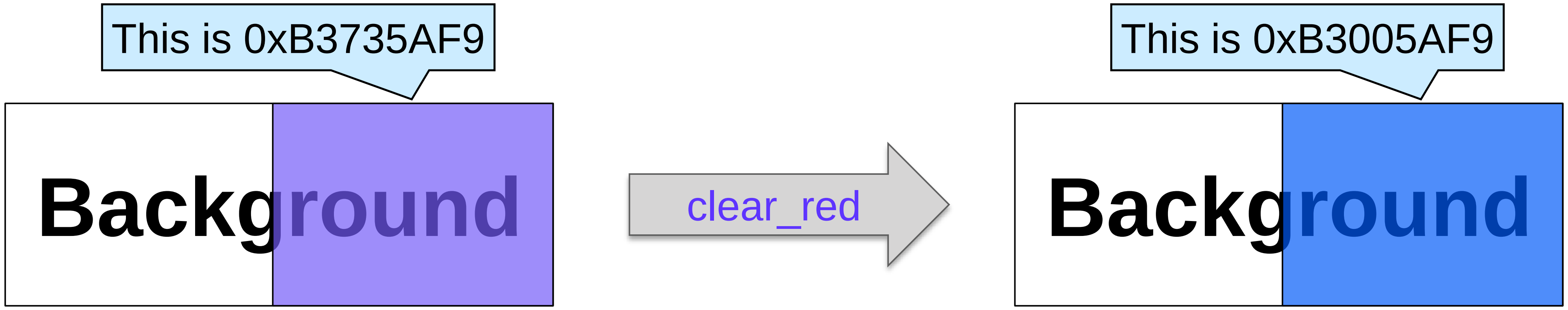
b & 1 = b

- We want to write a function that returns a pixel identical to p but with no red in it
  - zero out red component of p – bits 16-23
  - preserve the all other bits
- We can use the **mask** 0xFF00FFFF
  - bits 16-23 are 0
  - all other bits are 1

Mask

```
int clear_red(int p) {
  return p & 0xFF00FFFF;
}
```

- Here's how it looks on our example



&: Isolating Red

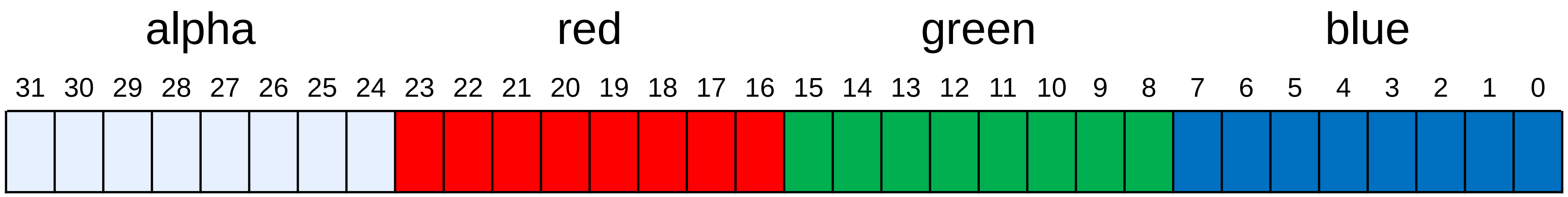
&	0	1
0	0	0
1	0	1

b

mask

b & 0 = 0

b & 1 = b



- We want to return a pixel with just the red component of p
  - ➢ preserve the red component of p – bits 16-23
  - ➢ zero out all other bits
  - ○ "and" p with the mask 0x00FF0000

