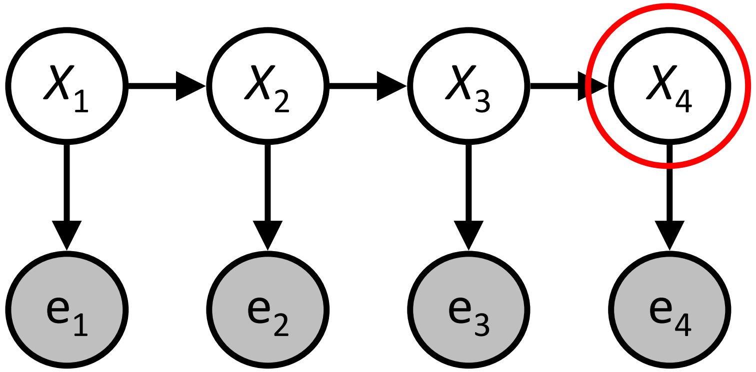
Warm-up as you walk in

Good morning!

- For the following Bayes net, write the query $P(X_4 \mid e_{1:4})$ in terms of the conditional probability tables associated with the Bayes net.

$P(X_4 \mid e_1, e_2, e_3, e_4) =$



Announcements

TA applications: www.ugrad.cs.cmu.edu/ta/S24/ (see Piazza post, also fill in Google form there by Nov 22)

Assignments

- HW9
    - Due ~~tonight~~ Tu Nov. 28, 10 pm
- HW10
    - Out next week, due 12/5, 10 pm
- P5
    - Out tonight, due Thursday 12/7, 10 pm

AI: Representation and Problem Solving

Hidden Markov Models

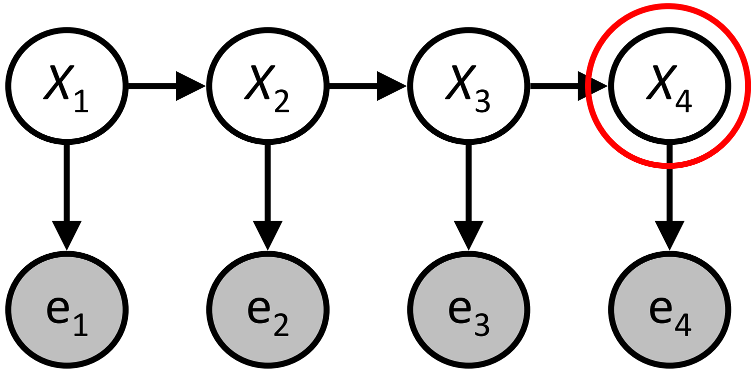
Instructors: Vincent Conitzer and Aditi Raghunathan

Slide credits: CMU AI and http://ai.berkeley.edu

Warm-up as you walk in

- For the following Bayes net, write the query $P(X_4 \mid e_{1:4})$ in terms of the conditional probability tables associated with the Bayes net.

$P(x_4) = P(X_4 = x_4)$



~~$P(X_4 \mid e_1, e_2, e_3, e_4)$~~ =

$$P(x_4 \mid e_1, e_2, e_3, e_4) = \frac{P(x_4, e_1, e_2, e_3, e_4)}{P(e_1, e_2, e_3, e_4)}$$

$$P(e_1, e_2, e_3, e_4) = \sum_{x_4} P(x_4, e_1, e_2, e_3, e_4)$$

$$P(x_4 \mid e_1, e_2, e_3, e_4) = \propto \underline{P(x_4, e_1, e_2, e_3, e_4)}$$

$$P(x_4, e_1, e_2, e_3, e_4) = \sum_{x_1, x_2, x_3} P(x_1, x_2, x_3, x_4, e_1, e_2, e_3, e_4)$$

$$= \sum_{x_1, x_2, x_3} P(x_1) \cdot P(e_1 \mid x_1) \cdot P(x_2 \mid x_1) P(e_2 \mid x_2) P(x_3 \mid x_2) \cdot P(e_3 \mid x_3) \cdot P(x_4 \mid x_3) \cdot P(e_4 \mid x_4)$$

$$= P(e_4 \mid x_4) \sum_{x_3} P(x_4 \mid x_3) P(e_3 \mid x_3) \sum_{x_2} P(x_3 \mid x_2) \cdot P(e_2 \mid x_2) \underbrace{\sum_{x_1} P(x_2 \mid x_1) P(e_1 \mid x_1) P(x_1)}_{f(x_2) = P(x_2, e_1)}$$

$$P(x_2, e_1) = \sum_{x_1} P(x_1) \cdot P(x_2 \mid x_1) \cdot P(e_1 \mid x_1)$$

Reasoning over Time or Space

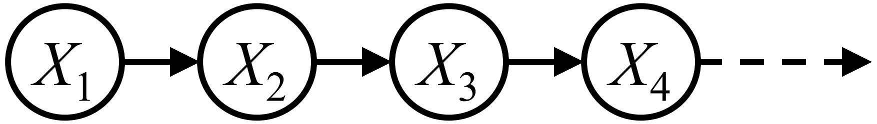
Often, we want to reason about a sequence of observations

- Speech recognition
- Robot localization
- User attention
- Medical monitoring

Need to introduce time (or space) into our models

Markov Chains

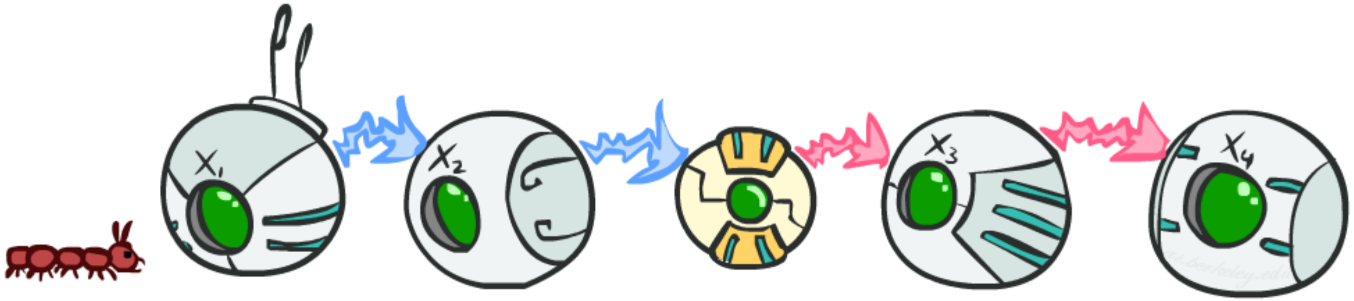
- Value of X at a given time is called the state



$P(X_1)$ $P(X_t|X_{t-1})$

- Parameters: called transition probabilities or dynamics, specify how the state evolves over time (also, initial state probabilities)
- Stationarity assumption: transition probabilities the same at all times
- Same as MDP transition model, but no choice of action

Conditional Independence



Basic conditional independence:

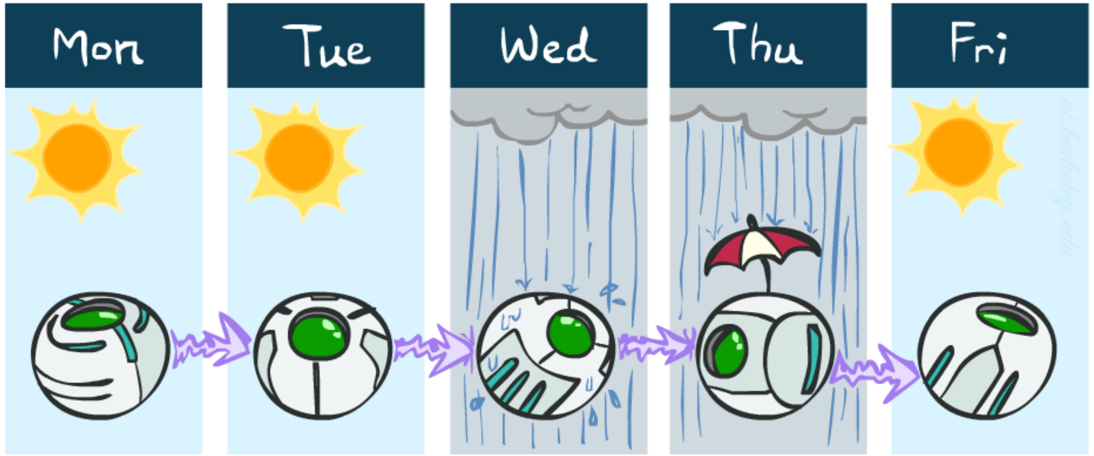
- Past and future independent given the present
- Each time step only depends on the previous
- This is called the (first order) Markov property

