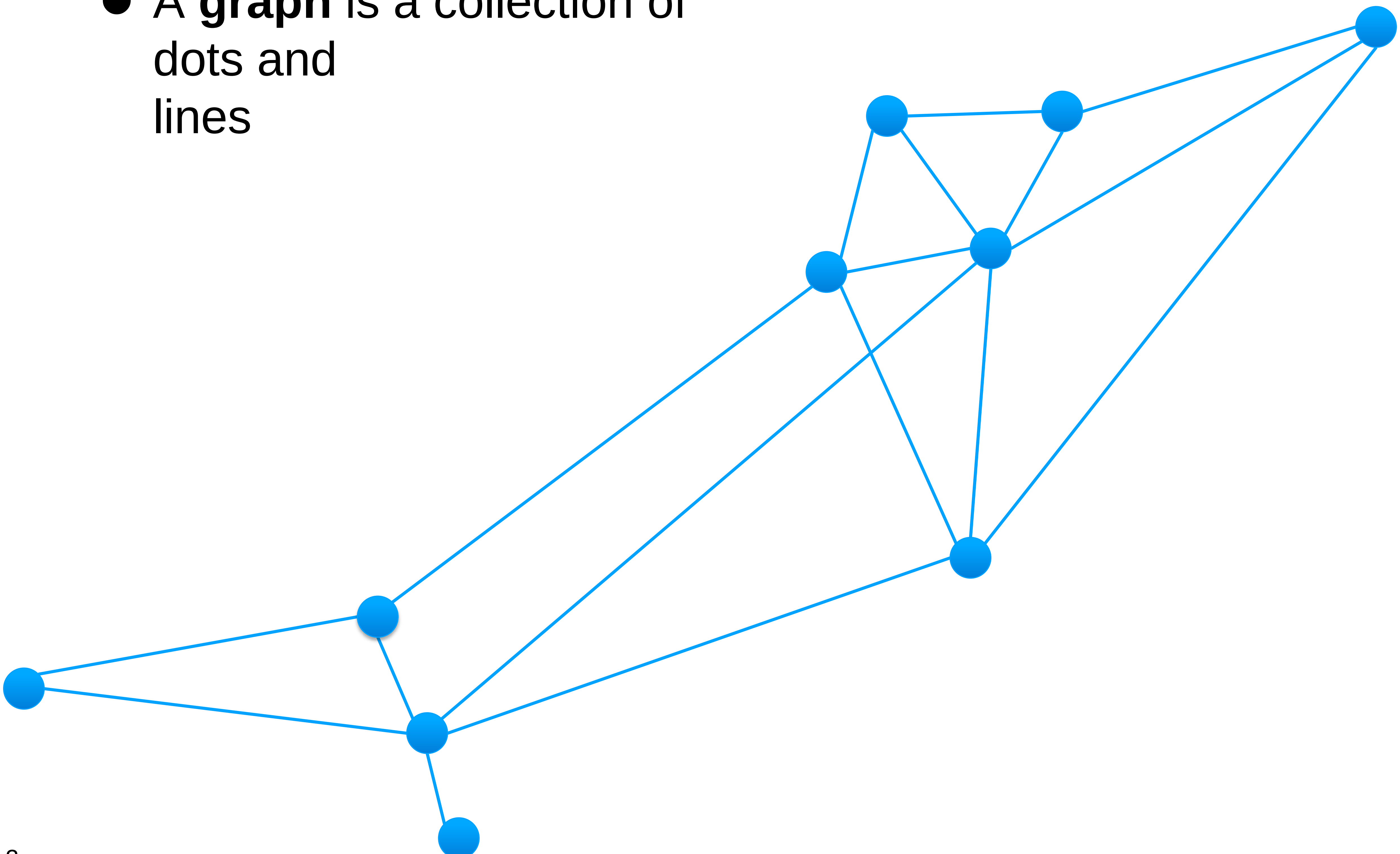
Graphs

Graphs

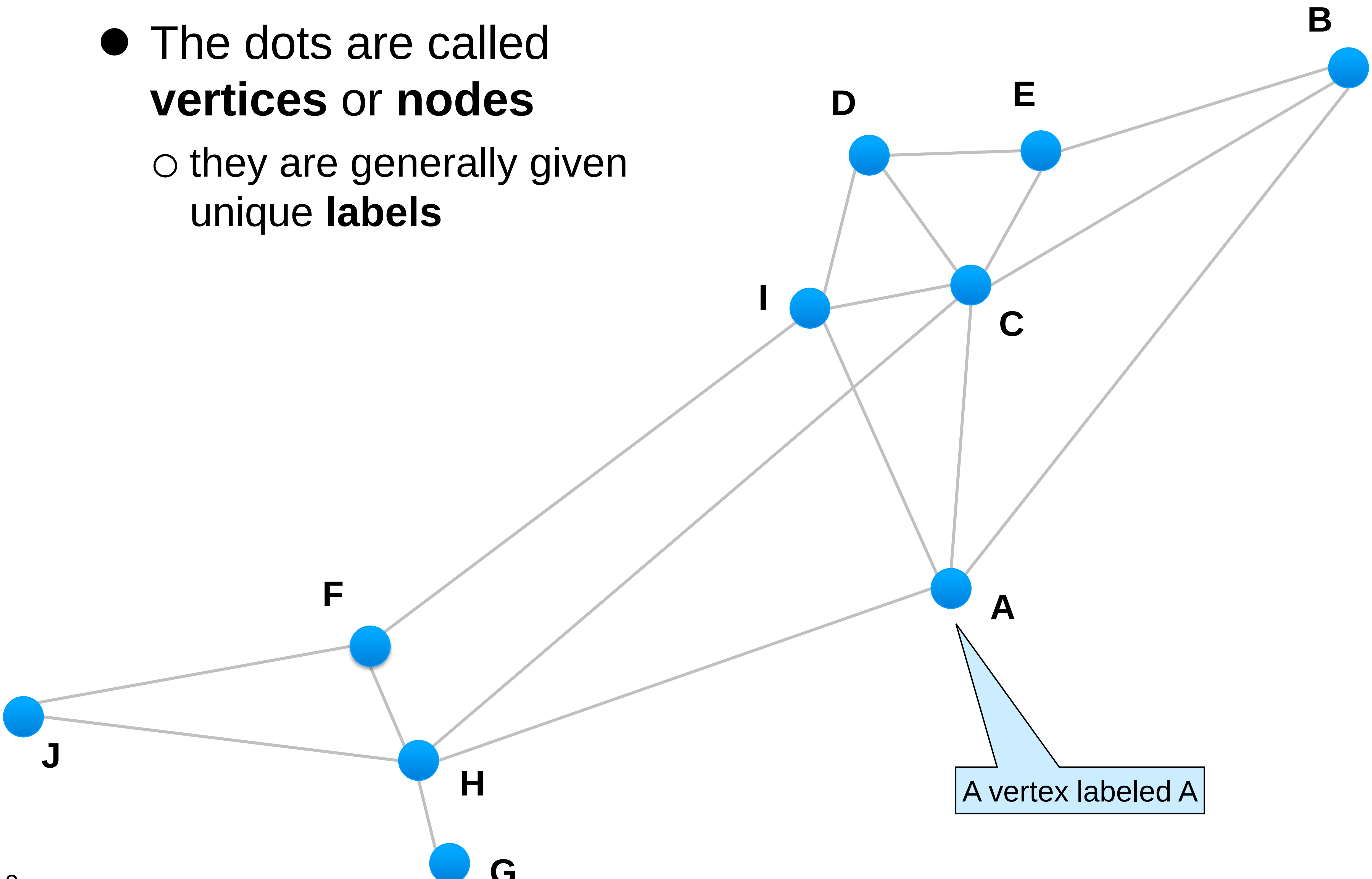
What is a Graph?

- A **graph** is a collection of dots and lines



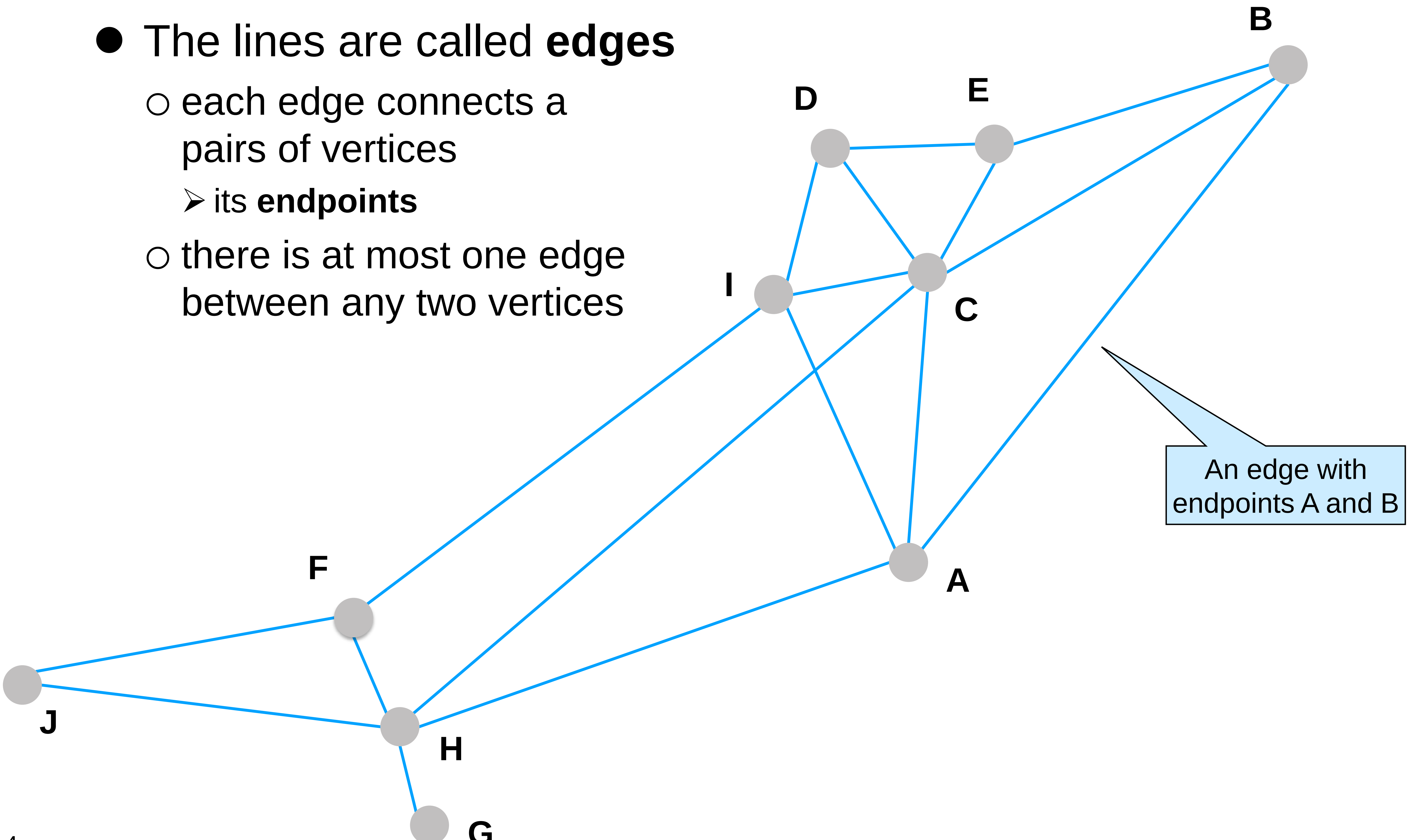
What is a Graph?

- The dots are called **vertices** or **nodes**
  - they are generally given unique **labels**



What is a Graph?

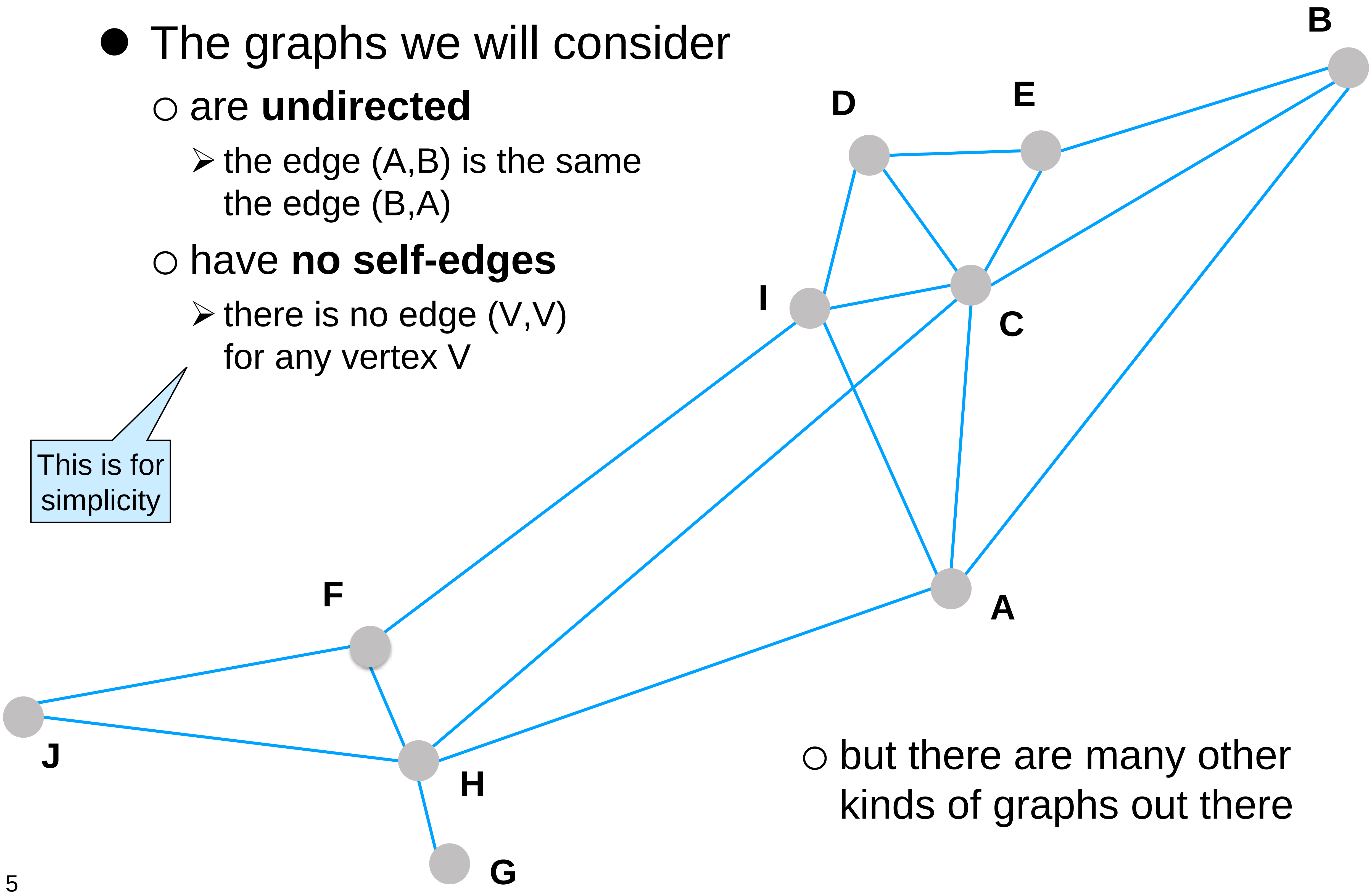
- The lines are called **edges**
  - each edge connects a pairs of vertices
    - its **endpoints**
  - there is at most one edge between any two vertices



What is a Graph?