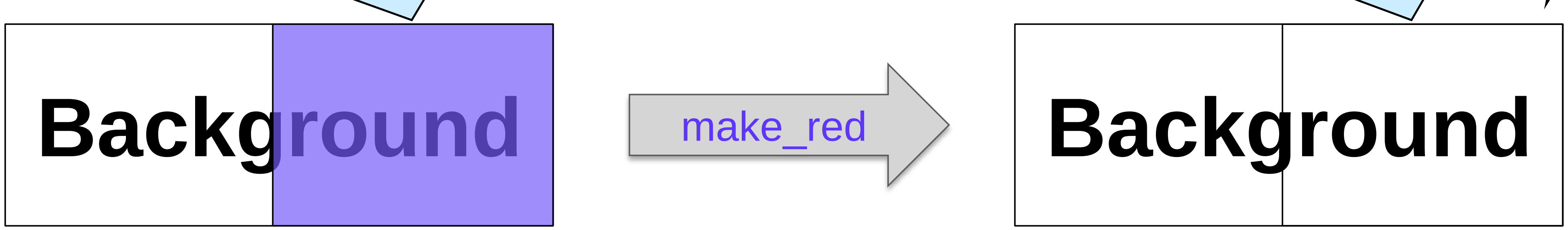
```
int make_red(int p) {
  int red = p & 0x00FF0000;
  return red;
}
```

Mask

Where's the red? The alpha channel is 00 so it is totally transparent

This is 0xB3735AF9

This is 0x00730000



Bitwise Or – |

- If we “or” any bit **b** with
  - **0**, we always get **b** back
    - b | 0 = b
  - **1**, we always get **1**
    - b | 1 = 1

b

<table>
<tr><th>|</th><th>0</th><th>1</th></tr>
<tr><th>0</th><td>0</td><td>1</td></tr>
<tr><th>1</th><td>1</td><td>1</td></tr>
</table>

b | 0 = b

b | 1 = 1

- Common uses of | are
  - setting bits to 1
  - constructing a bit pattern from parts