Note that the chain is just a (growable) BN

- We can always use generic BN reasoning on it if we truncate the chain at a fixed length

Example: Markov Chain Weather

States: $X = \{rain, sun\}$

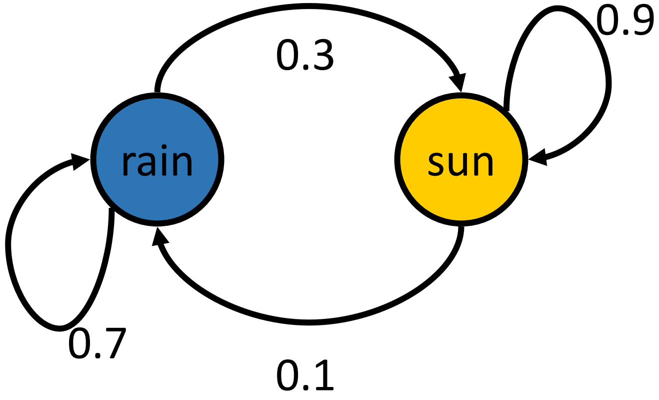


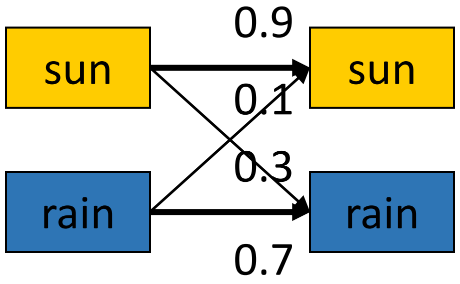
- Initial distribution: 1.0 sun

- CPT $P(X_t \mid X_{t-1})$:

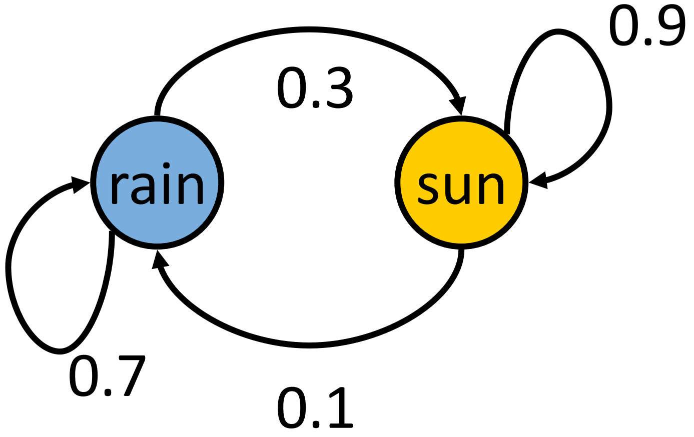
X_{t-1}	X_t	$P(X_t \mid X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

Two new ways of representing the same CPT





Example: Markov Chain Weather



Initial distribution: $P(X_1 = sun) = 1.0$

What is the probability distribution after one step?

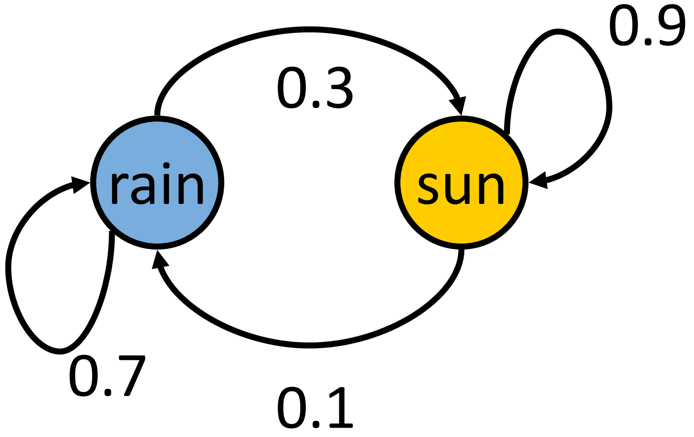
$P(X_2 = sun) = ?$

$\sum_{x_1} P(x_1) \cdot P(X_2 = sun \mid x_1) = P(X_1 = sun) \cdot P(X_2 = sun \mid X_1 = sun)$

$+ P(X_1 = rain) \cdot P(X_2 = sun \mid X_1 = rain)$

$= 1 \cdot .9 = .9$

Example: Markov Chain Weather



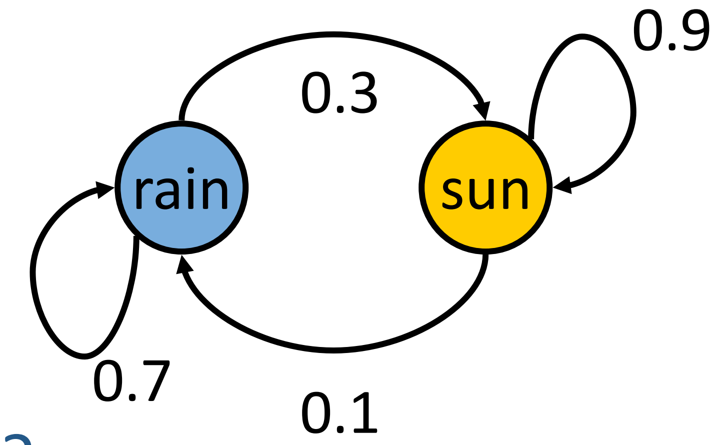
Initial distribution: $P(X_1 = sun) = 1.0$

What is the probability distribution after one step?

$P(X_2 = sun) = ?$

$$
\begin{aligned}
P(X_2 = sun) &= \sum_{x_1} P(X_1 = x_1, X_2 = sun) \\
&= \sum_{x_1} P(X_2 = sun \mid X_1 = x_1)P(X_1 = x_1) \\
&= P(X_2 = sun \mid X_1 = sun)P(X_1 = sun) + \\
&\quad P(X_2 = sun \mid X_1 = rain)P(X_1 = rain) \\
&= 0.9 \cdot 1.0 + 0.3 \cdot 0.0 = 0.9
\end{aligned}
$$

Poll 1



Initial distribution: $P(X_2 = sun) = 0.9$

What is the probability distribution after the next step?

$P(X_3 = sun) = ?$

$$= \sum_{x_2} P(X_2 = x_2) \cdot P(X_3 = sun \mid X_2 = x_2)$$

$$= P(X_2 = sun) P(X_3 = sun \mid X_2 = sun) + P(X_2 = rain) P(X_3 = sun \mid X_2 = rain)$$

$$= .9 \cdot .9 + .1 \cdot .3 = .84$$

A) 0.81

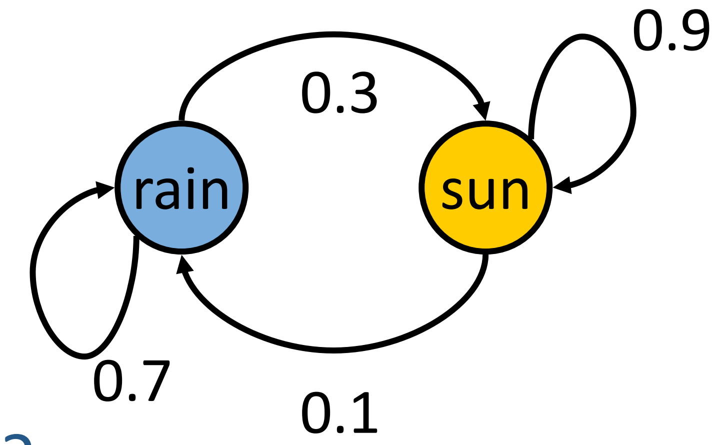
B) 0.84

C) 0.9

D) 1.0

E) 1.2

Poll 1



Initial distribution: $P(X_2 = sun) \ = 0.9$

What is the probability distribution after the next step?