- The graphs we will consider
  - are **undirected**
    - the edge (A,B) is the same the edge (B,A)
  - have **no self-edges**
    - there is no edge (V,V) for any vertex V

This is for simplicity

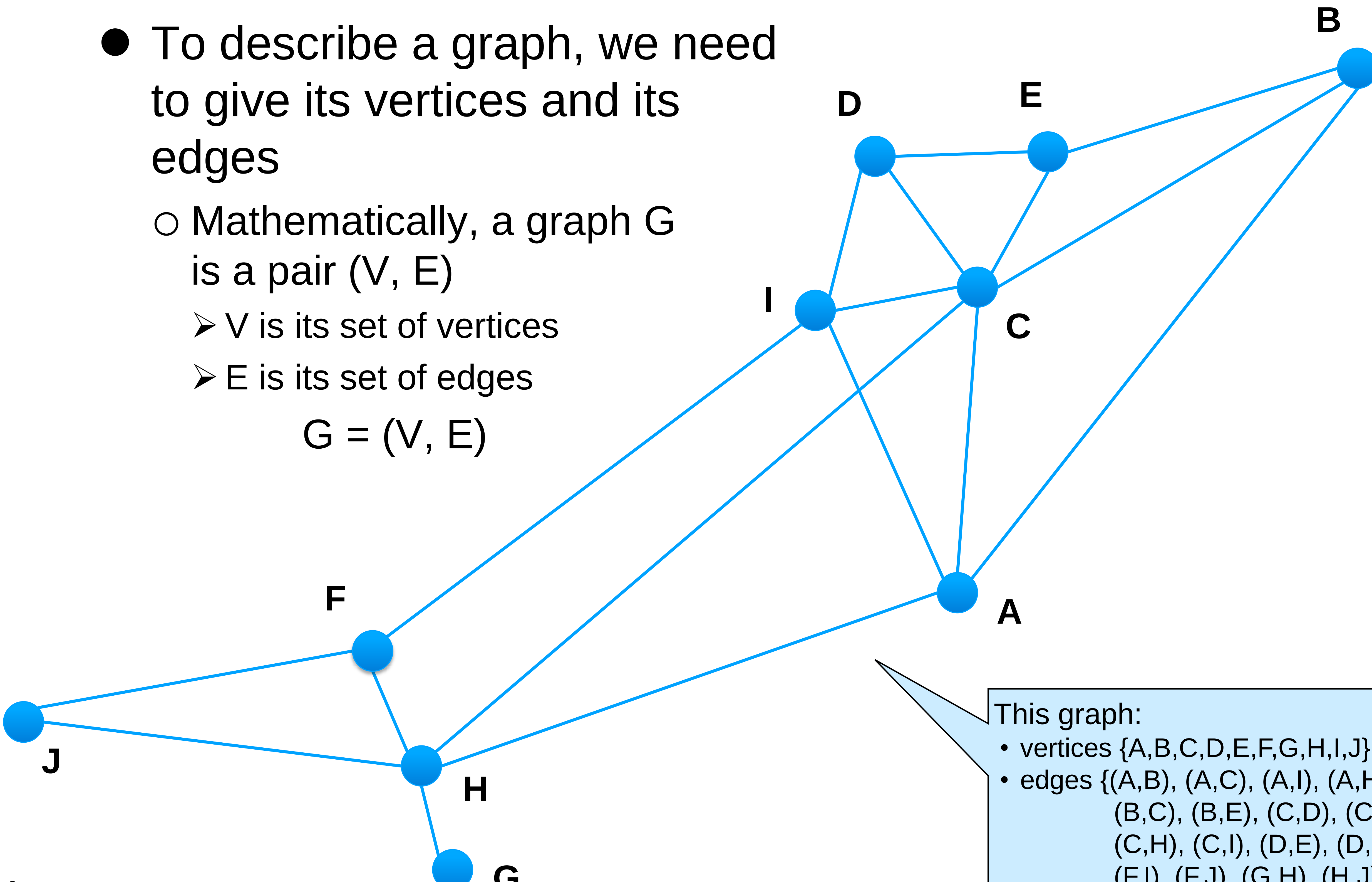


- but there are many other kinds of graphs out there

What is a Graph?

- To describe a graph, we need to give its vertices and its edges
  - Mathematically, a graph G is a pair (V, E)
    - V is its set of vertices
    - E is its set of edges

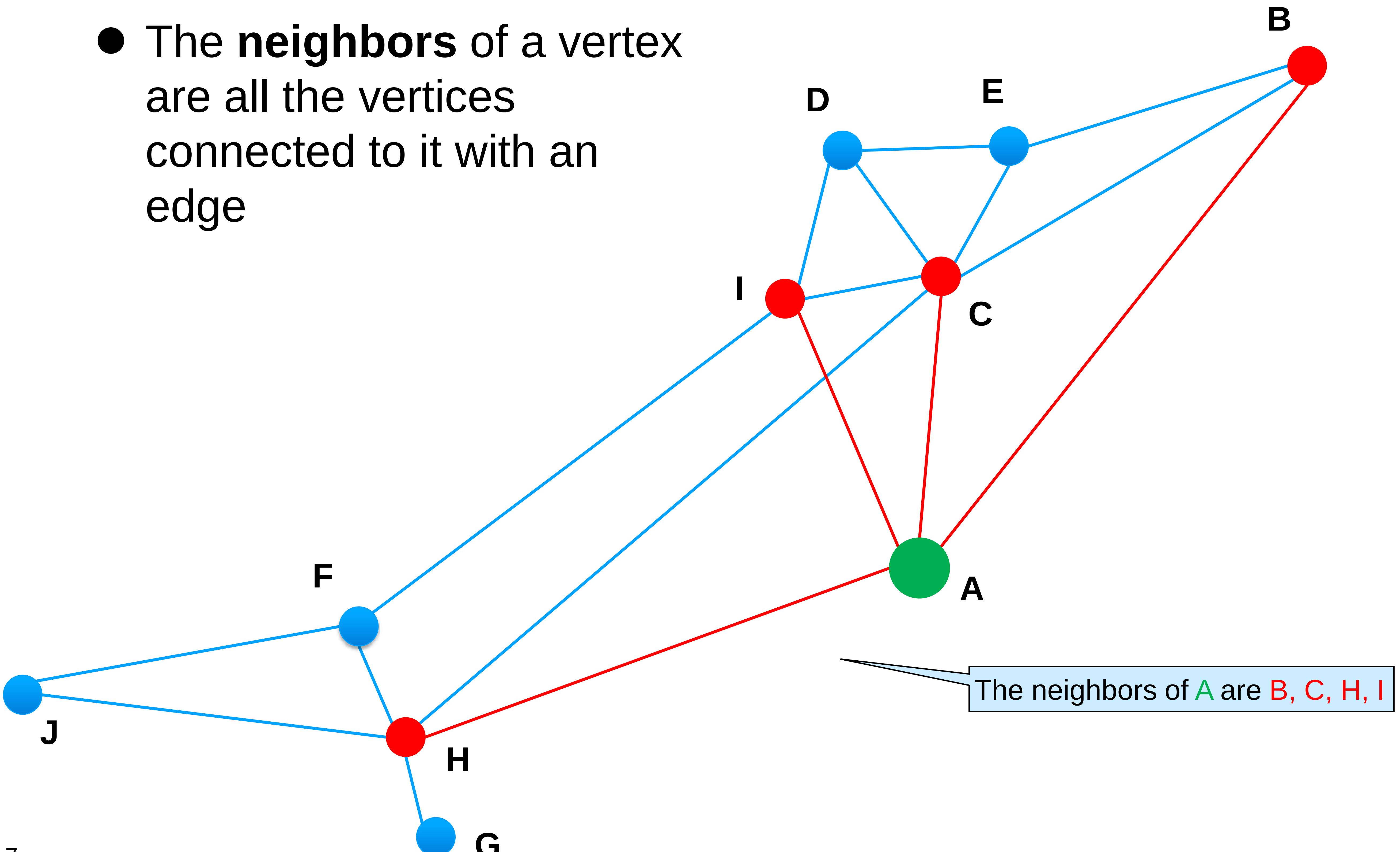
$$G = (V, E)$$



This graph:
- vertices {A,B,C,D,E,F,G,H,I,J}
- edges {(A,B), (A,C), (A,I), (A,H), (B,C), (B,E), (C,D), (C,E), (C,H), (C,I), (D,E), (D,I), (F,H), (F,I), (F,J), (G,H), (H,J)}

What is a Graph?

- The **neighbors** of a vertex are all the vertices connected to it with an edge



What are Graphs Good for?

- Graphs are a convenient **abstraction** that brings out commonalities between different domains

- Once we understand a problem in term of graphs, we can use **general graph algorithms** to solve it
  - no need to reinvent the wheel every time