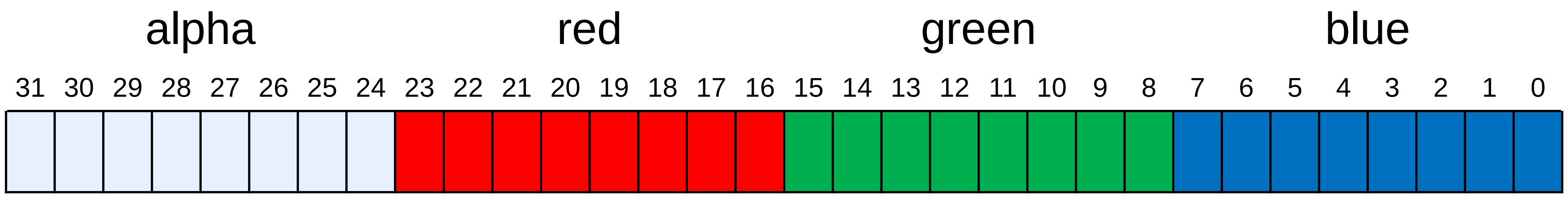
This is similar to clearing bits with &

|: Opacify

|	0	1
0	0	1
1	1	1

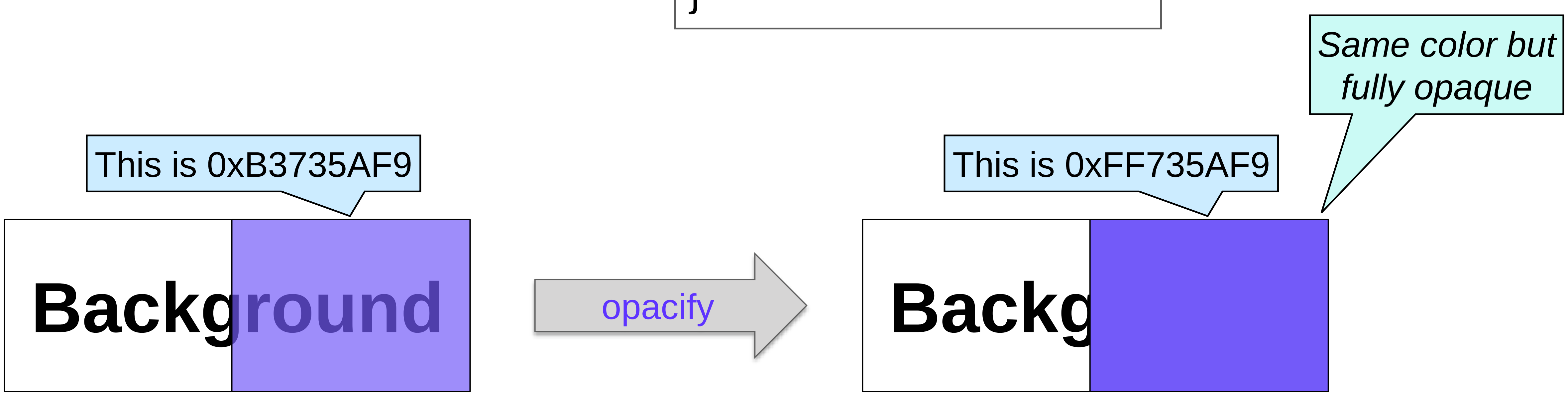
b | 0 = b

b | 1 = 1



- We want to make a pixel fully opaque
  - set the alpha bits to 1 – bits 24-31
  - preserve the other component of p
- We can "or" p with 0xFF000000
  - bits 24-31 become 1
  - all other bits stay as in p

```
int opacify(int p) {
  return p | 0xFF000000;
}
```



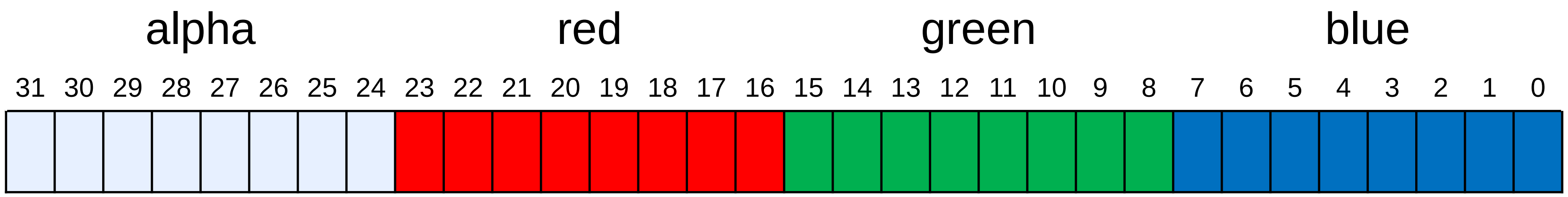
|: Constructing Pixels from Parts

|	0	1
0	0	1
1	1	1

b

b | 0 = b

b | 1 = 1



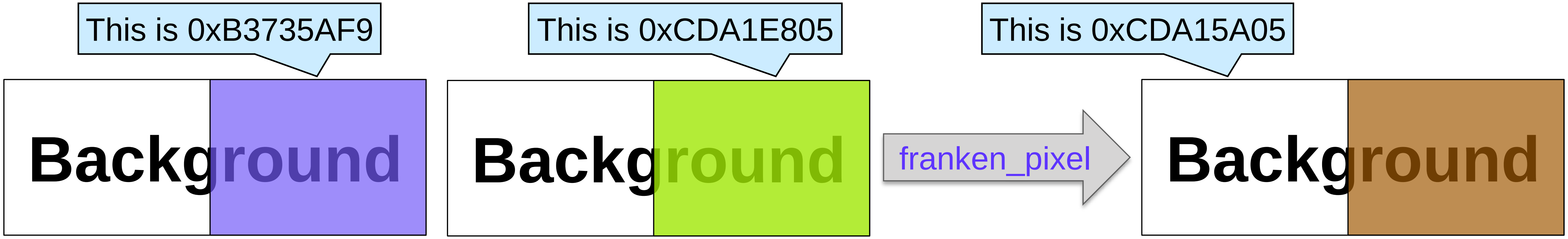
- Return a pixel with the same green component as p and the same alpha, red and blue components as q
  - isolate the green component of p using the mask 0x0000FF00
  - isolate the other components of q using the mask 0xFFFF00FF
  - combine them with “or”

```
int franken_pixel(int p, int q) {
  int p_green = p & 0x0000FF00;
  int q_others = q & 0xFFFF00FF;
  return p_green | q_others;
}
```