$P(X_3 = sun) \ = ?$

A) 0.81

B) 0.84

C) 0.9

D) 1.0

E) 1.2

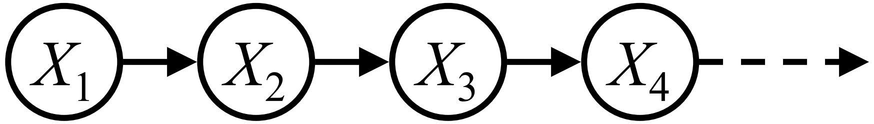
$$P(X_3 = sun) = \sum_{x_2} P(X_3 = sun, X_2 = x_2)$$

$$= \sum_{x_2} P(X_3 = sun \mid X_2 = x_2) \, P(X_2 = x_2)$$

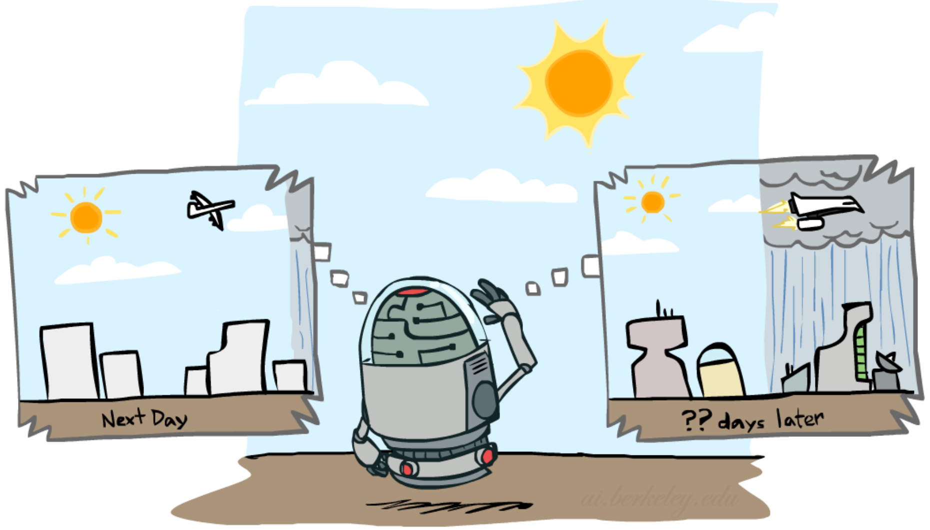
$$= 0.9 \cdot 0.9 + 0.3 \cdot 0.1$$

$$= 0.81 + 0.03 = 0.84$$

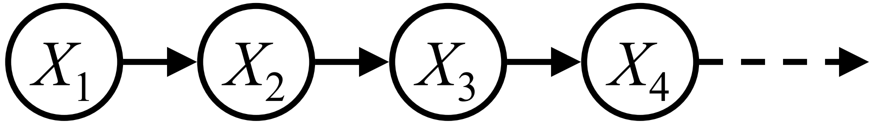
Markov Chain Inference



If you know the transition probabilities, $P(X_t \mid X_{t-1})$, and you know $P(X_4)$, write an equation to compute $P(X_5)$.



Markov Chain Inference

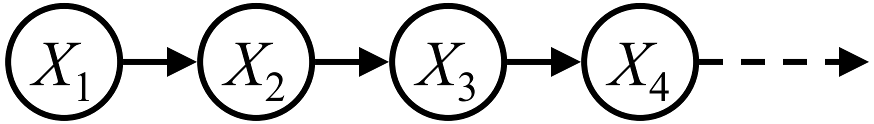


If you know the transition probabilities, $P(X_t \mid X_{t-1})$, and you know $P(X_4)$, write an equation to compute $P(X_5)$.

$$P(X_5) = \sum_{x_4} P(x_4, X_5)$$

$$= \sum_{x_4} P(X_5 \mid x_4) P(x_4)$$

Markov Chain Inference



If you know the transition probabilities, $P(X_t \mid X_{t-1})$, and you know $P(X_4)$, write an equation to compute $P(X_5)$.

$$
\begin{aligned}
P(X_5) &= \textstyle\sum_{x_1,x_2,x_3,x_4} P(x_1, x_2, x_3, x_4, X_5) \\
&= \textstyle\sum_{x_1,x_2,x_3,x_4} P(X_5 \mid x_4) P(x_4 \mid x_3) P(x_3 \mid x_2) P(x_2 \mid x_1) P(x_1) \\
&= \textstyle\sum_{x_4} P(X_5 \mid x_4) \sum_{x_1,x_2,x_3} P(x_4 \mid x_3) P(x_3 \mid x_2) P(x_2 \mid x_1) P(x_1) \\
&= \textstyle\sum_{x_4} P(X_5 \mid x_4) \sum_{x_1,x_2,x_3} P(x_1, x_2, x_3, x_4) \\
&= \textstyle\sum_{x_4} P(X_5 \mid x_4) P(x_4)
\end{aligned}
$$

Weather prediction

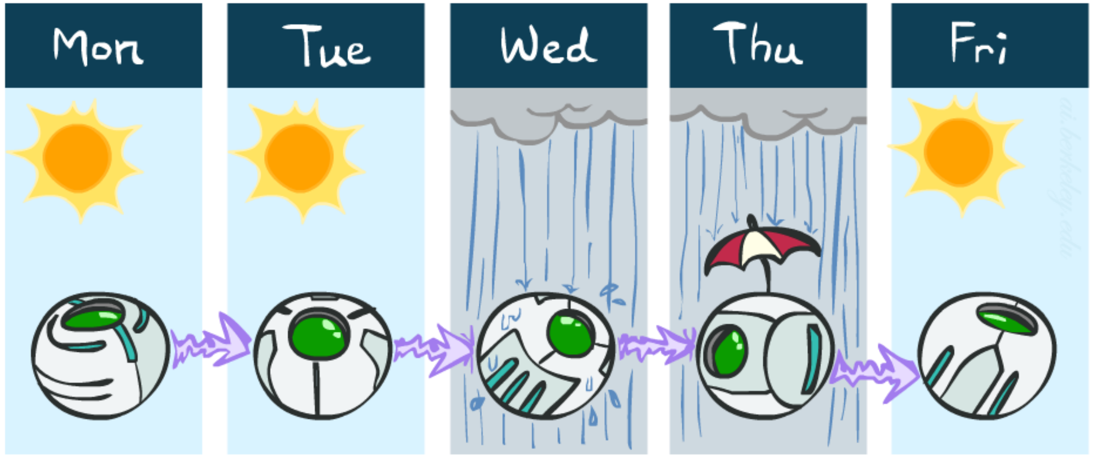
States {rain, sun}

- Initial distribution $P(X_0)$

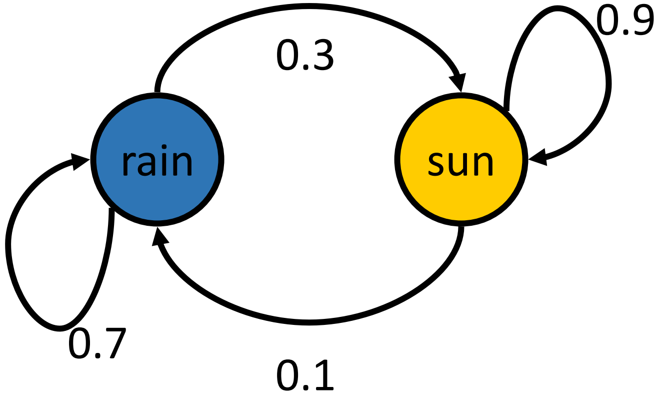
$P(X_0)$	
sun	rain
0.5	0.5

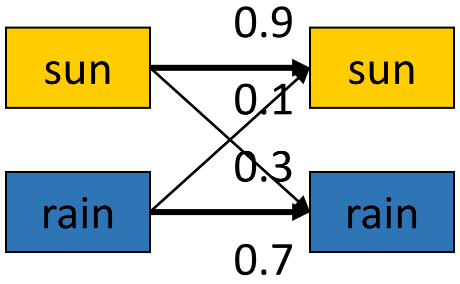
- Transition model $P(X_t|X_{t-1})$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7



Two new ways of representing the same CPT

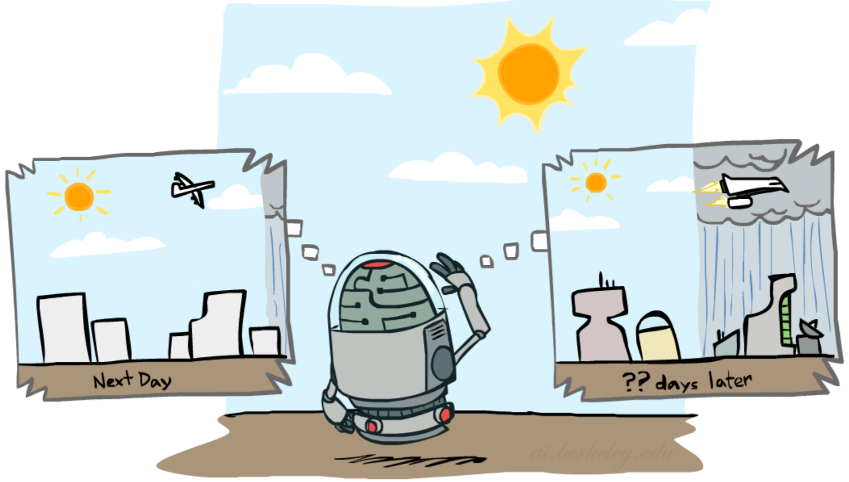




Weather prediction

Time 0: $P(X_0) = \langle 0.5, 0.5 \rangle$

X_{t-1}	**$P(X_t \mid X_{t-1})$**	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7



What is the weather like at time 1?