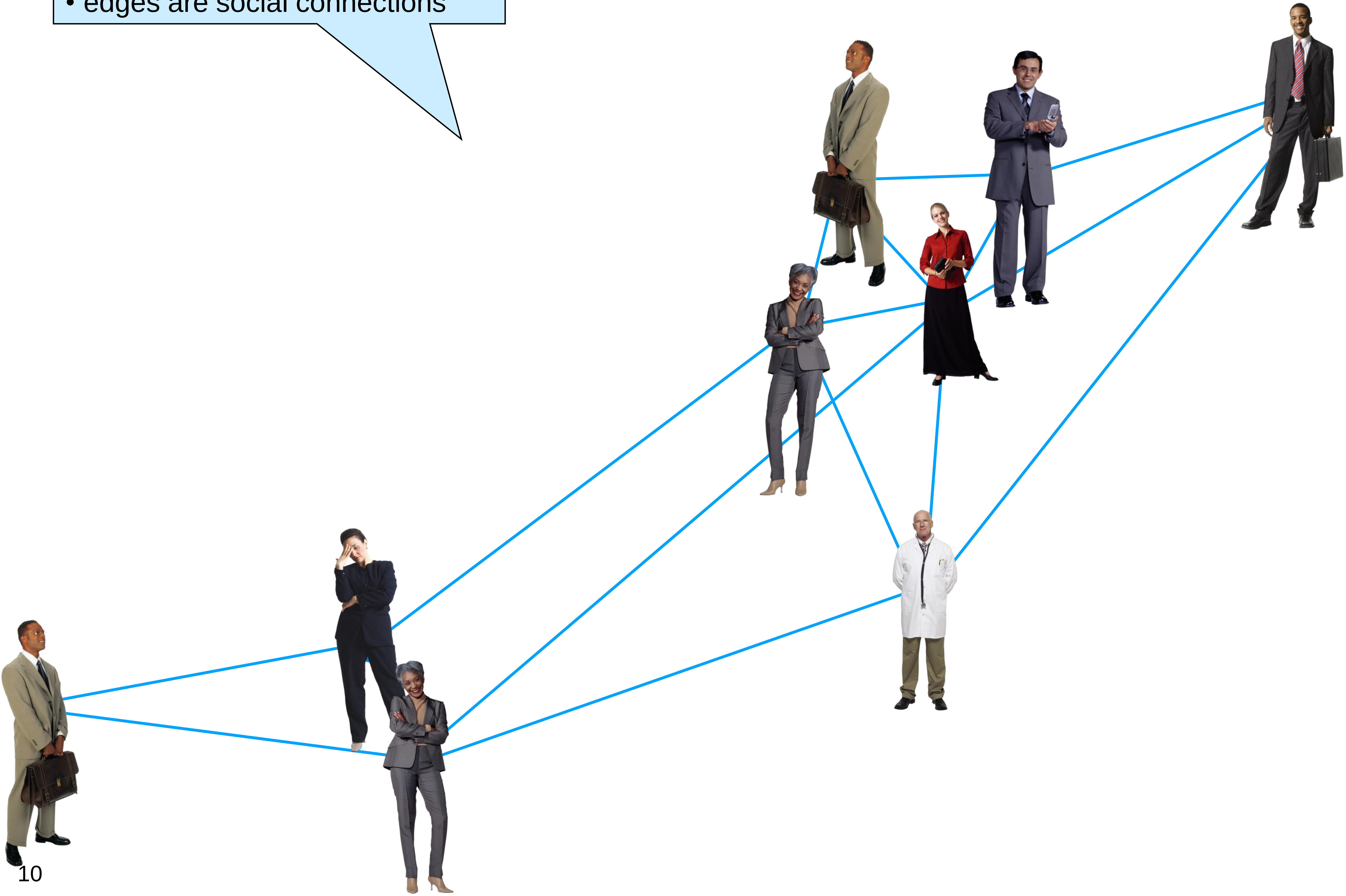
- Graphs are everywhere

Our graph could represent a road network

- vertices are cities
- edges are major highways

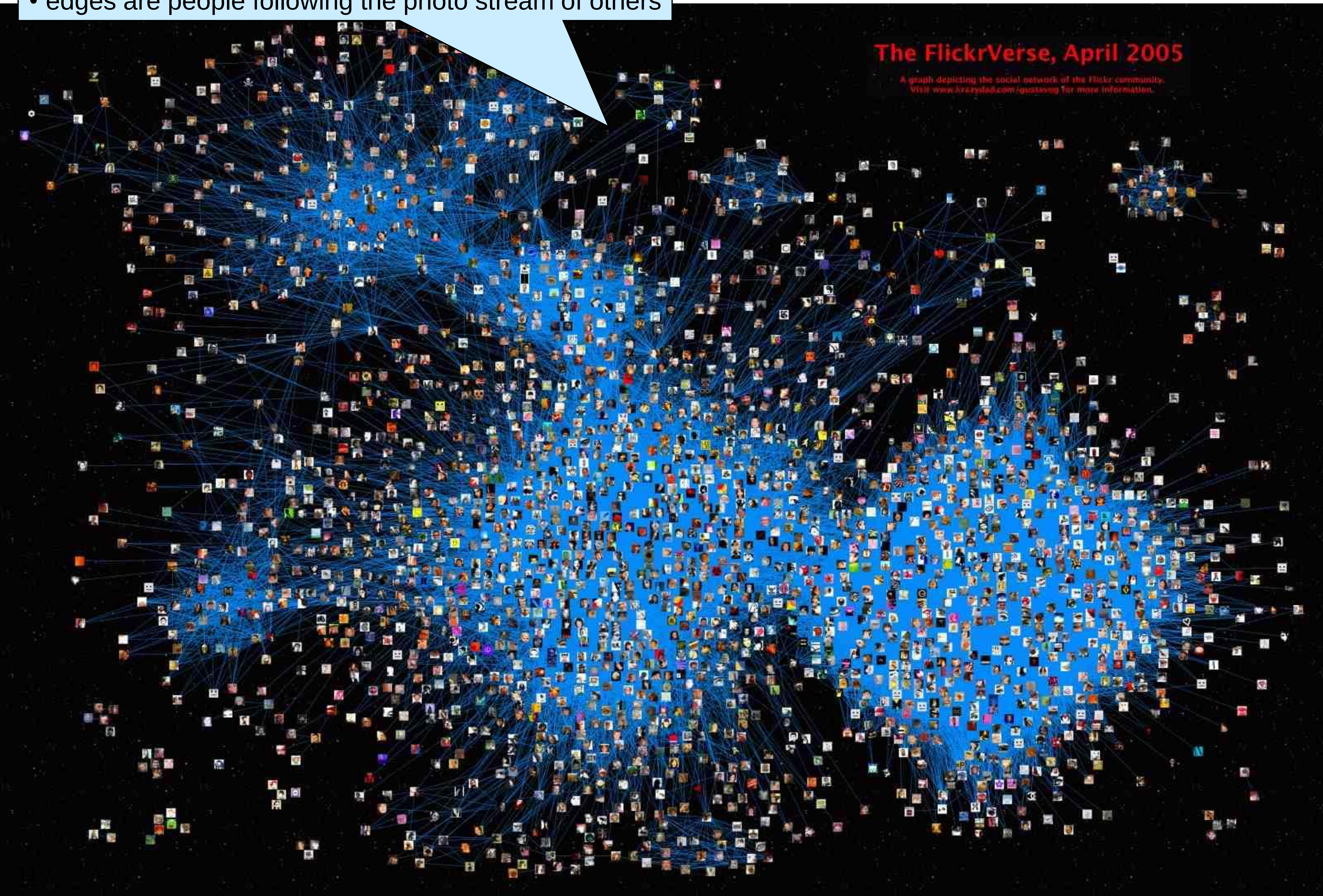
It could represent a social network

- vertices are people
- edges are social connections



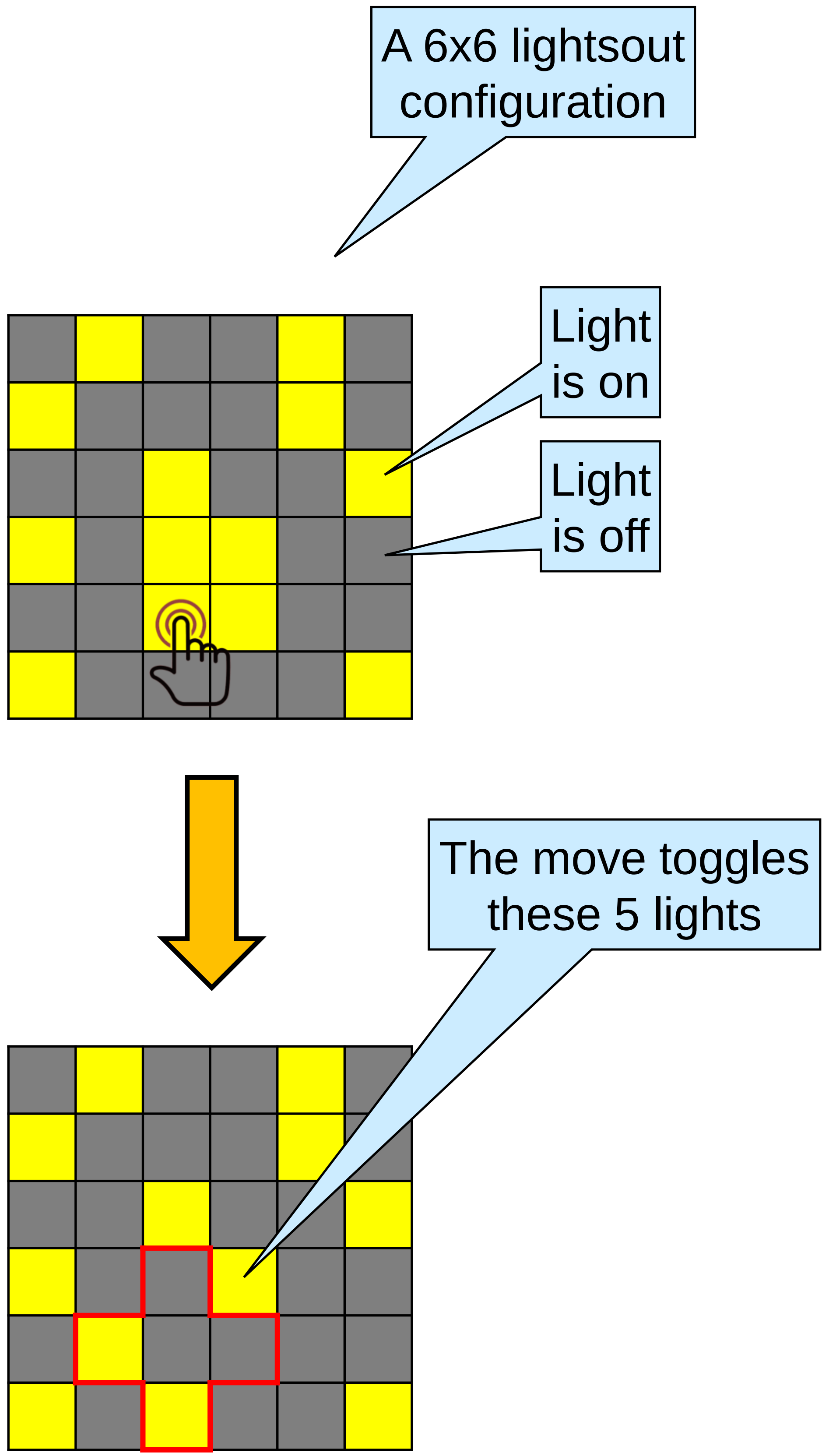
This is what a social network looked like … in 2005

- vertices are people posting photos
- edges are people following the photo stream of others



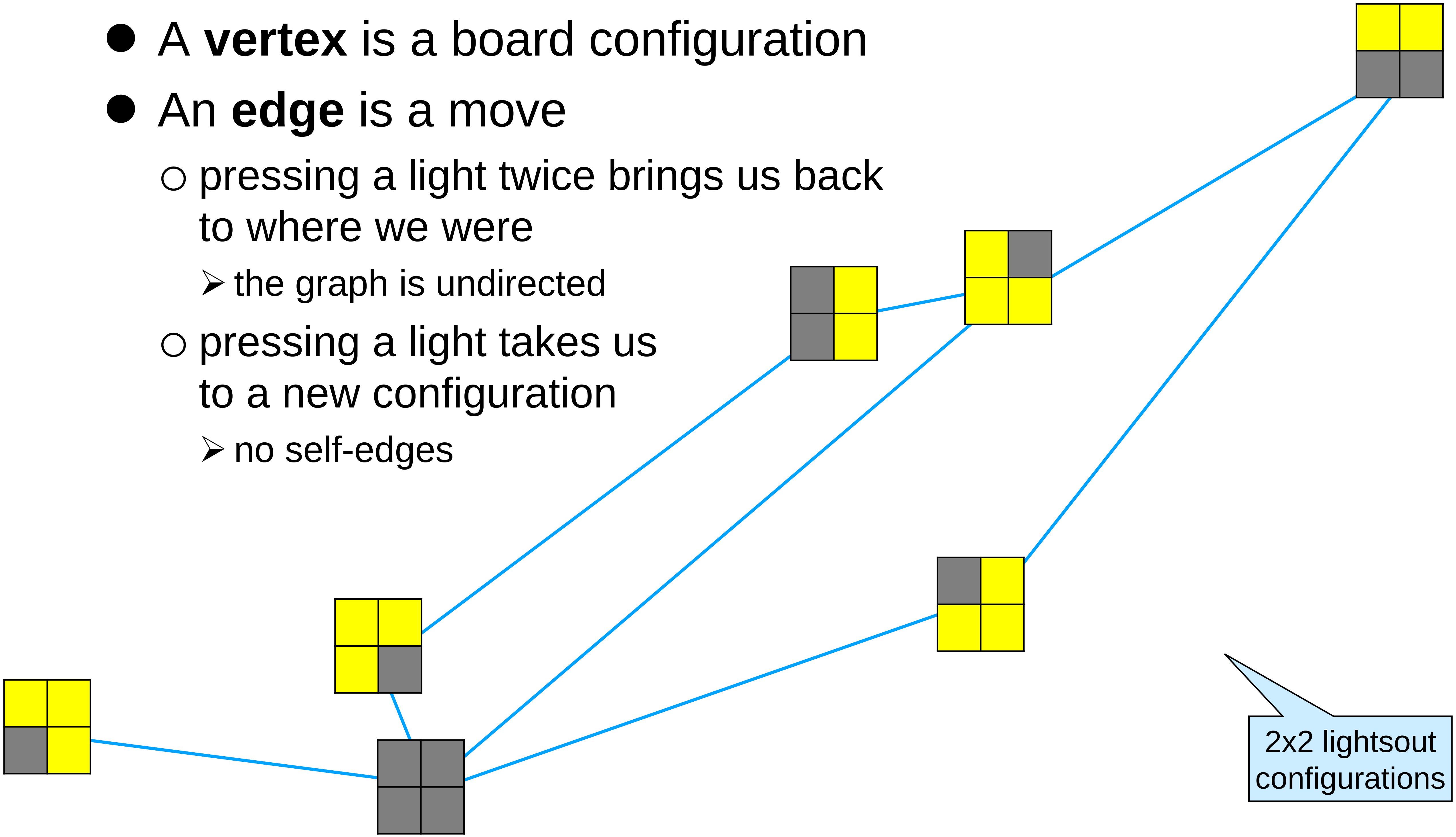
Lightsout

- Lightsout is a *game* played on boards consisting of *n x n* lights
  - each light can be either on or off
- We make a *move* by pressing a light, which toggles it and its cardinal neighbors
- From a given configuration, the *goal* of the game is to turn off all light



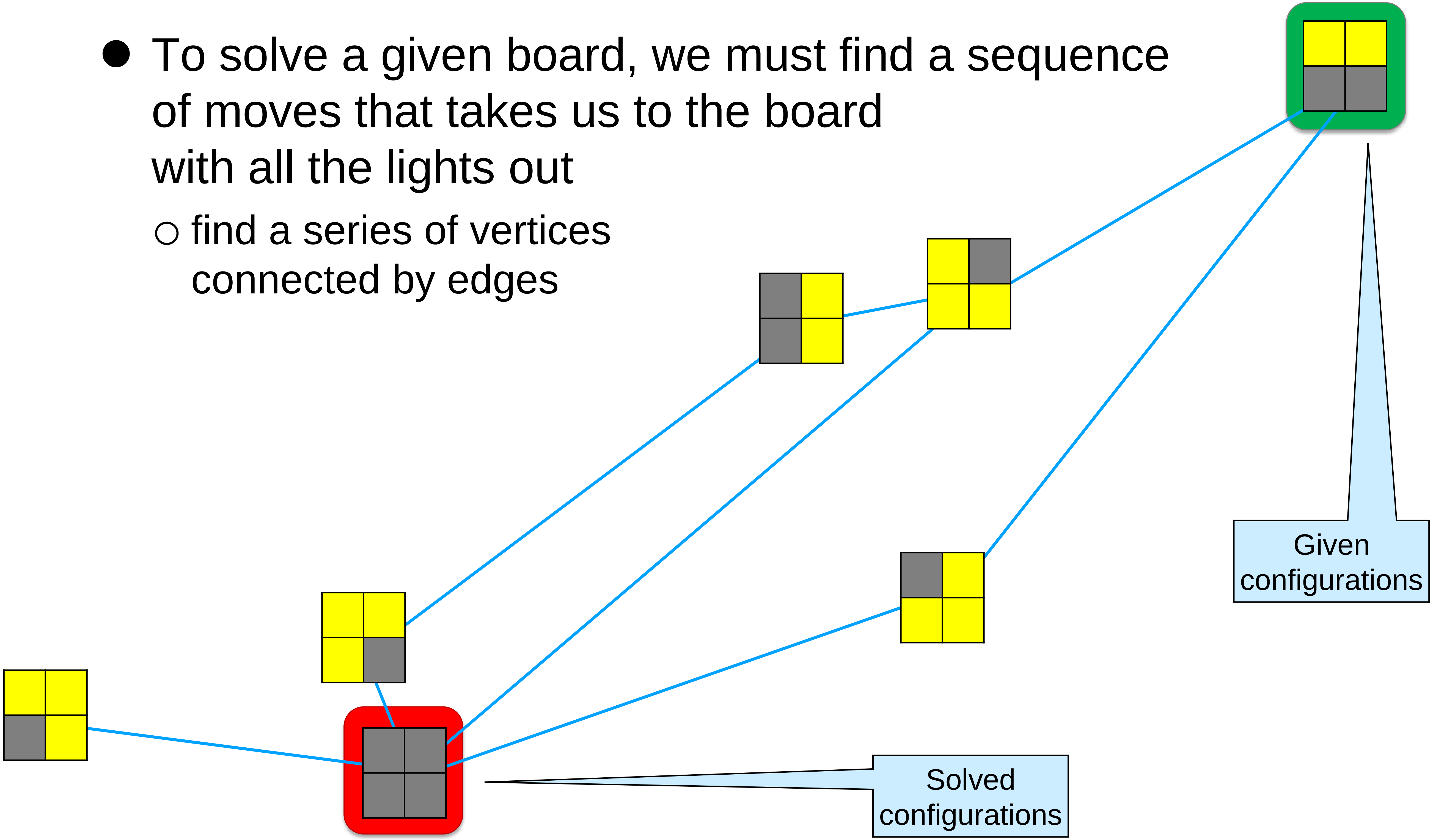
Lightsout as a Graph

- A **vertex** is a board configuration
- An **edge** is a move
  - pressing a light twice brings us back to where we were
    - ➢ the graph is undirected
  - pressing a light takes us to a new configuration
    - ➢ no self-edges



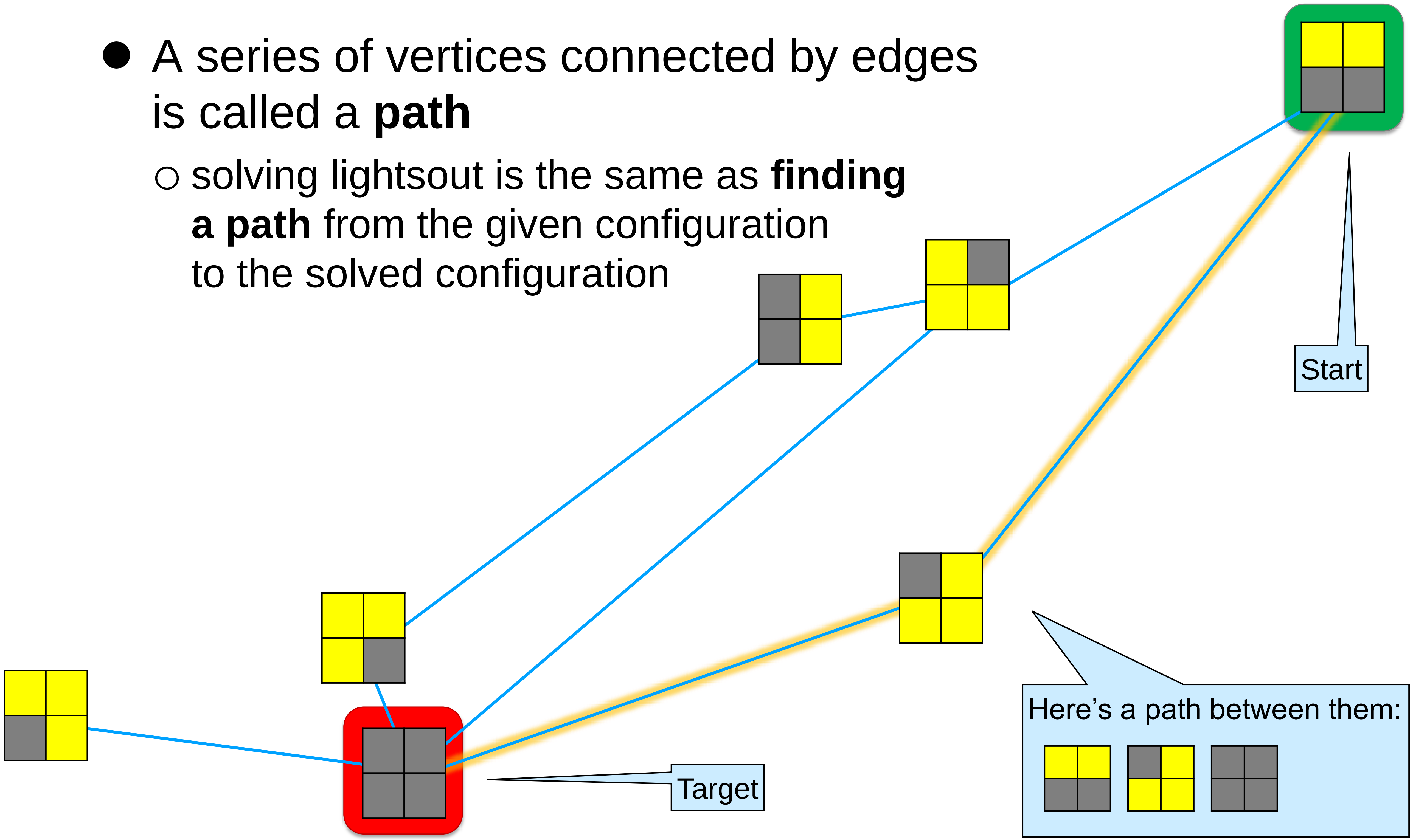
Lightsout as a Graph

- To solve a given board, we must find a sequence of moves that takes us to the board with all the lights out
  - find a series of vertices connected by edges