if p is 0xB3735AF9, then p_green is 0x00005A00

if q is 0xCDA1E805, then q_others is 0xCDA10005

0x00005A00
| 0xCDA10005
= 0xCDA15A05



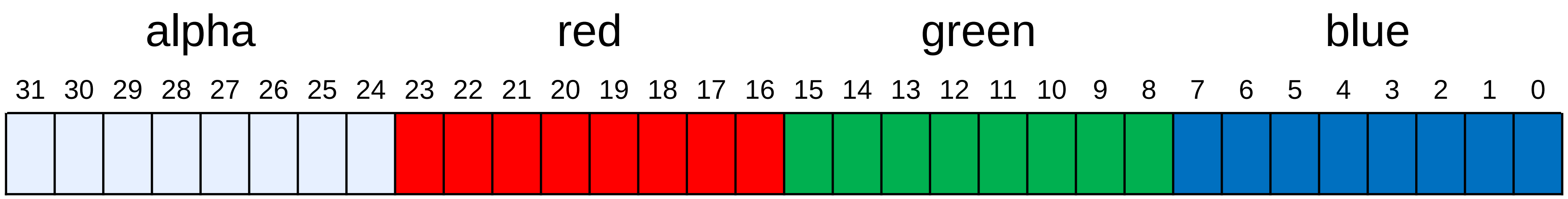
Bitwise Not – ~

- Bitwise negation flips bits

~	**0**	**1**
	1	0

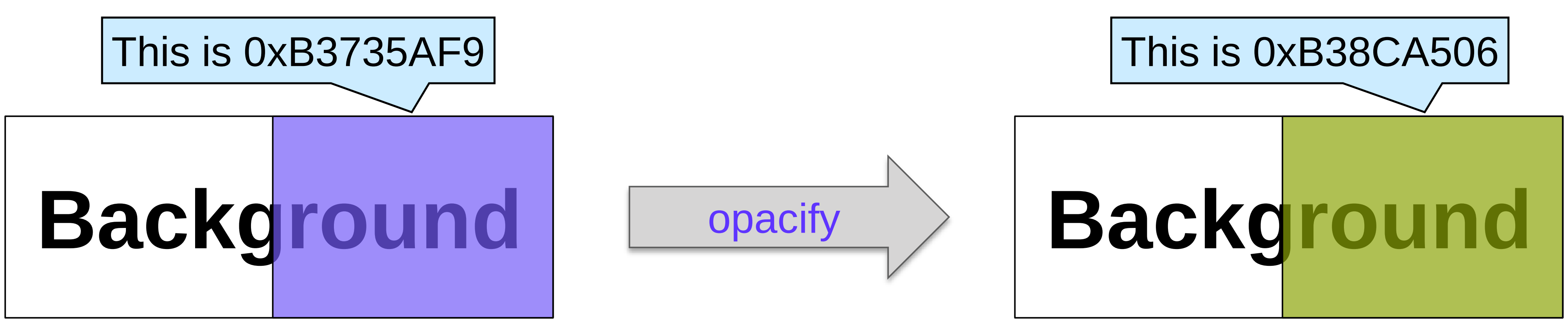
~: Flipping bits

~	0	1
	1	0



- Return the pixel with the same opacity but inverted colors
  - preserve the alpha channel
  - change the value of all other channels to 255 minus their original value
    - that's the same as flipping the bits of all channels

```
int invert(int p) {
  return (p & 0xFF000000) | (~p & 0x00FFFFFF);
}
```



Bitwise Xor – ^

- If we "xor" any bit **b** with
  - **0**, we always get **b** back
    - b ^ 0 = b
  - **b** itself, we always get **0**
    - b ^ b = 0
  - furthermore, "xor" is associative and commutative

b

^	**0**	**1**
0	0	1
1	1	0

b ^ 0 = b

b ^ b = 0

- One consequence is that **(m ^ k) ^ k = m**
  - if **m** is a *message* and **k** is a *key* then **x** = **(m ^ k)** is the *encryption* of **m** with **k**
  - to *decrypt* **x**, we do **x ^ k**, and **m** pops out