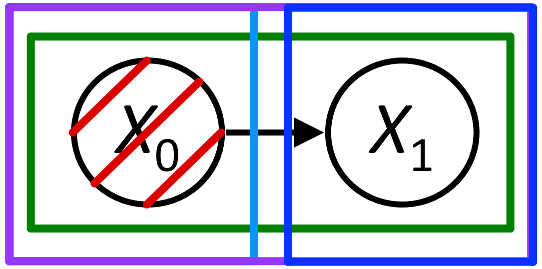
$$P(X_1) =$$

$$\sum_{x_0} P(X_0 = x_0, X_1)$$

$$= \sum_{x_0} P(X_1 \mid X_0 = x_0) P(X_0 = x_0)$$

$$= 0.5\langle 0.9, 0.1 \rangle + 0.5\langle 0.3, 0.7 \rangle$$

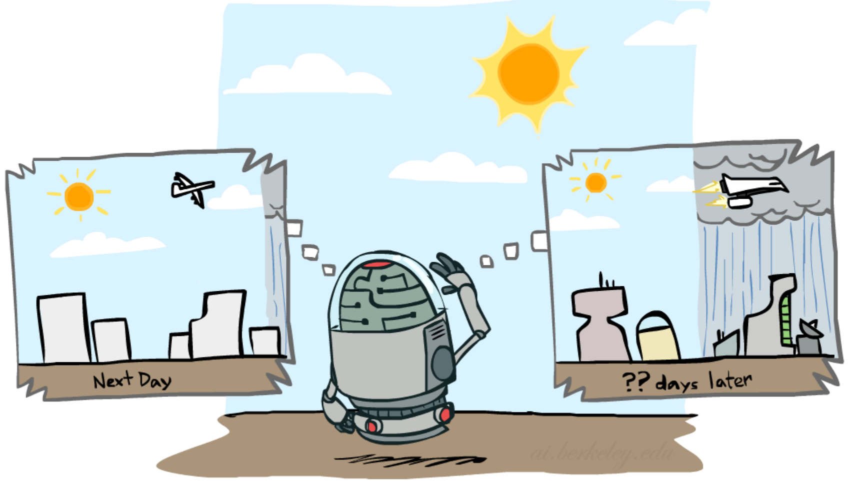
$$= \langle 0.6, 0.4 \rangle$$



Weather prediction, contd.

Time 1: $P(X_1) = \langle 0.6, 0.4 \rangle$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7



What is the weather like at time 2?

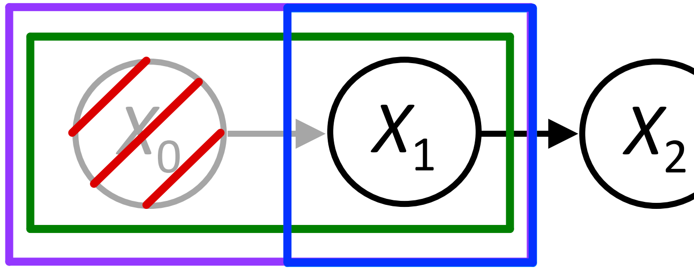
$P(X_2) =$

$$\sum_{x_1} P(X_1 = x_1, X_2)$$

$$= \sum_{x_1} P(X_2 \mid X_1 = x_1) P(X_1 = x_1)$$

$$= 0.6\langle 0.9, 0.1 \rangle + 0.4\langle 0.3, 0.7 \rangle$$

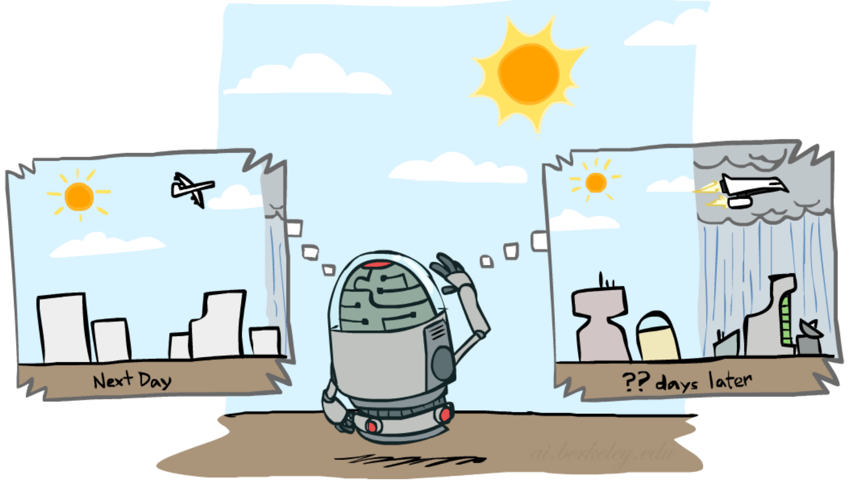
$$= \langle 0.66, 0.34 \rangle$$



Weather prediction, contd.

Time 2: $P(X_2) = \langle 0.66, 0.34 \rangle$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7



What is the weather like at time 3?

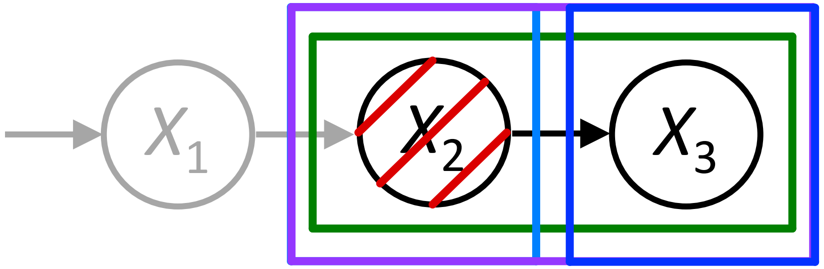
$P(X_3)$ =

$$\sum_{x_2} P(X_2 = x_2, X_3)$$

$$= \sum_{x_2} P(X_3 \mid X_2 = x_2) P(X_2 = x_2)$$

$$= 0.66\langle 0.9, 0.1 \rangle + 0.34\langle 0.3, 0.7 \rangle$$

$$= \langle 0.696, 0.304 \rangle$$



Forward algorithm (simple form)

What is the state at time *t*?

$$P(X_t) = \sum_{x_{t-1}} P(X_{t-1} = x_{t-1}, X_t)$$

$$= \sum_{x_{t-1}} P(X_t | X_{t-1} = x_{t-1}) P(X_{t-1} = x_{t-1})$$

Transition model

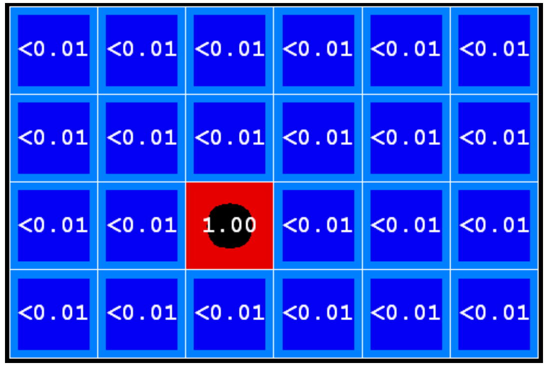
Probability from previous iteration

Iterate this update starting at *t*=0

Prediction with Markov chains

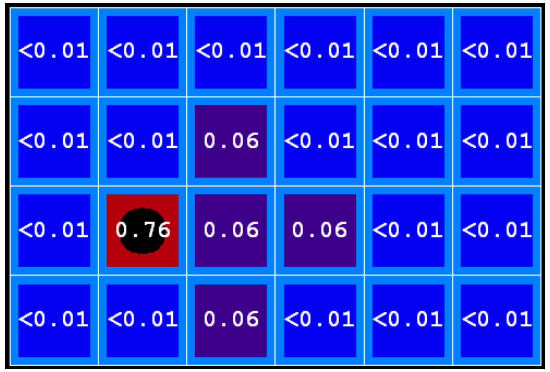
As time passes, uncertainty “accumulates”

(Transition model: ghosts usually go clockwise)



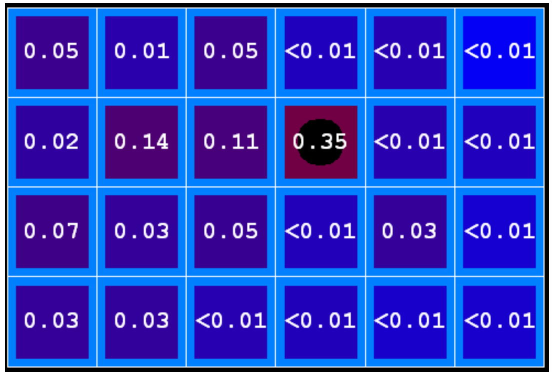
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	1.00	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01