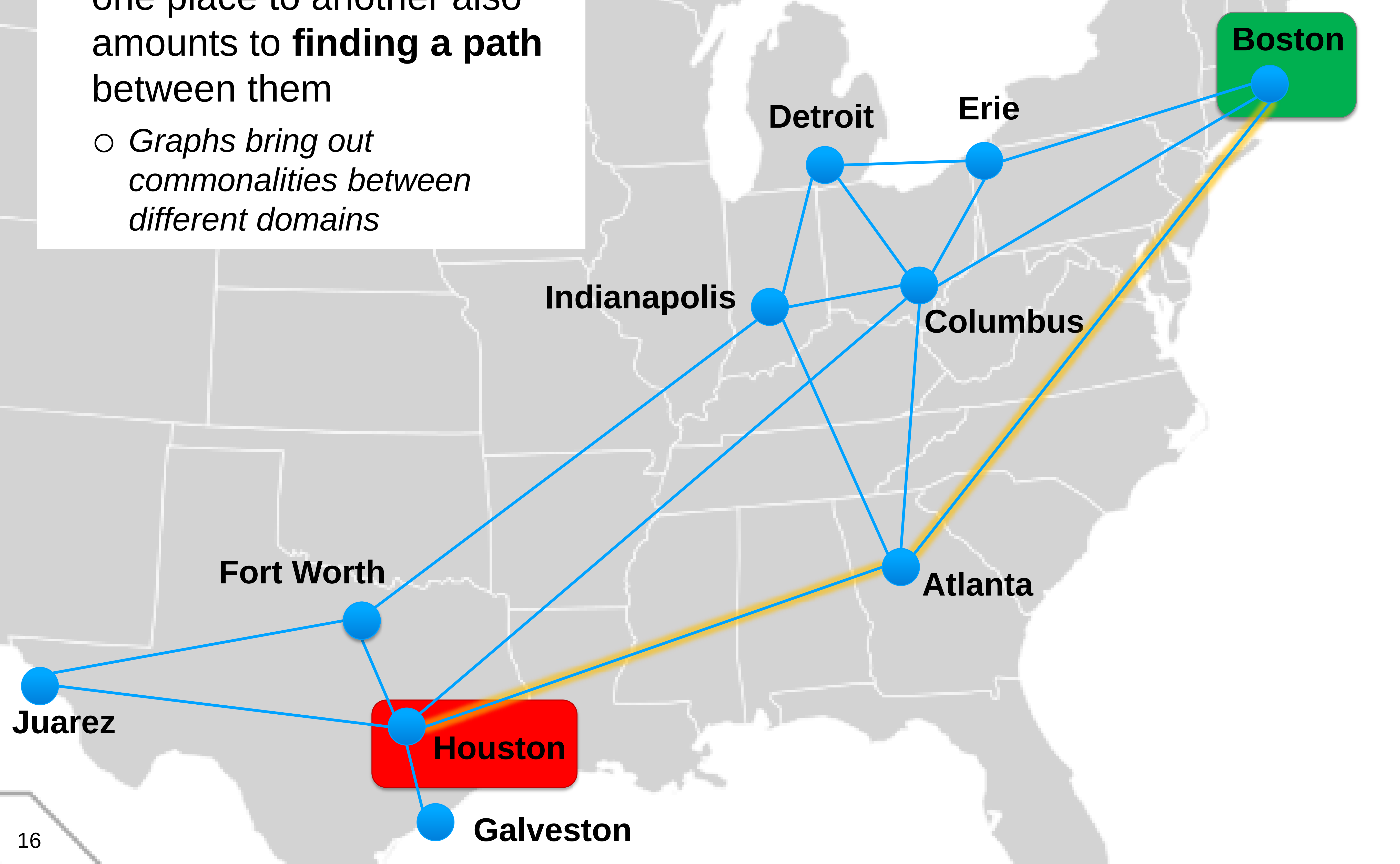
Lightsout as a Graph

- A series of vertices connected by edges is called a **path**
  - solving lightsout is the same as **finding a path** from the given configuration to the solved configuration



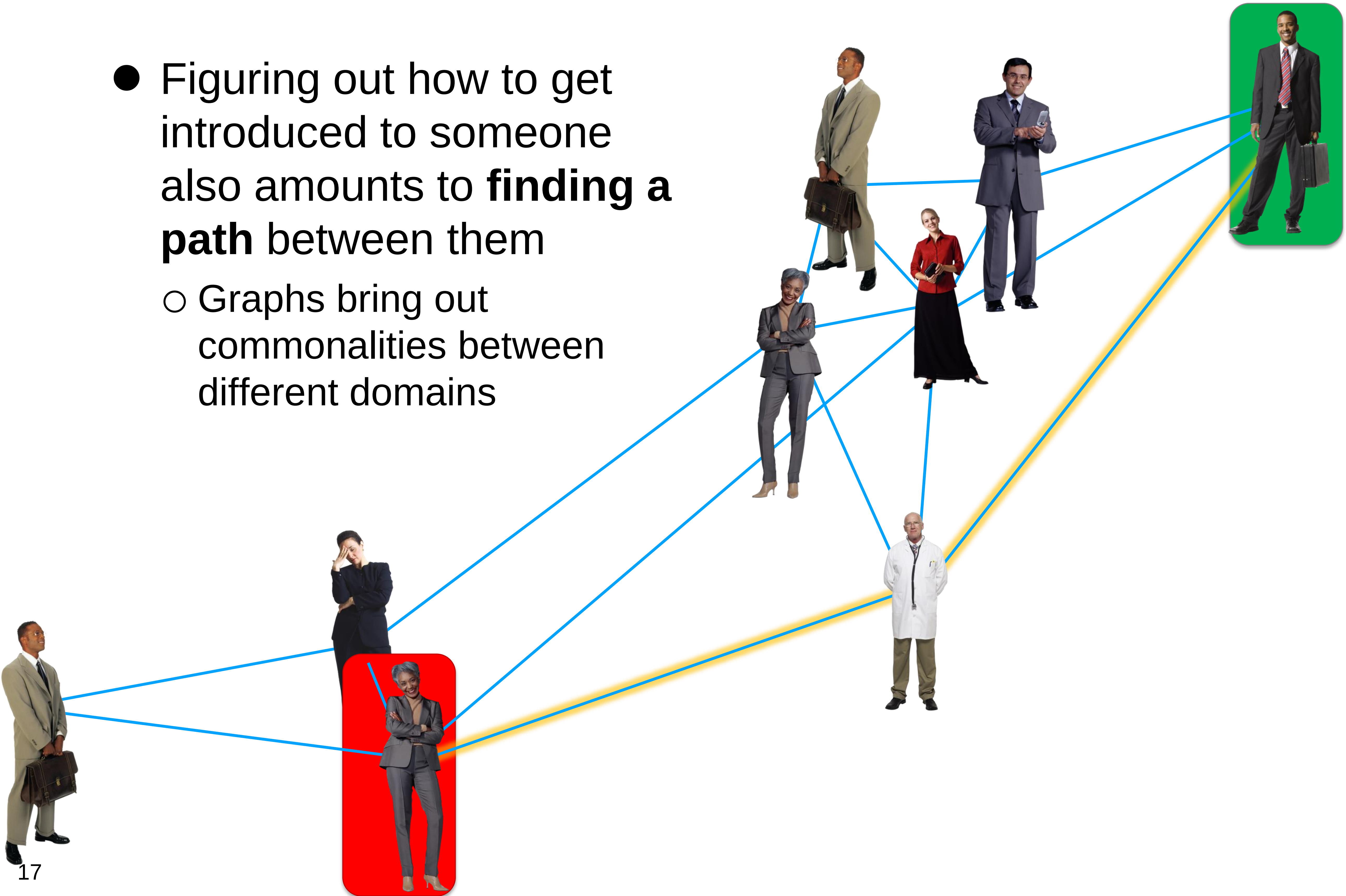
Getting Directions

- Figuring out how to go from one place to another also amounts to **finding a path** between them
    - *Graphs bring out commonalities between different domains*



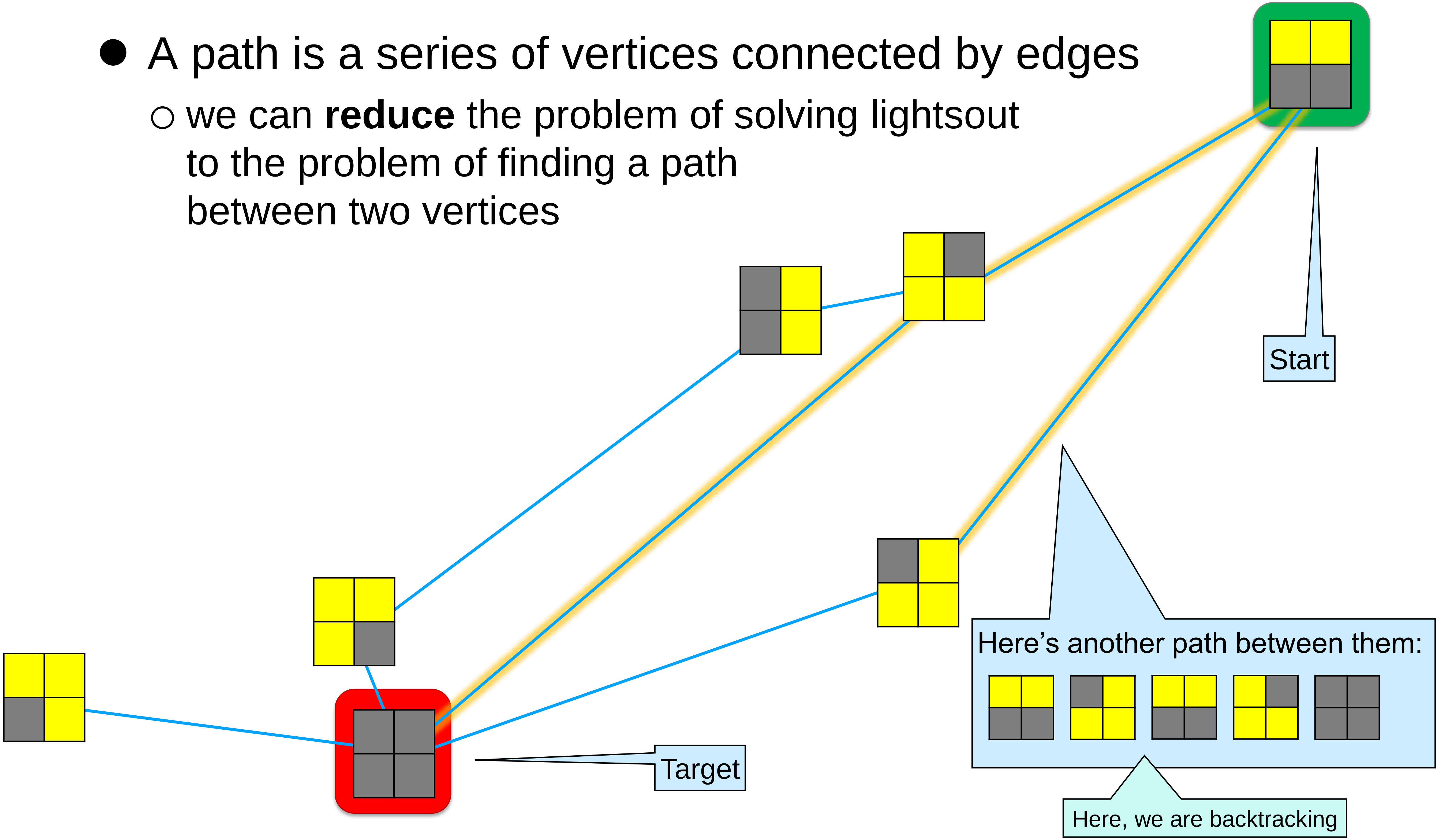
Getting Introduced

- Figuring out how to get introduced to someone also amounts to **finding a path** between them
  - Graphs bring out commonalities between different domains



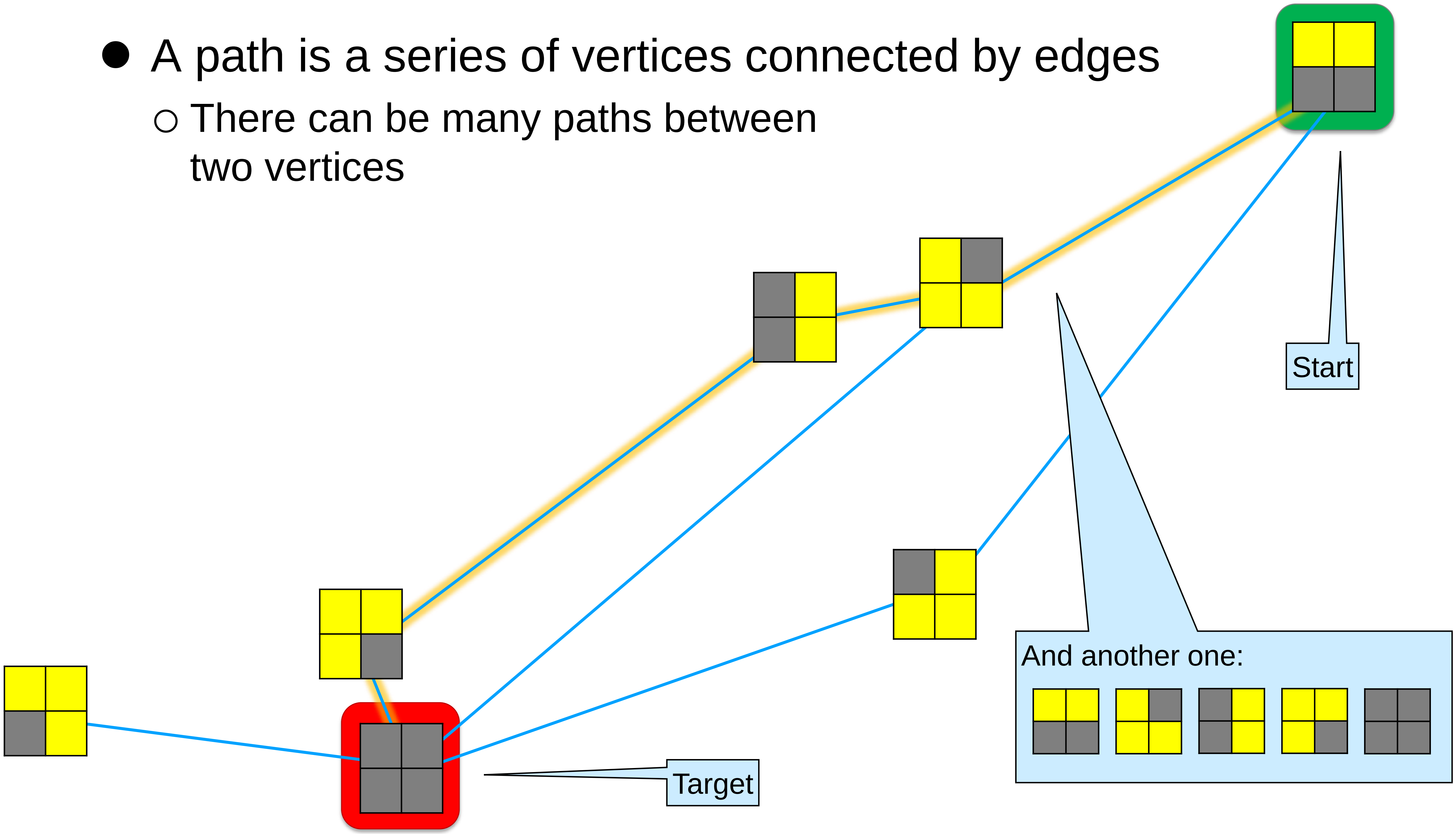
Lightsout as a Graph

- A path is a series of vertices connected by edges
  - we can **reduce** the problem of solving lightsout to the problem of finding a path between two vertices



Lightsout as a Graph

- A path is a series of vertices connected by edges
  - There can be many paths between two vertices