- "xor" is commonly used in cryptography

Shifts

Moving Bits Around

- The bitwise operations manipulate each position independently from all other positions in a bit pattern
  - We can't use them to move bits to new positions

- The **shift operations** enable us to move bits around
  - **left shift**: x << k moves the bits of x left by k positions
  - **right shift**: x >> k moves the bits of x right by k positions

The int x is understood as a **bit pattern**

The int k is understood as a **number**

- Since an int has 32 bits, k must be between 0 and 31

```
//@requires 0 <= k && k < 32;
```

Unsafe otherwise

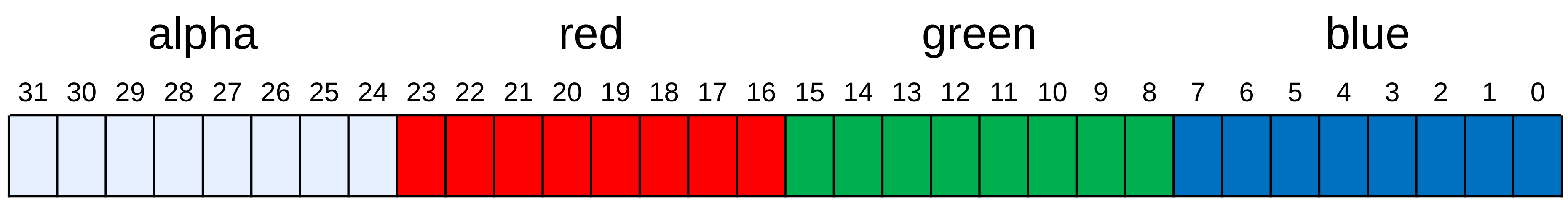
Left Shift

- **x << k** shifts the bits of x left by k positions
  - the leftmost k bits of x are dropped
  - the rightmost k bits of the result are set to 0

- So
  - 0101 << 1 evaluates to 1010:   0101 → 1010
  - 0101 << 3 evaluates to 1000:   0101 → 1000

4-bit examples

Blue Everywhere



- Return a pixel whose red and green components have the same intensity as p’s blue component
  - isolate the blue component of p
  - put it in the red, green and blue positions

Leave alpha unchanged

```
int blue_everywhere(int p) {
   int alpha = p & 0xFF000000;
   int blue   = p & 0x000000FF;
  return alpha | (blue << 16) | (blue << 8) | blue;
}
```

Why is it gray? Gray is when all colors are the same

Right Shift

- **x >> k** shifts the bits of x right by k positions
  - the rightmost k bits of x are dropped
  - the leftmost k bits of the result are a copy of the leftmost bit of x
    - This is called **sign extension**

That's because in two's complement, the leftmost bit is the sign bit

- So
  - 0101 >> 1 == 0010
  - 0101 >> 3 == 0000
  - 1010 >> 1 == 1101
  - 1010 >> 3 == 1111