T = 1



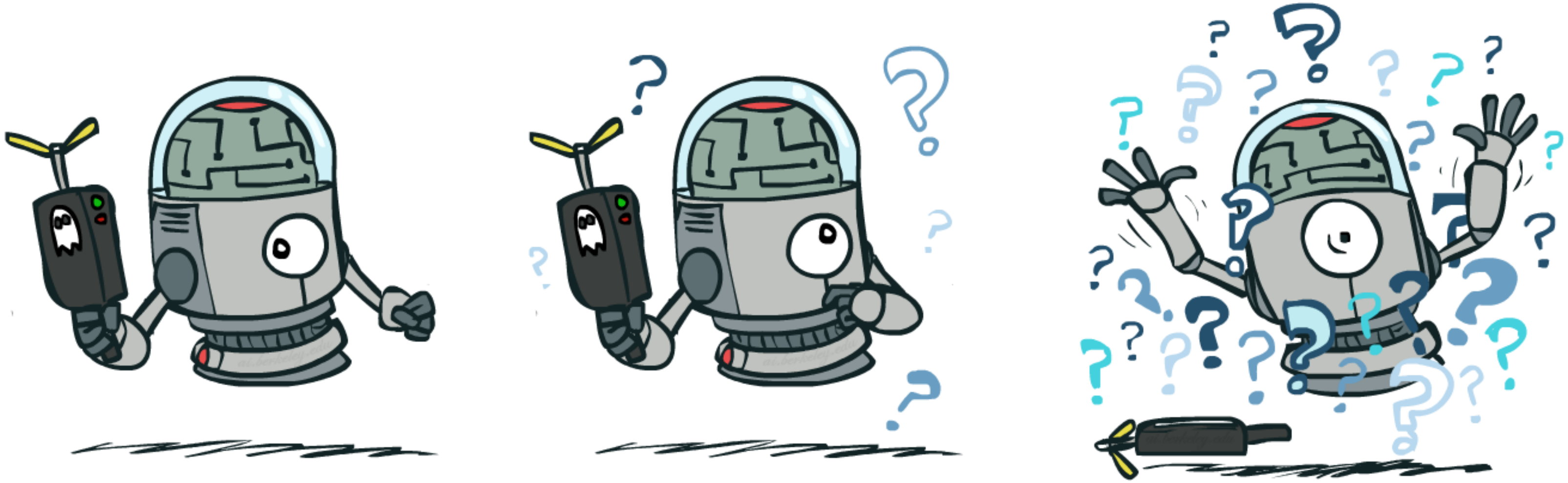
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	0.06	<0.01	<0.01	<0.01
<0.01	0.76	0.06	0.06	<0.01	<0.01
<0.01	<0.01	0.06	<0.01	<0.01	<0.01

T = 2



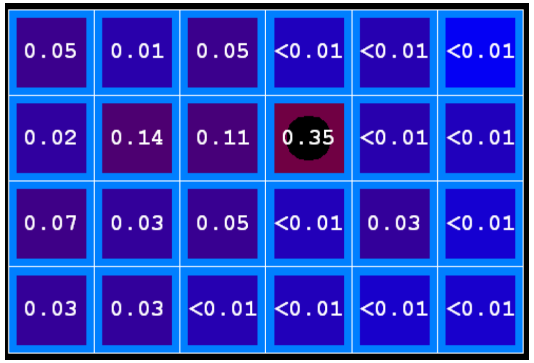
0.05	0.01	0.05	<0.01	<0.01	<0.01
0.02	0.14	0.11	0.35	<0.01	<0.01
0.07	0.03	0.05	<0.01	0.03	<0.01
0.03	0.03	<0.01	<0.01	<0.01	<0.01

T = 5



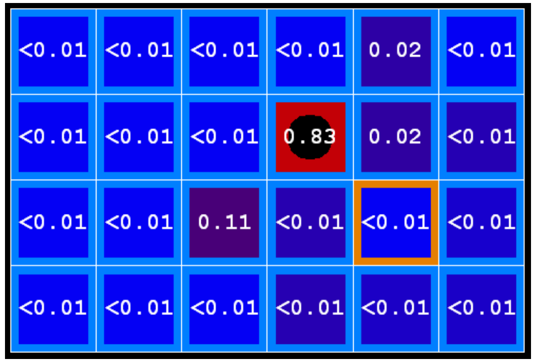
Observations Reduce Uncertainty

As we get observations, beliefs get reweighted, uncertainty “decreases”



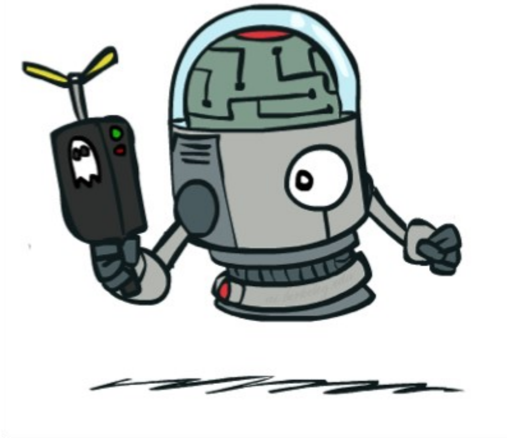
0.05	0.01	0.05	<0.01	<0.01	<0.01
0.02	0.14	0.11	0.35	<0.01	<0.01
0.07	0.03	0.05	<0.01	0.03	<0.01
0.03	0.03	<0.01	<0.01	<0.01	<0.01

Before observation



<0.01	<0.01	<0.01	<0.01	0.02	<0.01
<0.01	<0.01	<0.01	0.83	0.02	<0.01
<0.01	<0.01	0.11	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01

After observation



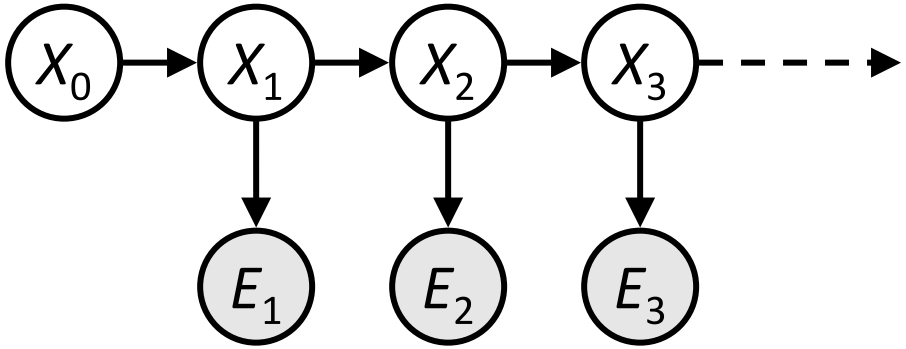
Hidden Markov Models

Hidden Markov Models

Usually the true state is not observed directly

Hidden Markov models (HMMs)

- Underlying Markov chain over states X
- You observe evidence E at each time step
- X_t is a single discrete variable; E_t may be continuous and may consist of several variables



Real HMM Examples

Speech recognition HMMs:

- Observations are acoustic signals (continuous valued)
- States are specific positions in specific words (so, tens of thousands)

Machine translation HMMs:

- Observations are words (tens of thousands)
- States are translation options

Robot tracking:

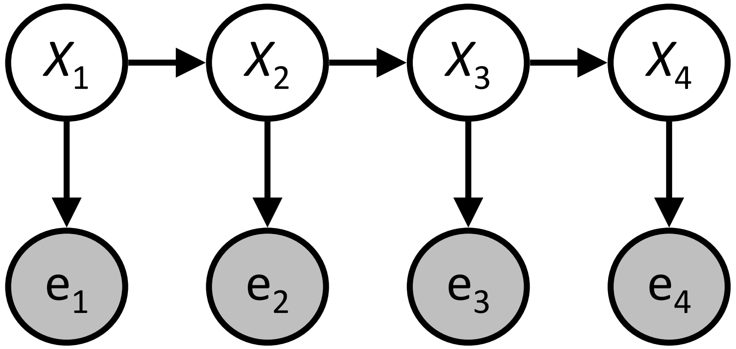
- Observations are range readings (continuous)
- States are positions on a map (continuous)

Molecular biology:

- Observations are nucleotides ACGT
- States are coding/non-coding/start/stop/splice-site etc.

HMM as a Bayes Net Warm-up

- For the following Bayes net, write the query $P(X_4 \mid e_{1:4})$ in terms of the conditional probability tables associated with the Bayes net.