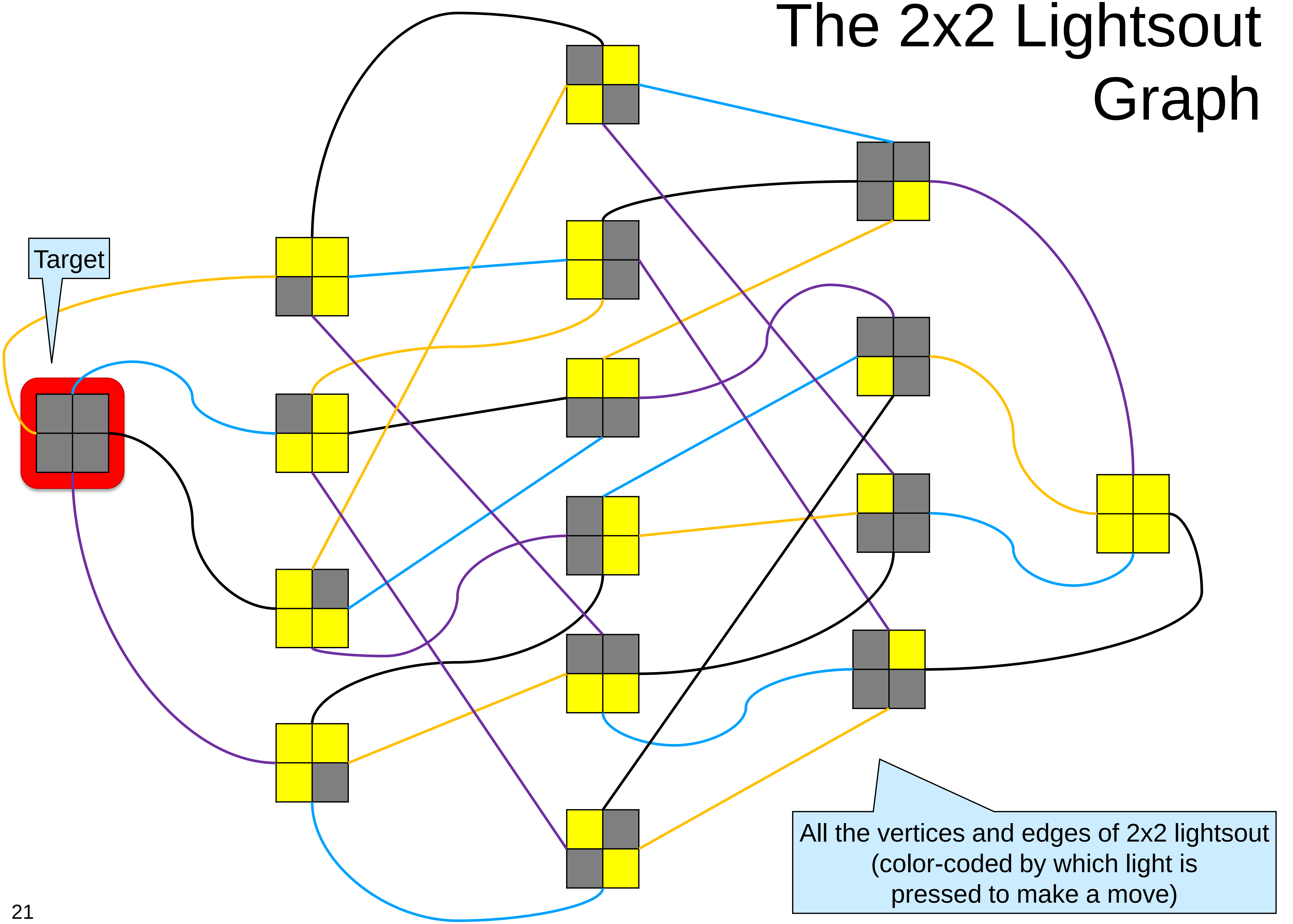
Lightsout as a Graph

- On *n x n* lightsout,
  - there are 2^{n*n} board configurations
    - each of the $n*n$ lights can be either on or off
  - from any board, we can make $n*n$ moves
    - by pressing any one of the $n*n$ lights

- The graph representing *n x n* lightsout has
  - 2^{n*n} vertices
  - $n*n * 2^{n*n} / 2$ edges
    - there are 2^{n*n} vertices
    - each has *n x n* neighbors
    - but this would count each edge (A,B) twice
      - from A to B and
      - from B to A

      so we divide by 2

The 2x2 Lightsout Graph



Models vs. Data Structures

- A graph can be
  - a conceptual **model** to understand a problem
  - a concrete **data structure** to solve it
- For 2x2 lightsout, it is both
  - Conceptually, it brings out the structure of the problem and highlights what it has in common with other problems
  - Concretely, we can traverse a data structure that represents it in search of a path to the solved board
- Turning 6x6 lightsout into a data structure is not practical
  - each board requires 36 bits
  - we need over 64GB to represent its 2^{36} vertices
  - we need over 2TB to represent its $36 * 2^{36} / 2$ edges

That's more memory than most computers have

Implicit Graphs

- We don't need a graph data structure to solve *n x n* lightsout
  - from each board we can **algorithmically** generate all boards that can be reached in one move
  - pick one of them and repeat until
    - we reach the solved board
    - or we reach a previously seen board
      - from it try a different move
- In the process, we are building an **implicit graph**
  - a small portion of the graph exists in memory at any time
    - the boards we have previously seen
      - vertices
    - the moves we still need to try from them
      - edges

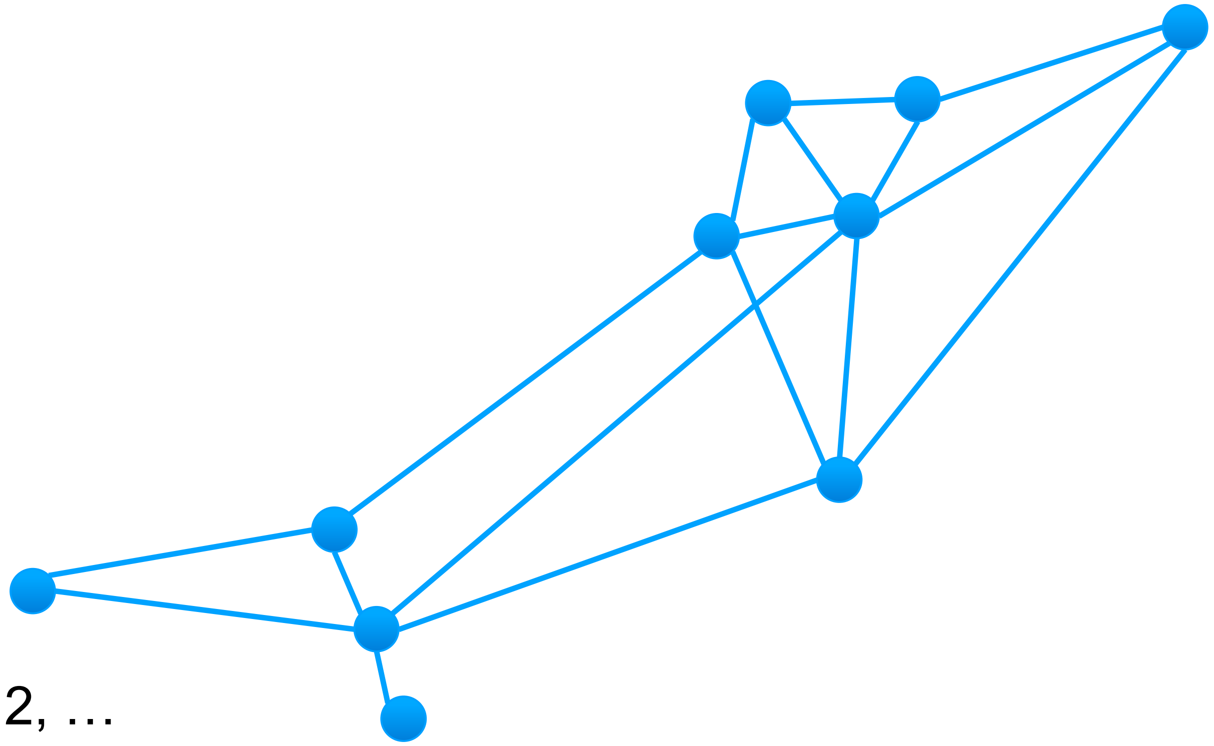
Explicit Graphs

- For many graphs, there is no algorithmic way to generate their edges
  - roads between cities
  - social networks
  - …
- We must represent them explicitly as a data structure in memory
- We will now develop a small library for solving problems with these **explicit graphs**

A Graph Interface

A Minimal Graph Data Structure

- What we need to represent
  - graphs themselves
    - type graph_t
  - the vertices of a graph
    - type vertex
      - we label vertices with the numbers 0, 1, 2, …
        - consecutive integers starting at 0
      - vertex is defined as unsigned int
  - the edges of the graph
    - we represent an edge as its endpoints
      - *no need for an edge type*



A Minimal Graph Data Structure

- Basic operations on graphs
  - graph_new(n) create a new graph with n vertices
    - we fix the number of vertices at creation time
      - we cannot add vertices after the fact
  - graph_size(G) returns the number of vertices in G
  - graph_hasedge(G, v, w) checks if the graph G contains the edge (v,w)
  - graph_addedge(G, v, w) adds the edge (v,w) to the graph G
  - graph_free(G) disposes of G

- A realistic graph library would provide a much richer set of operations
  - we can define most of them on the basis of these five

A Minimal Graph Interface – I

File graph.h

```
typedef unsigned int vertex;
typedef struct graph_header *graph_t;

graph_t graph_new(unsigned int numvert);
//@ensures \result != NULL;

void graph_free(graph_t G);
//@requires G != NULL;

unsigned int graph_size(graph_t G);
//@requires G != NULL;

bool graph_hasedge(graph_t G, vertex v, vertex w);
//@requires G != NULL;
//@requires v < graph_size(G) && w < graph_size(G);

void graph_addedge(graph_t G, vertex v, vertex w);
//@requires G != NULL;
//@requires v < graph_size(G) && w < graph_size(G);
//@requires v != w && !graph_hasedge(G, v, w);
  …
```

vertex is a concrete type

In a C header file,
we must define abstract types
… but we don't need to give the details

This says that v and w
must be valid vertices

No self-edges

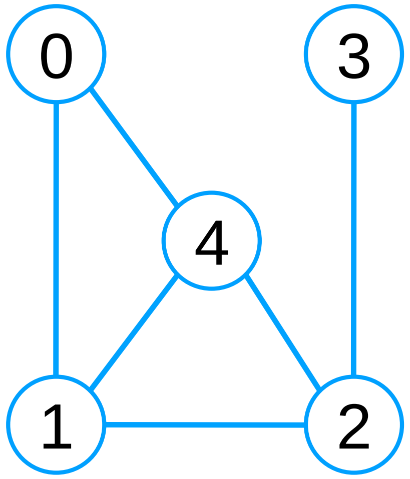
For simplicity,
only add **new** edges

Example

- We create this graph as

```
graph_t G = graph_new(5);
graph_addedge(G, 0, 1);
graph_addedge(G, 0, 4);
graph_addedge(G, 1, 2);
graph_addedge(G, 1, 4);
graph_addedge(G, 2, 3);
graph_addedge(G, 2, 4);
```

in any order



We sometimes write the labels inside the vertices

- Then
  - graph_hasedge(G, 3, 2) returns true, but
  - graph_hasedge(G, 3, 1) return false
    - there is a path from 3 to 1, but no direct edge

Neighbors

- It is convenient to handle neighbors explicitly
    - this is not strictly necessary
    - but graph algorithms get better complexity if we do so inside the library

- Abstract type of neighbors
    - neighbors_t
- Operations on neighbors
    - graph_get_neighbors(G, v)
        - returns the neighbors of vertex v in G
    - graph_hasmore_neighbors(nbors)
        - checks if there are additional neighbors
    - graph_next_neighbor(nbors)
        - returns the next neighbor
    - graph_free_neighbors(nbors)
        - dispose of unexamined neighbors

These allow us to iterate through the neighbors of a vertex

This is called an **iterator**

A Minimal Graph Interface – II

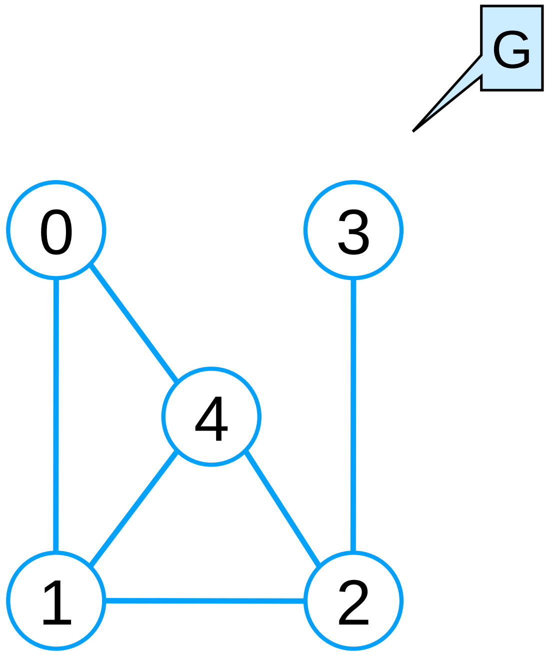
File graph.h

```
…

typedef struct neighbor_header *neighbors_t;

neighbors_t graph_get_neighbors(graph_t G, vertex v);
//@requires G != NULL && v < graph_size(G);
//@ensures \result != NULL;

bool graph_hasmore_neighbors(neighbors_t nbors);
//@requires nbors  != NULL;

vertex graph_next_neighbor(neighbors_t nbors);
//@requires nbors  != NULL;
//@requires graph_hasmore_neighbors(nbors);

void graph_free_neighbors(neighbors_t nbors);
//@requires nbors  != NULL;
```

These declarations are part of the same header file

There must be additional neighbors to retrieve the next neighbor

Example



- We grab the neighbors of vertex 4 as

  neighbors_t n4 = graph_get_neighbors(G, 4);
  - n4 contains vertices 0, 1, 2 in some order

  vertex a = graph_next_neighbor(n4);
  - say a is vertex 1
    - it could also be 0 or 2

  vertex b = graph_next_neighbor(n4);
  - say b is vertex 0
    - it cannot be 1 because we already got that neighbor
    - but it could be 2

  vertex c = graph_next_neighbor(n4);
  - c has to be vertex 2
    - it cannot be 0 or 1 because we already got those neighbors

  graph_hasmore_neighbor(n4)
  - returns false because we have exhausted all the neighbors of 4

Implementing Graphs

Implementing Graphs

- How to implement graphs based on what we studied?
  - The main operations are
    - adding an edge to the graph
    - checking if an edge is contained in the graph
      - These are the operations we had for **sets**
    - iterating through the neighbors of a vertex

- Implement graphs as
  - a linked list of edges
  - a hash set

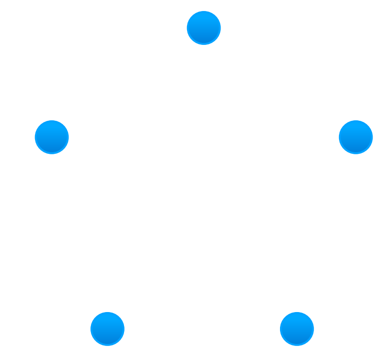
We could also use AVL trees
if we are able to sort the edges

- How much would the operations cost?

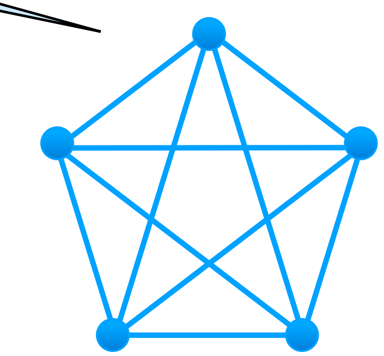
Measuring the Cost of Graph Operations

- If a graph has **v** vertices, the number **e** of edges ranges between
  - 0, and
  - v(v-1)/2
    - there is an edge between each of the v vertices and the other v-1 vertices, but we divide by 2 so that we don't double-count edges

The graph has no edges



This is a complete graph



- So, e $\in O(v^2)$
  - we could do with just v as a cost parameter,
  - but many graphs have far fewer than v(v-1)/2 edges
    - using only v would be overly pessimistic
- Use **both** v and e as cost parameters

Naïve Graph Implementations

- For implementations based on known data structures, the cost of the basic graph operations are

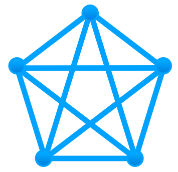
	Linked list of edges	**Hash set of edges**
graph_hasedge	O(e)	O(1) avg
graph_addedge	O(1)	O(1) avg+amt

- What about iterating through the neighbors of a vertex?