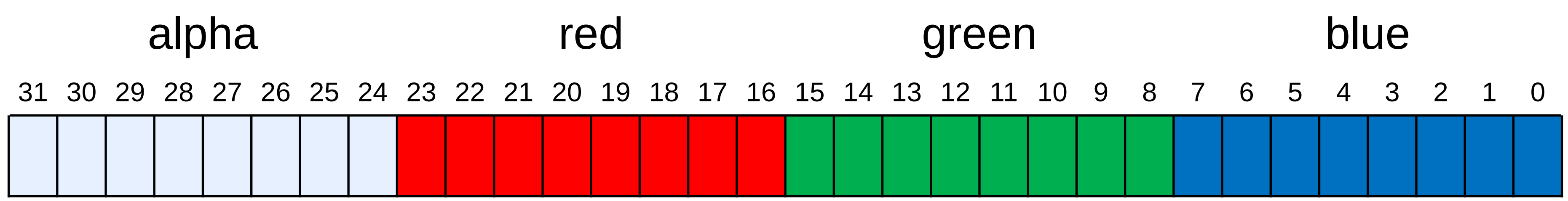
The sign bit is 0, so we add 0's

The sign bit is 1, so we add 1's

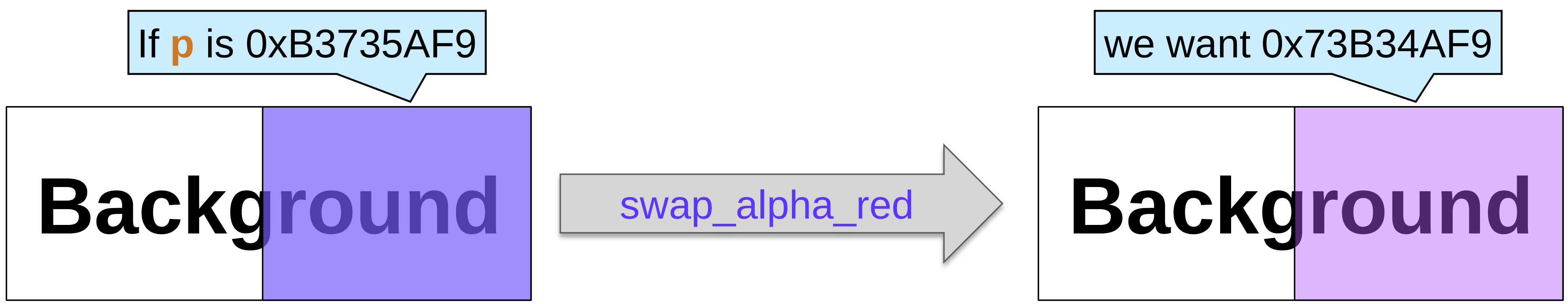
Sign bit

4-bit examples

Swapping the Alpha and Red Channels

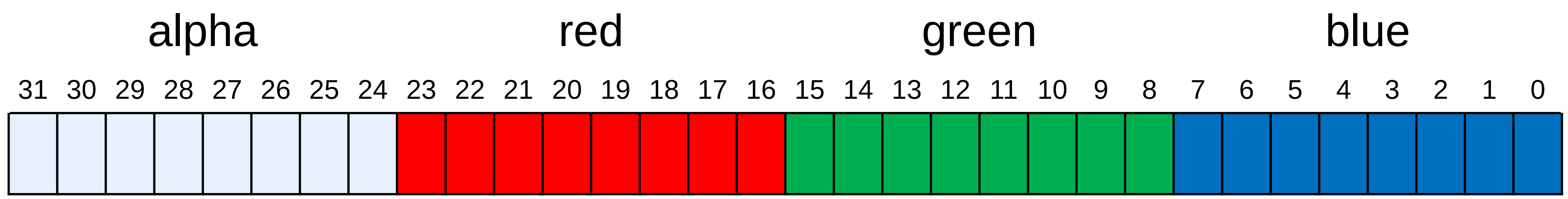


- Return a pixel identical to p, but where the red and alpha channel are swapped



- isolate the channels of p
- shift alpha right by 8 bits — so that its bits are in the red position
- shift red left by 8 bits — so that its bits are in the alpha position
- combine the parts and return