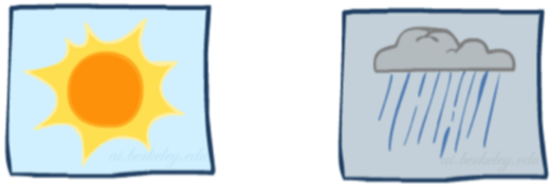


$P(X_4 \mid e_1, e_2, e_3, e_4) =$

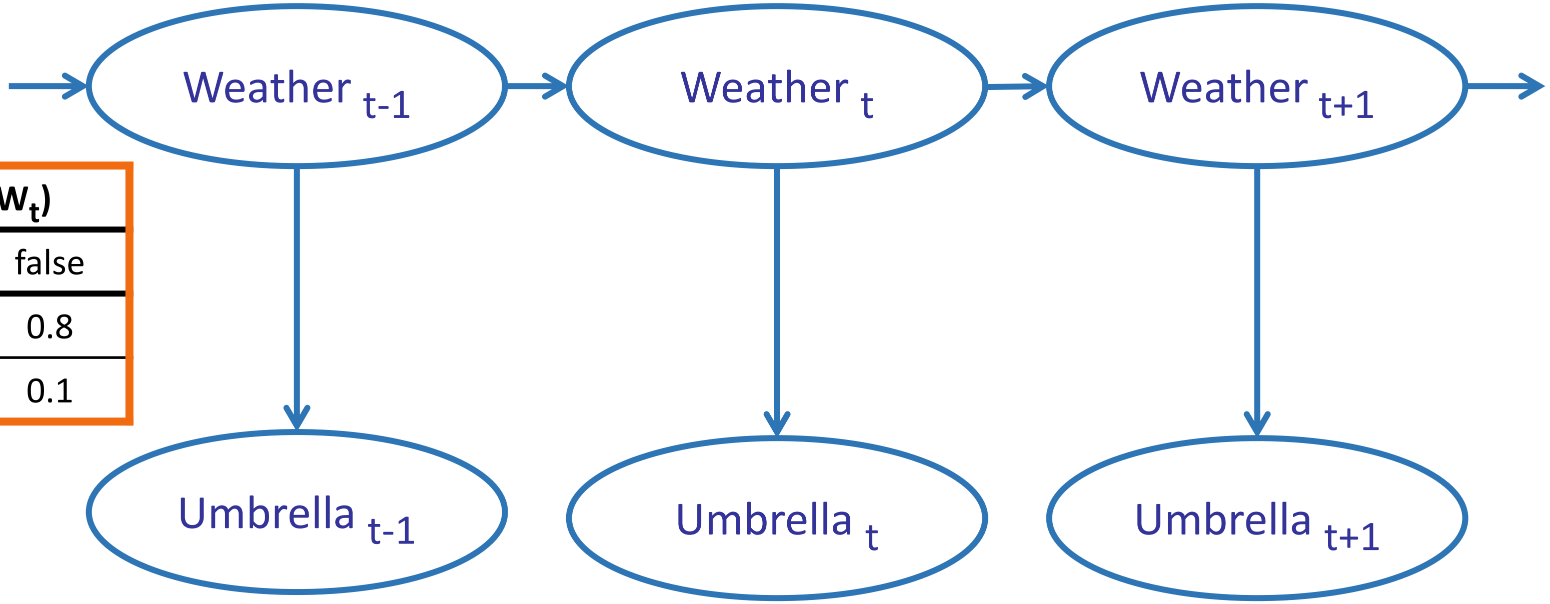
Example: Weather HMM

An HMM is defined by:

- Initial distribution: $P(X_0)$
- Transition model: $P(X_t \mid X_{t-1})$
- Sensor model: $P(E_t \mid X_t)$



W_{t-1}	**$P(W_t \mid W_{t-1})$**	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7



Weather $_{t-1}$

Weather $_{t}$

Weather $_{t+1}$

Umbrella $_{t-1}$

Umbrella $_{t}$

Umbrella $_{t+1}$

W_t	**$P(U_t \mid W_t)$**	
	true	false
sun	0.2	0.8
rain	0.9	0.1

HMM as Probability Model

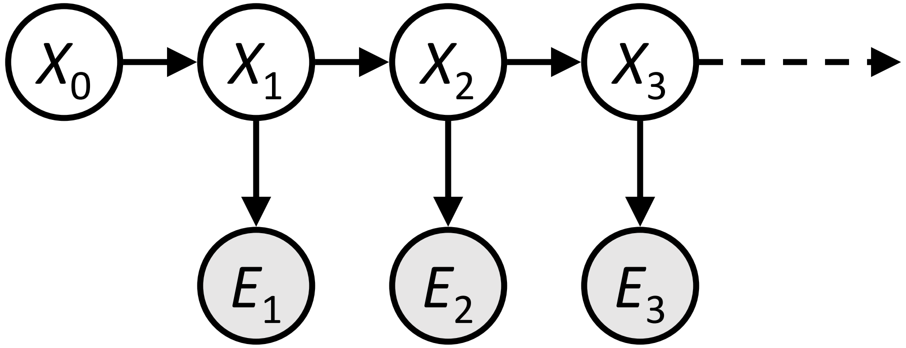
- Joint distribution for Markov model:

$$P(X_0,\ldots, X_T) = P(X_0) \prod_{t=1:T} P(X_t \mid X_{t-1})$$

- Joint distribution for hidden Markov model:

$$P(X_0,\ X_1,E_1,\ \ldots,\ X_T,E_T) = P(X_0) \prod_{t=1:T} P(X_t \mid X_{t-1})\ P(E_t \mid X_t)$$

- Future states are independent of the past given the present
- Current evidence is independent of everything else given the current state
- Are evidence variables independent of each other?

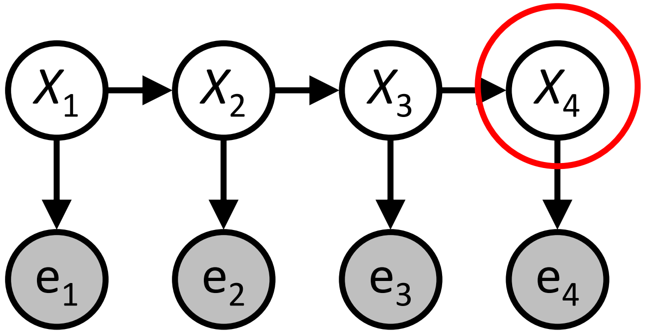


Useful notation: $X_{a:b} = X_a , X_{a+1}, \ldots, X_b$

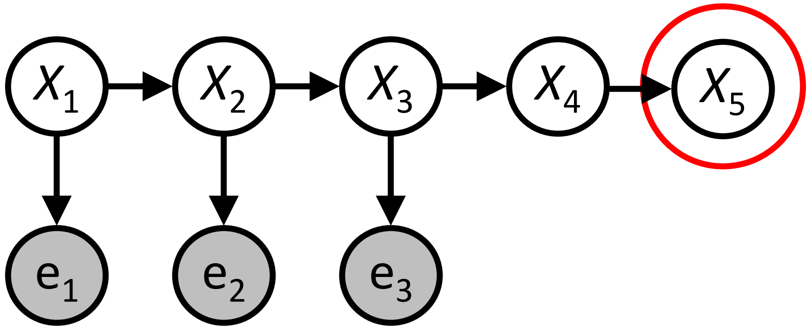
For example: $P(X_{1:2} \mid e_{1:3}) = P(X_1, X_2, \mid e_1 , e_2, e_3)$