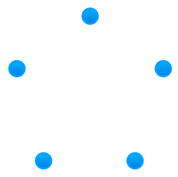
Naïve Graph Implementations

- Finding the neighbors of a vertex requires going over all the edges
  - graph_get_neighbors has cost O(e) and O(v) avg

- How many neighbors are there?
  - at most v-1
    - if this vertex has an edge to all other vertices



  - at most e
    - there cannot be more neighbors than edges in the graph



- A vertex has O(min(v,e)) neighbors
  - iterating through the neighbors costs O(min(v,e))
    - times the cost of the operation being performed

Naïve Graph Implementations

- In summary

	Linked list of edges	**Hash set of edges**
graph_hasedge	O(e)	O(1) avg
graph_addedge	O(1)	O(1) avg + amt
graph_get_neighbors	O(e)	O(v) avg
Iterating through neighbors	O(min(v,e))	O(min(v,e))

Classic Graph Implementations

- Can we do better?

- Two representations of graphs are commonly used
  - the adjacency matrix representation
  - the adjacency list representation

“adjacency” is just a fancy word for neighbors

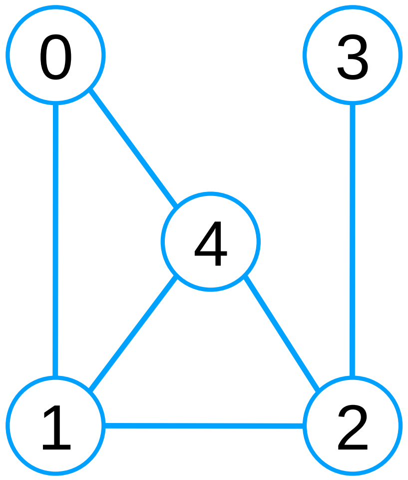
- Both give us better cost

  … in different situations …

The Adjacency Matrix Representation

- Represent the graph as a v*v matrix of booleans
  - M[i,j] == true if there is an edge between i and j
  - M[i,j] == false otherwise

  M is called the **adjacency matrix**



- Cost of the operations
  - graph_hasedge(G, v, w): O(1)
    - just return M[v,w]
  - graph_addedge(G, v, w): O(1)
    - just set M[v,w] to true
  - graph_get_neighbors(G, v): O(v)
    - go through the row for v in M

	0	1	2	3	4
0		✓			✓
1	✓		✓		✓
2		✓		✓	✓
3			✓		
4	✓	✓	✓		

M[2,4] == true because G contains edge (2,4)

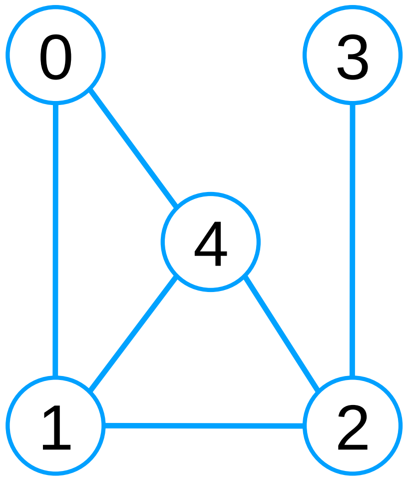
For undirected graphs, M is symmetric: M[i,j] == M[j,i]

No self-edges, so M[i,i] == false

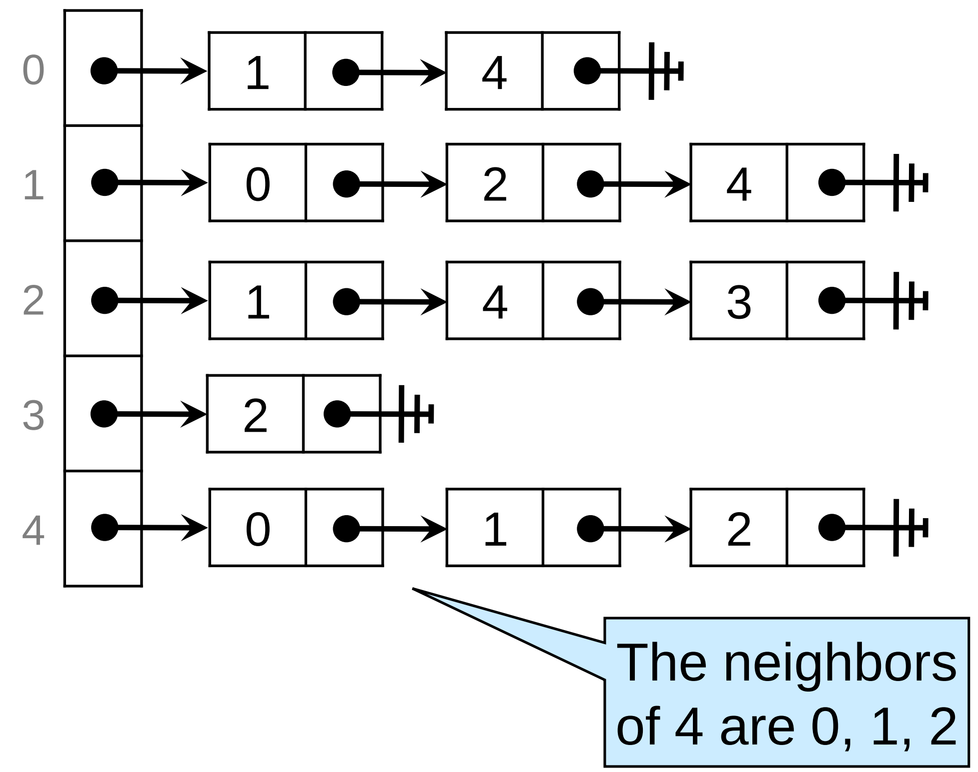
- Space needed: $O(v^2)$

The Adjacency List Representation

- For each vertex v, keep track of its neighbors in a list
  - the **adjacency list** of v
- Store the adjacency lists in a vertex-indexed array

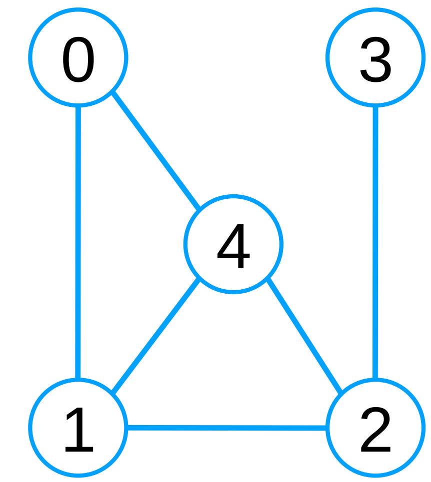


- Cost of the operations
  - graph_hasedge(G, v, w): O(min(v,e))
    - each vertex has O(min(v,e)) neighbors
    - each adjacency list has length O(min(v,e))
  - graph_addedge(G, v, w): O(1)
    - add v in w's list and w in v's list
  - graph_get_neighbors(G, v): O(1)
    - just grab v's adjacency list

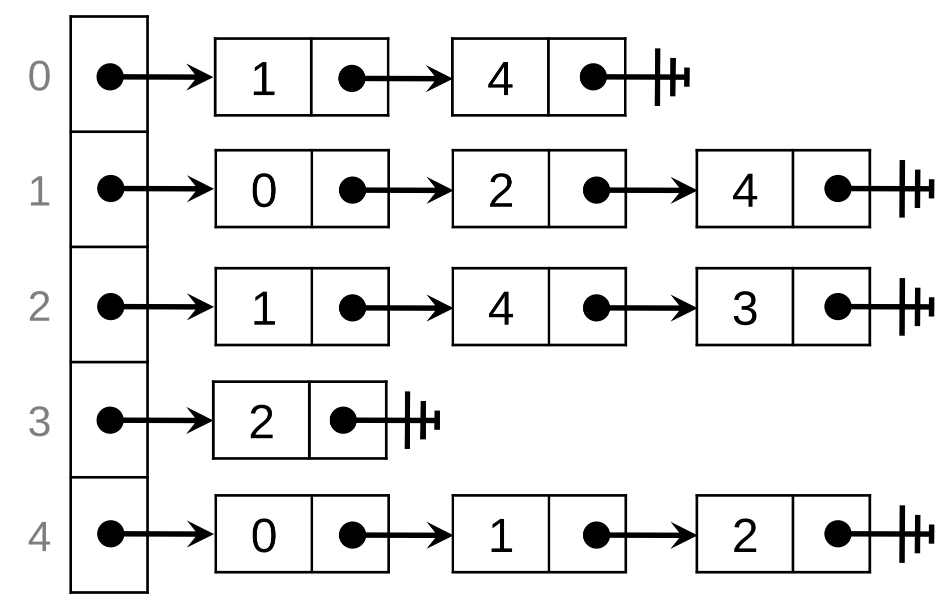


The Adjacency List Representation

- *For each vertex v, keep track of its neighbors in a list*
  - *the **adjacency list** of v*
- *Store the adjacency lists in a vertex-indexed array*



- Space needed: O(v + e)
  - a v-element array
  - 2e list items
    - each edge corresponds to exactly 2 list items



- O(v + e) is conventionally written O(max(v,e))

Why? Note that $\max(v,e) \le v+e \le 2\max(v,e)$

Adjacency Matrix vs. List

	Adjacency matrix	**Adjacency list**
Space	$O(v^2)$	O(v + e)
graph_hasedge	O(1)	O(min(v,e))
graph_addedge	O(1)	O(1)
graph_get_neighbors	O(v)	O(1)
Iterating through neighbors	O(min(v,e))	O(min(v,e))

When to Use What Representation?

- Recall that $0 \leq e \leq v(v-1)/2$
- A graph is **dense** if it has lots of edges
  - e is on the order of v^2
- A graph is **sparse** if it has relatively few edges
  - e is in $O(v)$
    - at most $O(v \log v)$
    - but definitely not $O(v^2)$
  - lots of graphs are sparse
    - social networks
    - roads between cities
    - …

Cost in Dense Graphs

- We replace e with v^2 and simplify

	Adjacency matrix	Adjacency list	
Space	$O(v^2)$	$O(v + e) \rightarrow O(v^2)$	Same
graph_hasedge	O(1)	O(min(v,e)) → O(v)	AM
graph_addedge	O(1)	O(1)	Same
graph_get_neighbors	O(v)	O(1)	AL
Iterating through neighbors	O(min(v,e)) → O(v)	O(min(v,e)) → O(v)	Same

Cost in Dense Graphs

- graph_hasedge is faster with AM
- graph_get_neighbors is faster with AL
  - but we typically iterate through the neighbors after grabbing them
- All other operations are the same
- The space requirements are the same

- For dense graphs
  - the two representations have about the same cost
  - but graph_hasedge is faster with AM

  the adjacency matrix representation is preferable

Cost in Sparse Graphs

- We replace e with v and simplify

Assume e $\in$ O(v)

	Adjacency matrix	Adjacency list	
Space	$O(v^2)$	O(v + e) → O(v)	AL
graph_hasedge	O(1)	O(min(v,e)) → O(v)	AM
graph_addedge	O(1)	O(1)	Same
graph_get_neighbors	O(v)	O(1)	AL
Iterating through neighbors	O(min(v,e)) → O(v)	O(min(v,e)) → O(v)	Same

Cost in Sparse Graphs

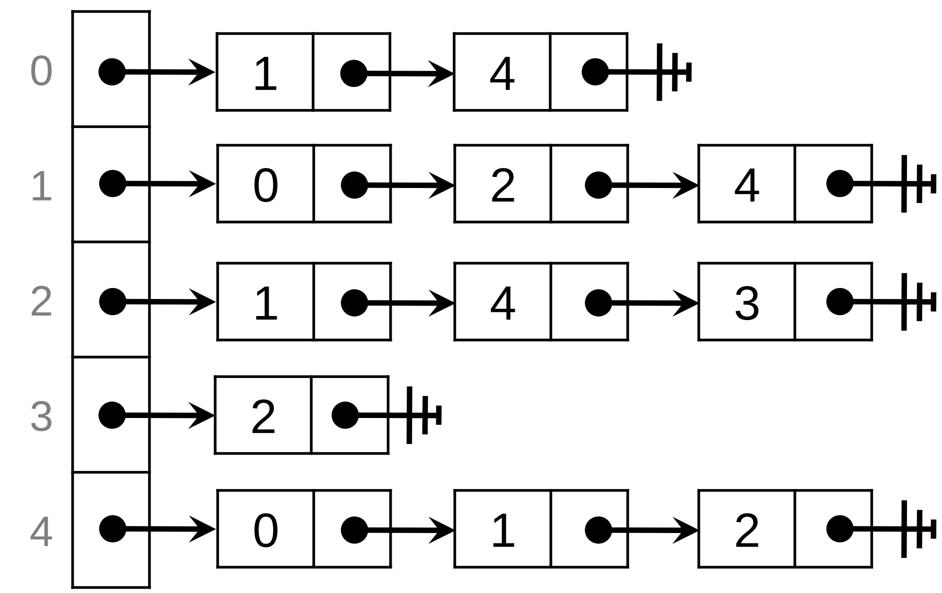
- AL requires **a lot less space**
- graph_hasedge is faster with AM
- graph_get_neighbors is faster with AL
  - but we typically iterate through the neighbors after grabbing them
- All other operations are the same

- For sparse graphs
  - AL uses substantially less space
  - the two representations have about the same cost
  - but graph_hasedge is faster with AM

  the adjacency list representation is preferable because it doesn’t require as much space

Adjacency List Implementation

Graph Types



- An adjacency list is just a NULL-terminated linked list of vertices
- The graph data structure consists of
  - the number v of vertices in the graph
    - field size
  - a v-element array of adjacency lists
    - field adjlist

```
typedef struct adjlist_node adjlist;
struct adjlist_node {
  vertex vert;
  adjlist *next;
};

typedef struct graph_header graph;
struct graph_header {
  unsigned int size;
  adjlist **adj;
};
```

adjlist*[] adj in C0

Representation Invariants

- The interface defines

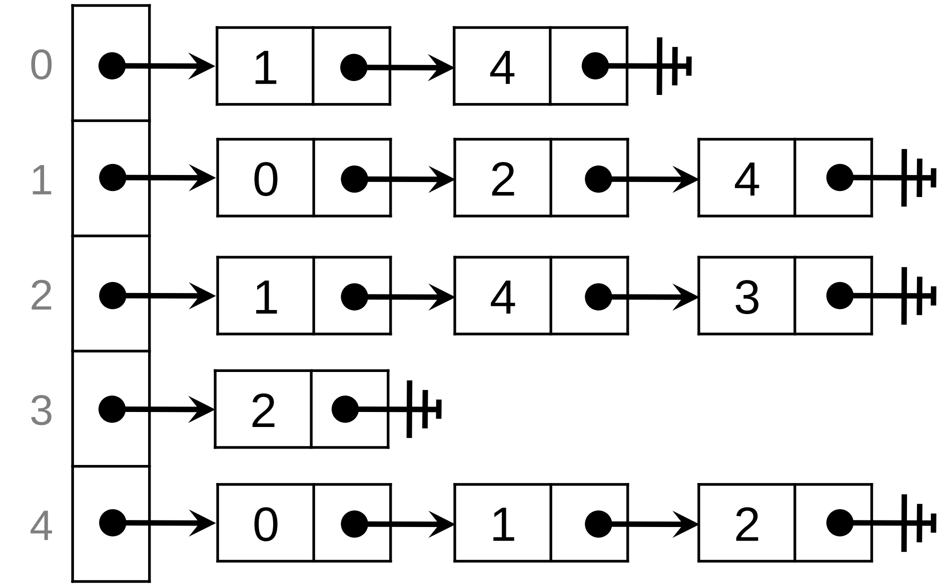
  typedef unsigned int vertex;

- A vertex is valid if its value is between 0 and the size of the graph

```
bool is_vertex(graph *G, vertex v) {
  REQUIRES(G != NULL);
  return v < G->size;
}
```