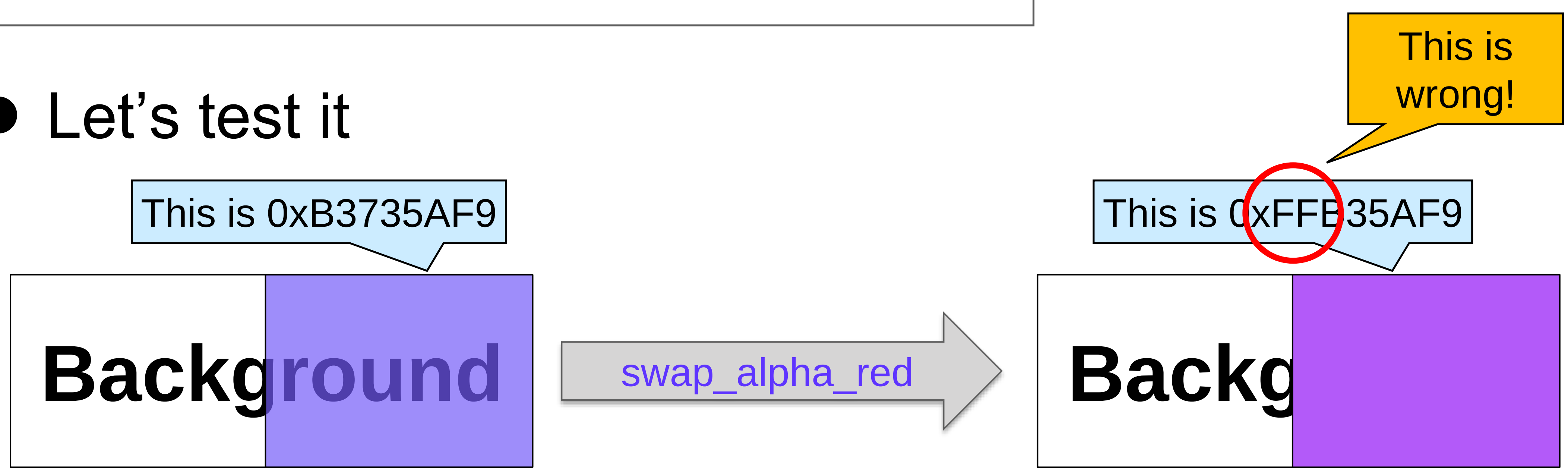
Swapping the Alpha and Red Channels



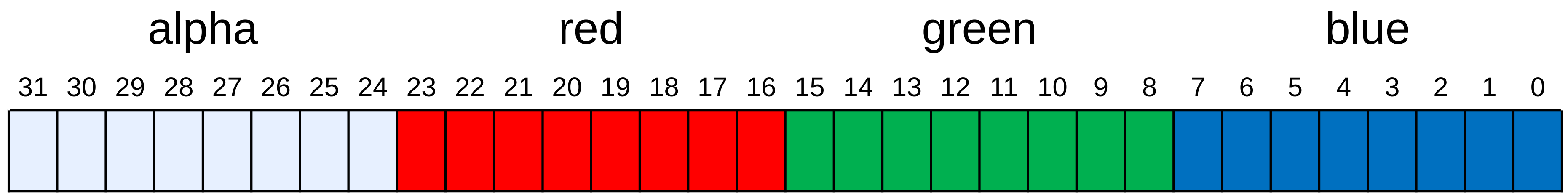

```
int swap_alpha_red(int p) {
   int new_alpha = (p & 0x00FF0000) << 8;
   int new_red   = (p & 0xFF000000) >> 8;
   int old_green = p & 0x0000FF00;
   int old_blue  = p & 0x000000FF;
 return new_alpha | new_red | old_green | old_blue;
}
```

- isolate the channels of p
- shift alpha right by 8 bits
- shift red left by 8 bits
- combine the parts and return

- Let's test it



Swapping the Alpha and Red Channels



- We have a bug!

If p is 0xB3735AF9,

```
int swap_alpha_red(int p) {
   int new_alpha = (p & 0x00FF0000) << 8;
   int new_red   = (p & 0xFF000000) >> 8;
   int old_green = p & 0x0000FF00;
   int old_blue  = p & 0x000000FF;
 return new_alpha | new_red | old_green | old_blue;
}
```