HMM Queries

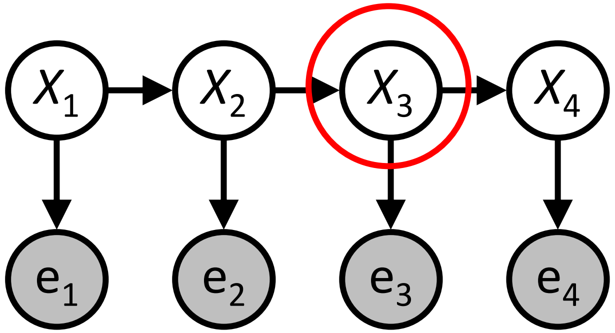
Filtering: $P(X_t | e_{1:t})$



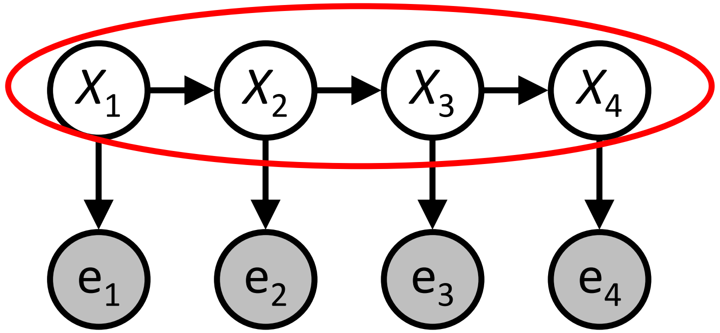
Prediction: $P(X_{t+k} | e_{1:t})$



Smoothing: $P(X_k | e_{1:t})$, $k<t$



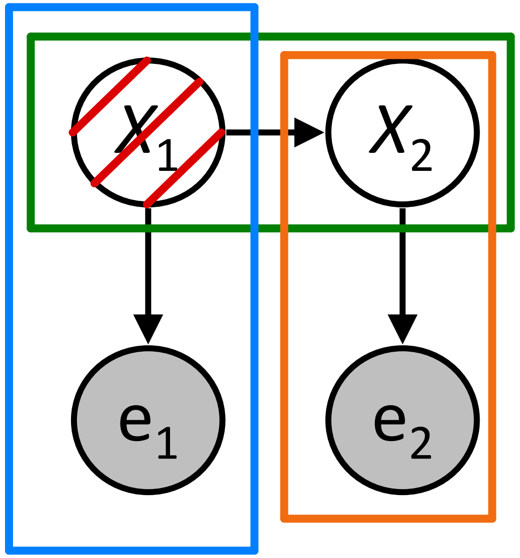
Explanation: $P(X_{1:t} | e_{1:t})$



Filtering **Algorithm**

Query: What is the current state, given all of the current and past evidence?

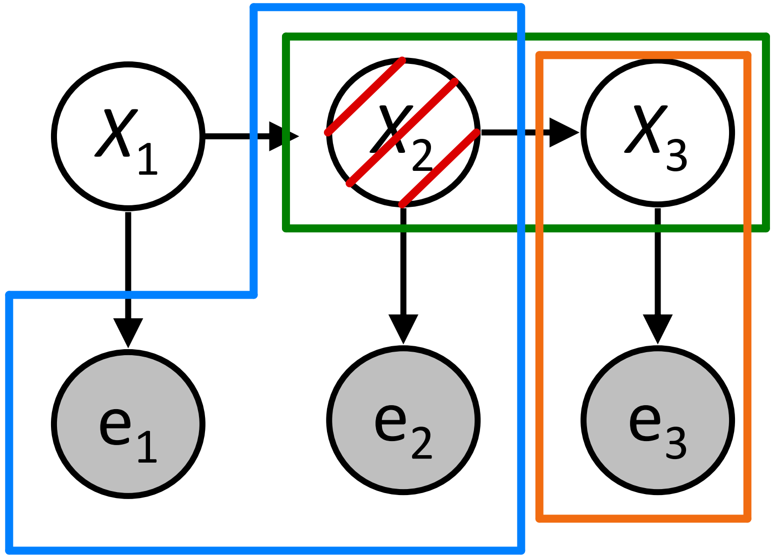
Marching **forward** through the HMM network



Filtering **Algorithm**

Query: What is the current state, given all of the current and past evidence?

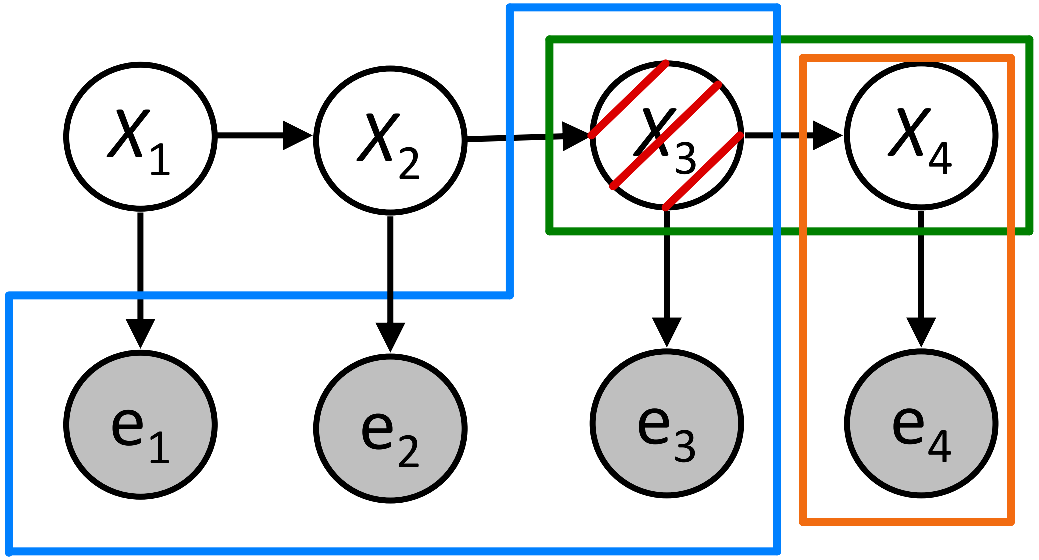
Marching **forward** through the HMM network



Filtering **Algorithm**

Query: What is the current state, given all of the current and past evidence?

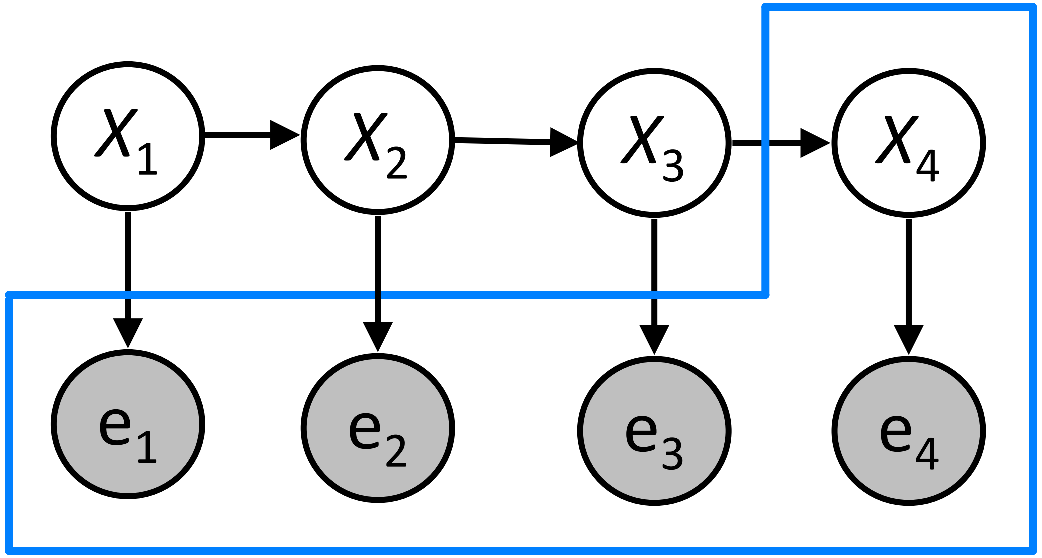
Marching **forward** through the HMM network



Filtering **Algorithm**

Query: What is the current state, given all of the current and past evidence?

Marching **forward** through the HMM network



Filtering **Algorithm**

$$P(X_{t+1} \mid e_{1:t+1}) = \underbrace{\alpha}_{\text{Normalize}} \underbrace{P(e_{t+1} \mid X_{t+1})}_{\text{Update}} \underbrace{\sum_{x_t} P(X_{t+1} \mid x_t)\, P(x_t \mid e_{1:t})}_{\text{Predict}}$$

$$\boldsymbol{f}_{1:t+1} = \textsc{Forward}(\boldsymbol{f}_{1:t}, e_{t+1})$$

Filtering **Algorithm**

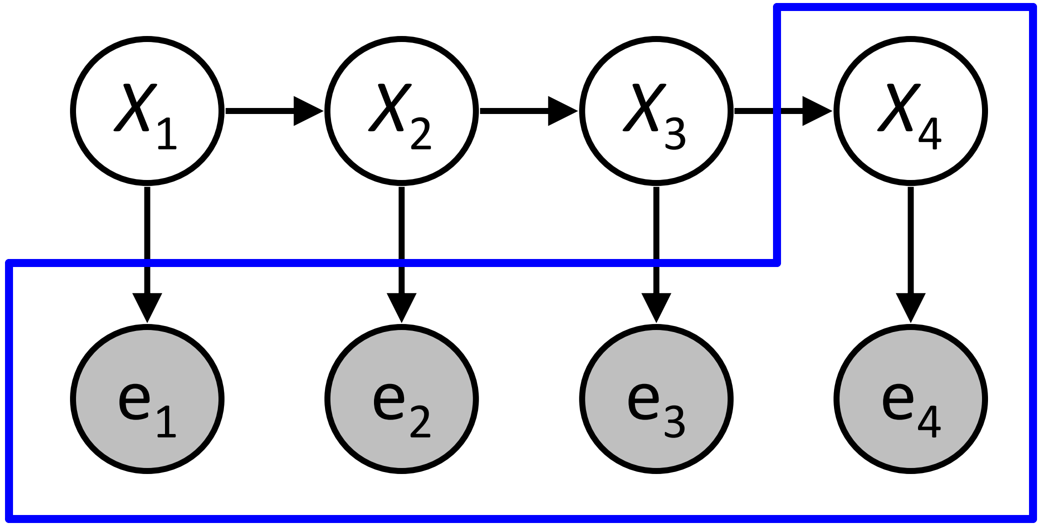
$P(x \mid y, z) = \frac{P(x, y \mid z)}{P(y \mid z)}$

Query: What is the current state, given all of the current and past evidence?