0 <= v
is automatic since v has
type unsigned int

Representation Invariants

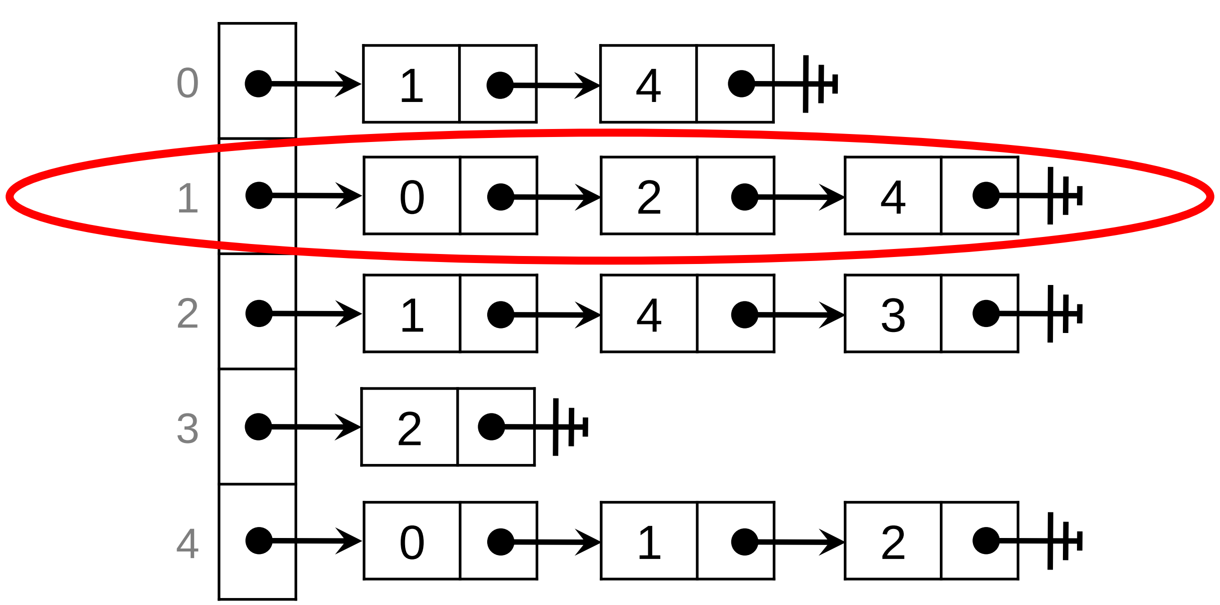


- A graph is valid if
  - it is non-NULL
  - the length of the array of adjacency lists is equal to it size
    - **but we can’t check this in C**
  - each adjacency list is valid

```
bool is_graph(graph *G) {
  if (G == NULL) return false;
  //@assert(G->size == \length(G->adj));
  for (unsigned int i = 0; i < G->size; i++) {
    if (!is_adjlist(G, i, G->adj[i])) return false;
  }
  return true;
}
```

Representation Invariants

- An adjacency list is valid if
  - it is NULL-terminated
  - each vertex is valid
  - there are not self-edges
  - every outgoing edge has a corresponding edge coming back in
  - there are no duplicate edges



```
bool is_adjlist(graph *G, vertex v, adjlist *L) {
  REQUIRES(G != NULL);
  //@requires(G->size == \length(G->adj));
  if (!is_acyclic(L)) return false;

  while (L != NULL) {
    vertex w = L->vert;      // w is a neighbor of v

    // Neighbors are legal vertices
    if (!is_vertex(G, wt)) return false;

    // No self-edges
    if (v == w) return false;

    // Every outgoing edge has a corresponding
    //    edge coming back to it
    if (!is_in_adjlist(G->adj[w], v)) return false;

    // Edges aren't duplicated
    if (is_in_adjlist(L->next, w)) return false;

    L = L->next;
  }
  return true;
}
```

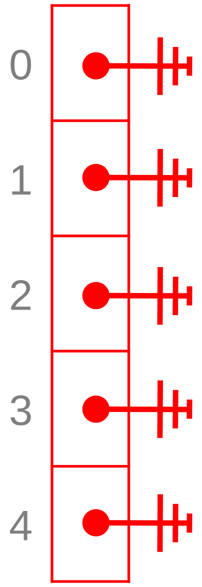
Basic operations

- graph_size returns the stored size
  - Cost O(1)

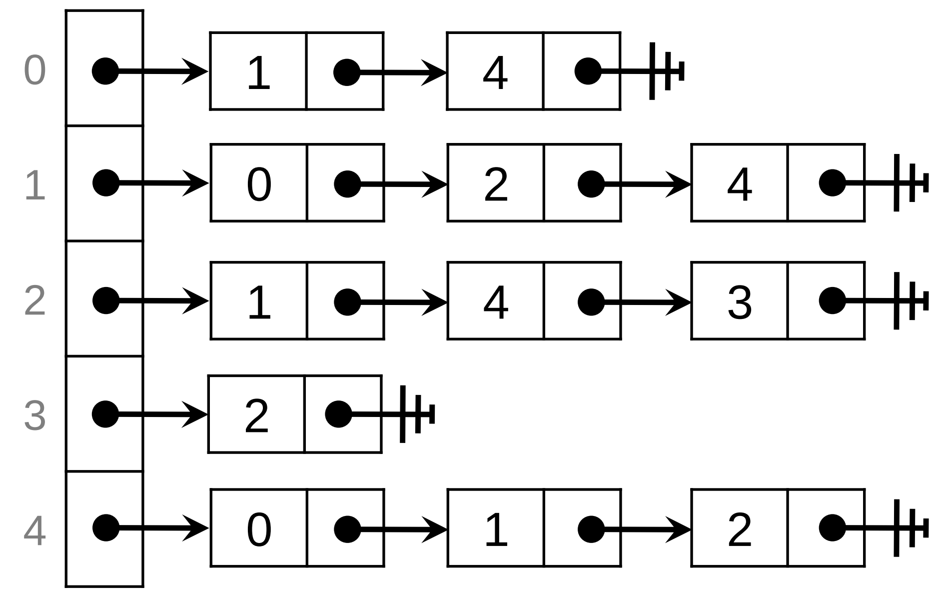
```
unsigned int graph_size(graph *G) {
  REQUIRES(is_graph(G));
  return G->size;
}
```

- graph_new creates an array of empty adjacency lists
  - calloc makes it convenient
  - Cost O(v)
    - calloc needs to zero out all v positions

```
graph *graph_new(unsigned int size) {
  graph *G = xmalloc(sizeof(graph));
  G->size = size;
  G->adj = xcalloc(size, sizeof(adjlist*));
  ENSURES(is_graph(G));
  return G;
}
```



Freeing a Graph



- graph_free must free
  - all adjacency lists
  - the array
  - the graph header

- Cost: O(v + e)
  - there are 2e nodes to free in the adjacency lists
  - v array positions need to be accessed for that

```
void graph_free(graph *G) {
  REQUIRES(is_graph(G));
  for (unsigned int i = 0; i < G->size; i++) {
    adjlist *L = G->adj[i];
    while (L != NULL) {
      adjlist *tmp = L->next;
      free(L);
      L = tmp;
    }
  }
  free(G->adj);
  free(G);
}
```

Free the adjacency list nodes

Free the array

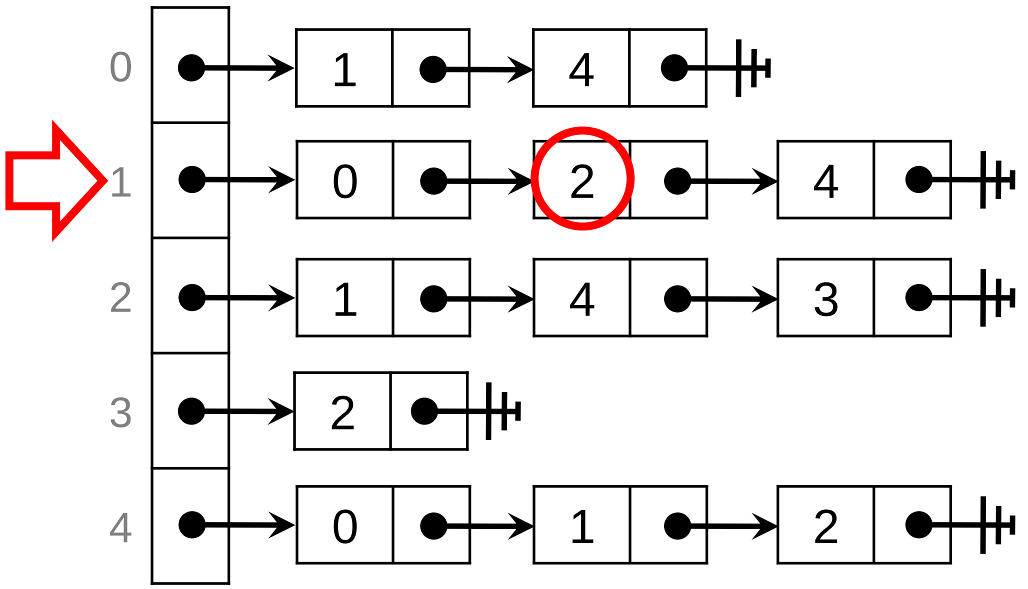
Free the header

Checking Edges

- graph_hasedge(G, v, w) does a linear search for w in the adjacency list of v
  - we could implement it the other way around as well

```
bool graph_hasedge(graph *G, vertex v, vertex w) {
  REQUIRES(is_graph(G));
  REQUIRES(is_vertex(G, v) && is_vertex(G, w));

  for (adjlist *L = G->adj[v]; L != NULL; L = L->next) {
    if (L->vert == w) return true;
  }
  return false;
}
```

- Its cost is O(min(v,e))
  - the maximum length of an adjacency list
  - the maximum number of neighbors of a vertex



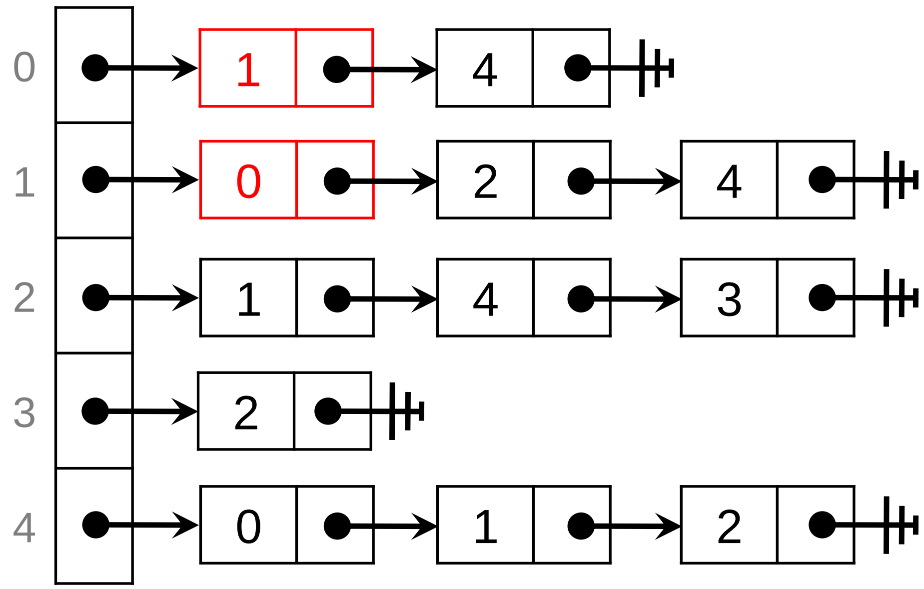
Adding an Edge

- The preconditions exclude
  - self-edges
  - edges already contained in the graph
- graph_addedge(G, v, w)
  - adds w as a neighbor of v
  - and v as a neighbor of w

```
void graph_addedge(graph *G, vertex v, vertex w) {
  REQUIRES(is_graph(G));
  REQUIRES(is_vertex(G, v) && is_vertex(G, w));
  REQUIRES(v != w && !graph_hasedge(G, v, w));

  adjlist *L;

  L = xmalloc(sizeof(adjlist));
  L->vert = w;
  L->next = G->adj[v];
  G->adj[v] = L;

  L = xmalloc(sizeof(adjlist));
  L->vert = v;
  L->next = G->adj[w];
  G->adj[w] = L;

  ENSURES(is_graph(G));
  ENSURES(graph_hasedge(G, v, w));
}
```

add w as a neighbor of v

add v as a neighbor of w

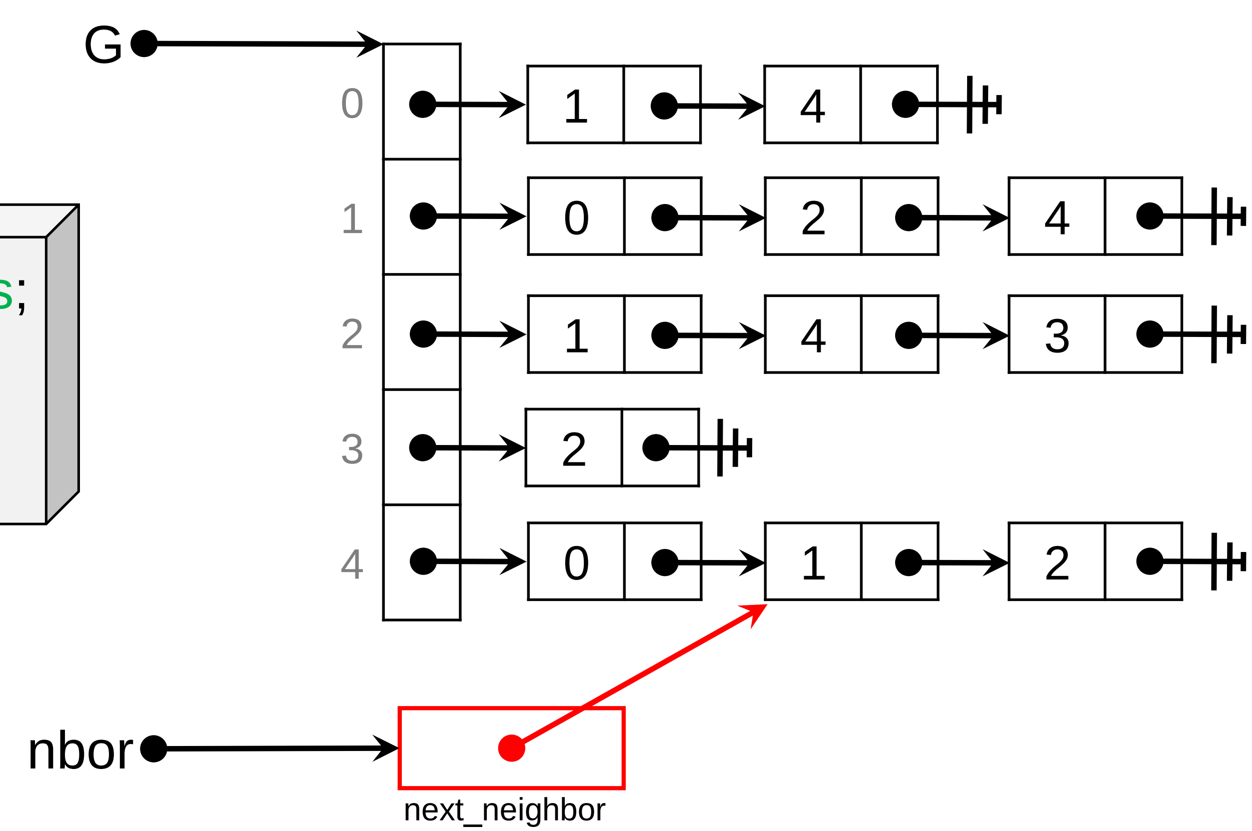


- Constant cost

Neighbors

- We can use the adjacency list of a vertex as a representation of its neighbors
  - We must be careful however not to modify the graph as we iterate through the neighbors
  - Define a struct with a single field
    - ➢ a pointer to the next neighbor to examine