this is 0x73000000

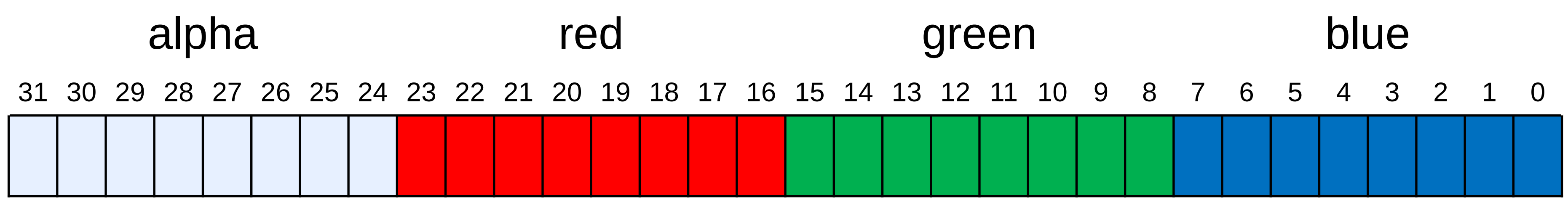
this is 0xFFB30000

this is 0x00005A00

this is 0x000000F9

- (p & 0xFF000000) >> 8 extends p's sign bit over the 8 leftmost bits
  - Beware of sign extension!

Swapping the Alpha and Red Channels

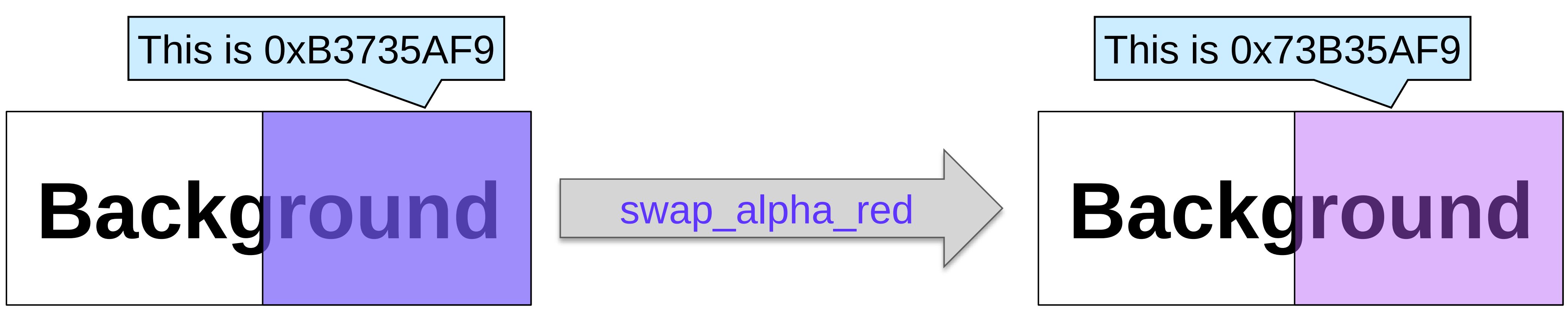


- To fix the bug, get rid of the sign-extended bits
  - **mask after shifting**

```
int swap_alpha_red(int p) {
   int new_alpha = (p << 8) & 0xFF000000;
   int new_red     = (p >> 8) & 0x00FF0000
   int old_green   = p & 0x0000FF00;
   int old_blue     = p & 0x000000FF;
 return new_alpha | new_red | old_green | old_blue;
}
```

This solves the issue

This is equivalent to what we had, but better be consistent



int Summary

The type int is used to

- represent integers
  - it uses modular arithmetic and two's complement
  - it manipulates them using the **arithmetic operations**
    - +, -, &, /, %, >, >=, <, <=
- encode bit patterns
  - it manipulates them using the **bitwise operations** and the **shifts**
    - &, |, ~, ^
    - <<, >>

NEVER mix and match operations

- it does not make sense to multiply pixels
- nor to & two numbers

Arithmetic vs. Bitwise Operations

NEVER mix and match arithmetic and bitwise operations

- Exceptions
  - -x = ~x + 1
  - $x << k = x * 2^k$
    - in particular, $1 << k = 2^k$
  - $x >> k = x$ divided by 2^k (*Python* division, not C0's)

Inside a processor chip,
- this is an efficient way to compute -x
- it avoids the need for circuitry for subtraction

x << k is a very efficient way to computer $x * 2^k$.
You are very likely to use it

x >> k is a very efficient too, but you are unlikely to use it: it's the "wrong" division