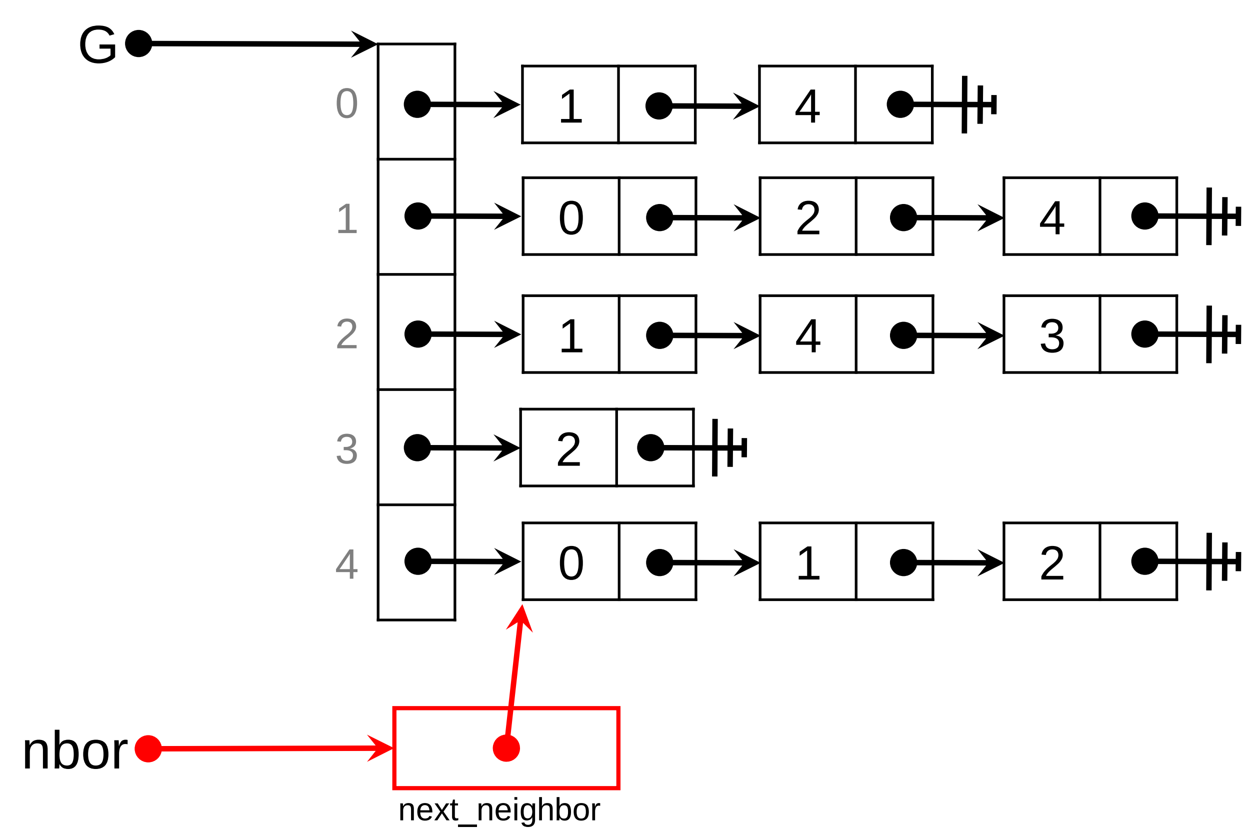
```
typedef struct neighbor_header neighbors;
struct neighbor_header {
  adjlist *next_neighbor;
};
```



Neighbors

- graph_get_neighbors(G, v)
  - creates a neighbors struct
  - points the next_neighbor fields to the adjacency list of v
  - returns this struct

```
neighbors *graph_get_neighbors(graph *G, vertex v) {
  REQUIRES(is_graph(G) && is_vertex(G, v));

  neighbors *nbors = xmalloc(sizeof(neighbors));
  nbors->next_neighbor = G->adj[v];
  ENSURES(is_neighbors(nbors));
  return nbors;
}
```



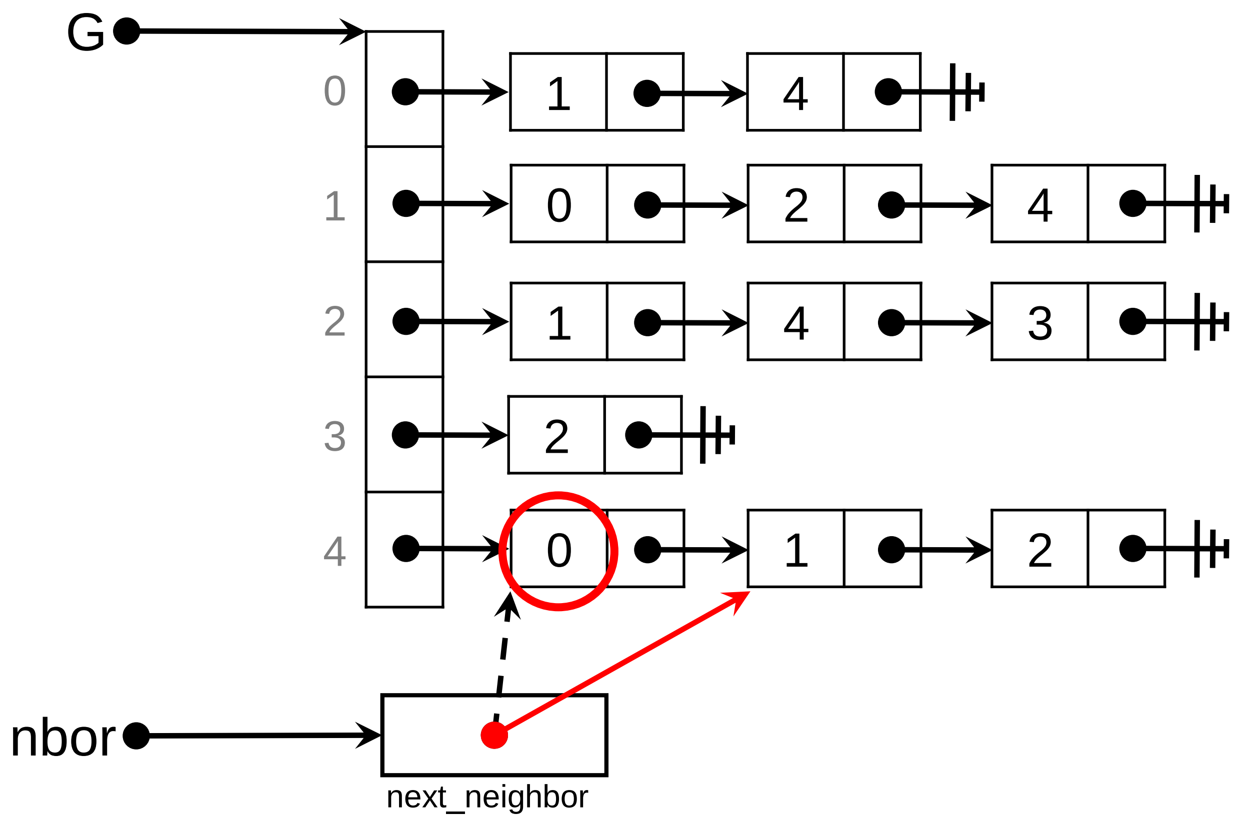
- Constant cost

Neighbors

- graph_next_neighbor
  - returns the next neighbor
  - advances the next_neighbor field along the adjacency list

It **must not** free that adjacency list node since it is owned by the graph

```
vertex graph_next_neighbor(neighbors *nbors) {
  REQUIRES(is_neighbors(nbors));
  REQUIRES(graph_hasmore_neighbors(nbors));

  vertex v = nbors->next_neighbor->vert;
  nbors->next_neighbor = nbors->next_neighbor->next;
  return v;
}
```



- Constant cost

Neighbors

- graph_hasmore_neighbors checks whether the end of the adjacency list has been reached

```
bool graph_hasmore_neighbors(neighbors *nbors) {
  REQUIRES(is_neighbors(nbors));
  return nbors->next_neighbor != NULL;
}
```

- graph_free_neighbors frees the neighbor header
  - and **only** the header

It must not free the rest of the adjacency list since it is owned by the graph

```
void graph_free_neighbors(neighbors *nbors) {
  REQUIRES(is_neighbors(nbors));
  free(nbors);
}
```

- Constant time

Cost Summary

	Adjacency list
Space	O(v + e)
graph_new	O(v)
graph_free	O(v + e)
graph_size	O(1)
graph_hasedge	O(min(v,e))
graph_addedge	O(1)
graph_get_neighbors	O(1)
graph_hasmore_neighbors	O(1)
graph_next_neighbor	O(1)
graph_free_neighbors	O(1)

Using the Graph Interface

Printing a Graph

- Using the graph interface, write a client function that prints a graph
  - for every vertex
    - print it
    - print every neighbor of this node

```
void graph_print(graph_t G) {
  for (vertex v = 0; v < graph_size(G); v++) {
    printf("Vertices connected to %u: ", v);
    neighbors_t nbors = graph_get_neighbors(G, v);
    while (graph_hasmore_neighbors(nbors)) {
      vertex w = graph_next_neighbor(nbors);
      printf(" %u,", w);
    }
    graph_free_neighbors(nbors);
    printf("\n");
  }
}
```

w is a neighbor of v

graph.h

```
typedef unsigned int vertex;
typedef struct graph_header *graph_t;

graph_t graph_new(unsigned int numvert);
//@ensures \result != NULL;

void graph_free(graph_t G);
//@requires G != NULL;

unsigned int graph_size(graph_t G);
//@requires G != NULL;

bool graph_hasedge(graph_t G, vertex v, vertex w);
//@requires G != NULL;
//@requires v < graph_size(G) && w < graph_size(G);

void graph_addedge(graph_t G, vertex v, vertex w);
//@requires G != NULL;
//@requires v < graph_size(G) && w < graph_size(G);
//@requires v != w && !graph_hasedge(G, v, w);

typedef struct neighbor_header *neighbors_t;

neighbors_t graph_get_neighbors(graph_t G, vertex v);
//@requires G != NULL && v < graph_size(G);
//@ensures \result != NULL;

bool graph_hasmore_neighbors(neighbors_t nbors);
//@requires nbors  != NULL;

vertex graph_next_neighbor(neighbors_t nbors);
//@requires nbors  != NULL;
//@requires graph_hasmore_neighbors(nbors);
//@ensures is_vertex(\result);

void graph_free_neighbors(neighbors_t nbors);
//@requires nbors  != NULL;
```

- We will see other algorithms that follow this pattern

graph_get_neighbors	O(1)
graph_hasmore_neighbors	O(1)
graph_next_neighbor	O(1)
graph_free_neighbors	O(1)

What is the Cost of graph_print?

- For a graph with v vertices and e edges
- using a library based on the **adjacency list** representation

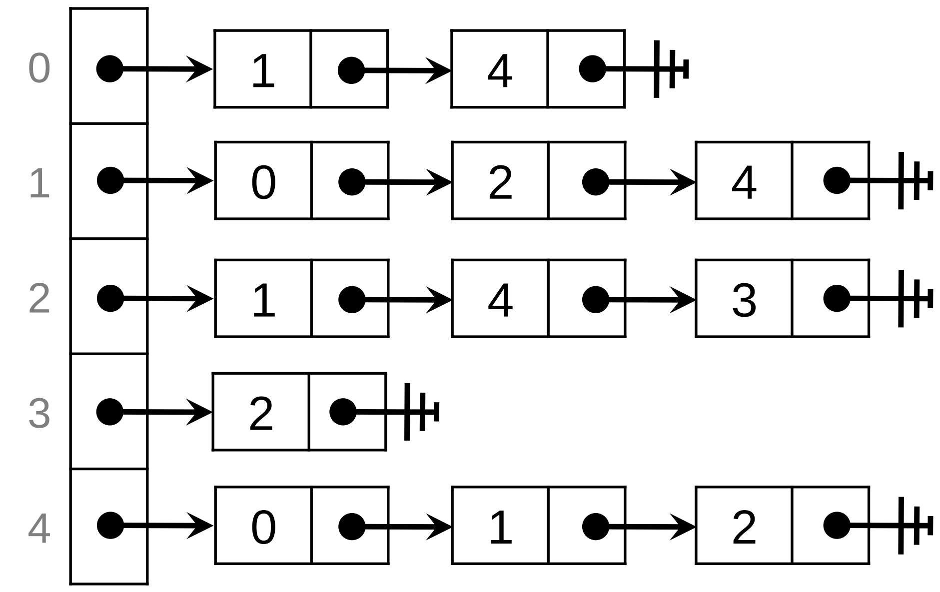
	Cost	Tally
`void graph_print(graph_t G) {`		
`  for (vertex v = 0; v < graph_size(G); v++) {`	v times	
`    printf("Vertices connected to %u: ", v);`	O(1)	O(v)
`    neighbors_t nbors = graph_get_neighbors(G, v);`	O(1)	O(v)
`    while (graph_hasmore_neighbors(nbors)) {`	O(min(v,e)) times	O(v min(v,e))
`      vertex w = graph_next_neighbor(nbors);`	O(1)	O(v min(v,e))
`      printf(" %u,", w);`	O(1)	O(v min(v,e))
`    }`		
`    graph_free_neighbors(nbors);`	O(1)	O(v min(v,e))
`    printf("\n");`	O(1)	O(v min(v,e))
`  }`		
`}`		

- So the cost of graph_print is O(v min(v, e))

What is the Cost of graph_print?

- The cost of graph_print is O(v min(v, e))
  - for a graph with v vertices and e edges using adjacency lists

- Is that right?
  - We assumed every vertex has O(min(v,e)) neighbors
  - But **overall** graph_print visits every edge exactly twice
    - once from each endpoint
    - the body of the inner loop runs 2e times over all iterations of the outer loop
    - the entire inner loop costs O(e)



What is the Cost of graph_print?

- The entire inner loop costs O(e)

```
void graph_print(graph_t G) {                                      Cost        Tally
  for (vertex v = 0; v < graph_size(G); v++) {                    v times
    printf("Vertices connected to %u: ", v);                        O(1)        O(v)
    neighbors_t nbors = graph_get_neighbors(G, v);                  O(1)        O(v)
    while (graph_hasmore_neighbors(nbors)) {                    ┐
      vertex w = graph_next_neighbor(nbors);                    ├ O(e)          O(v + e)
      printf(" %u,", w);                                        │
    }                                                           ┘
    graph_free_neighbors(nbors);                                    O(1)        O(v + e)
    printf("\n");                                                   O(1)        O(v + e)
  }
}
```

- The actual cost of graph_print is O(v + e)
  - for a graph with v vertices and e edges **using adjacency lists**

What is the Cost of graph_print?

- Using the adjacency matrix representation
- By the same argument, the entire inner loop costs O(e)
  - and graph_free_neighbors too

```
void graph_print(graph_t G) {
  for (vertex v = 0; v < graph_size(G); v++) {
    printf("Vertices connected to %u: ", v);
    neighbors_t nbors = graph_get_neighbors(G, v);
    while (graph_hasmore_neighbors(nbors)) {
      vertex w = graph_next_neighbor(nbors);
      printf(" %u,", w);
    }
    graph_free_neighbors(nbors);
    printf("\n");
  }
}
```

Line	Cost	Tally
for (vertex v = 0; v < graph_size(G); v++) {	v times	
printf("Vertices connected to %u: ", v);	O(1)	O(v)
neighbors_t nbors = graph_get_neighbors(G, v);	O(v)	$O(v^2)$
while … graph_free_neighbors(nbors);	O(e)	$O(v^2 + e)$
printf("\n");	O(1)	$O(v^2 + e)$

This is O(min(v,e)) by itself, but there are only 2e neighbors to free

- The actual cost of graph_print is $O(v^2 + e)$
  - This is $O(v^2)$ since $e \in O(v^2)$ always

What is the Cost of print_graph?

- Adjacency list representation: O(v + e)
- Adjacency matrix representation: O(v^2)

Same as space bounds

```
void graph_print(graph_t G) {
  for (vertex v = 0; v < graph_size(G); v++) {
    printf("Vertices connected to %u: ", v);
    neighbors_t nbors = graph_get_neighbors(G, v);
    while (graph_hasmore_neighbors(nbors)) {
      vertex w = graph_next_neighbor(nbors);
      printf(" %u,", w);
    }
    graph_free_neighbors(nbors);
    printf("\n");
  }
}
```

- For a dense graph
  - $e \in O(v^2)$

  they are the same

- For a sparse graph, AL is better