Matching math with Bayes net

$$P(X_t \mid e_{1:t}) = P(X_t \mid e_t, e_{1:t-1})$$
$$= \alpha\, P(X_t, e_t \mid e_{1:t-1})$$

Def. of cond. probability with X_t, e_t



Filtering **Algorithm**

Query: What is the current state, given all of the current and past evidence?

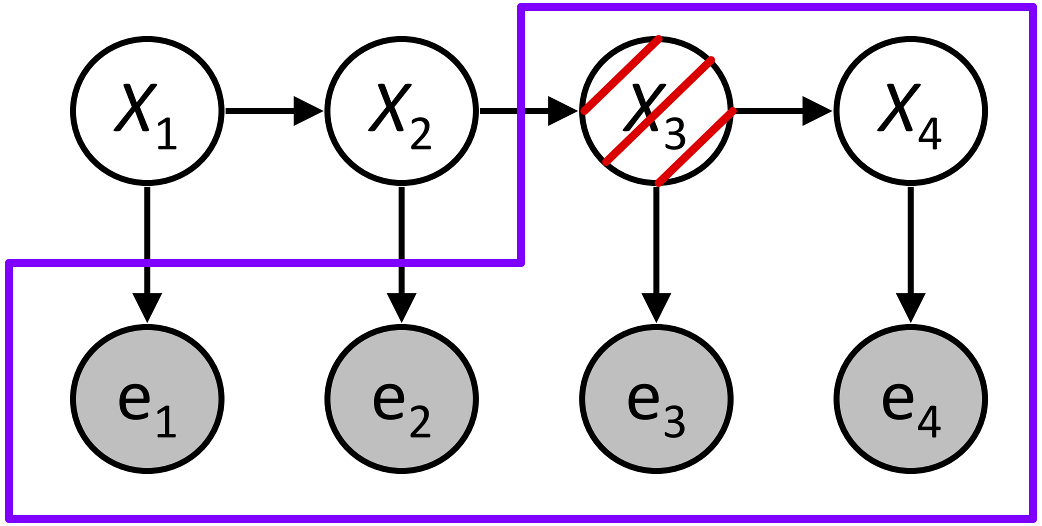
Matching math with Bayes net

$$P(X_t \mid e_{1:t}) = P(X_t \mid e_t, e_{1:t-1})$$

$$= \alpha\, P(X_t, e_t \mid e_{1:t-1})$$

$$= \alpha \sum_{x_{t-1}} P(x_{t-1}, X_t, e_t \mid e_{1:t-1})$$

Summation over variable X_{t-1}



Filtering **Algorithm**

Query: What is the current state, given all of the current and past evidence?

Matching math with Bayes net

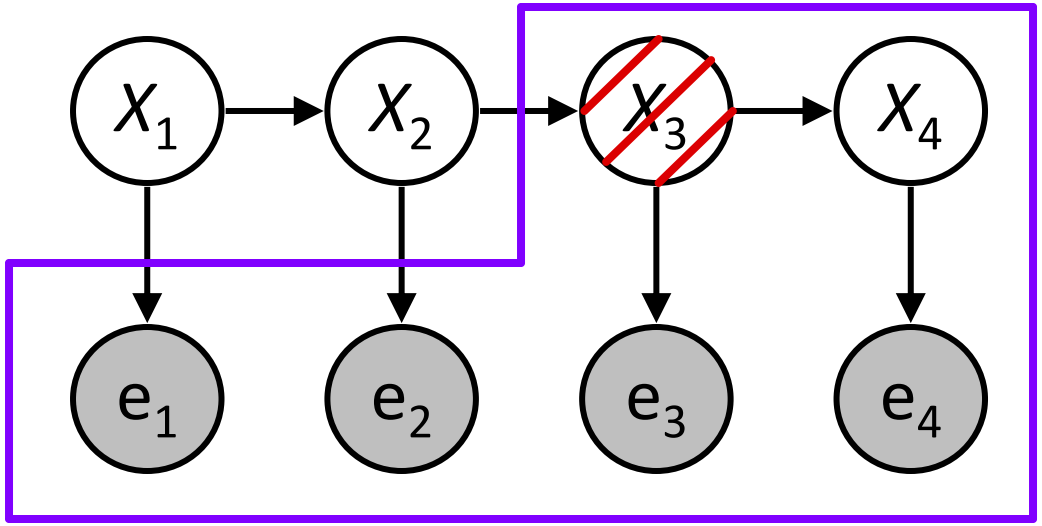
$$P(X_t \mid e_{1:t}) = P(X_t \mid e_t, e_{1:t-1})$$

$$= \alpha\, P(X_t, e_t \mid e_{1:t-1})$$

$$= \alpha \sum_{x_{t-1}} P(x_{t-1}, X_t, e_t \mid e_{1:t-1})$$

$$= \alpha \sum_{x_{t-1}} P(x_{t-1} \mid e_{1:t-1})\, P(X_t \mid x_{t-1}, e_{1:t-1})\, P(e_t \mid X_t, x_{t-1}, e_{1:t-1})$$

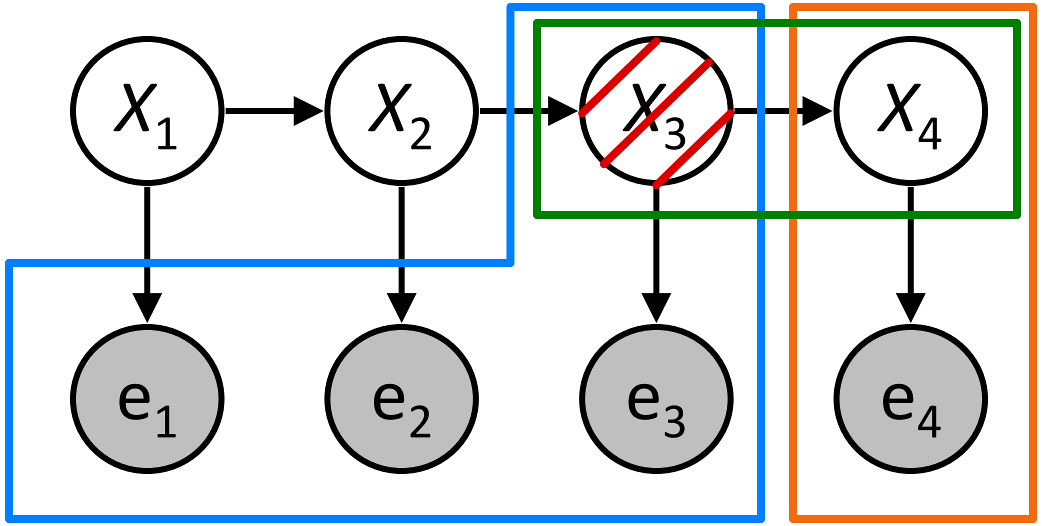
Chain rule with x_{t-1}, X_t, and e_t



Filtering **Algorithm**

Query: What is the current state, given all of the current and past evidence?

Matching math with Bayes net



$$P(X_t \mid e_{1:t}) = P(X_t \mid e_t, e_{1:t-1})$$

$$= \alpha \, P(X_t, e_t \mid e_{1:t-1})$$

$$= \alpha \sum_{x_{t-1}} P(x_{t-1}, X_t, e_t \mid e_{1:t-1})$$

$$= \alpha \sum_{x_{t-1}} P(x_{t-1} \mid e_{1:t-1}) \, P(X_t \mid x_{t-1}, e_{1:t-1}) \, P(e_t \mid X_t, x_{t-1}, e_{1:t-1})$$

Chain rule with x_{t-1}, X_t, and e_t

Filtering **Algorithm**

Query: What is the current state, given all of the current and past evidence?

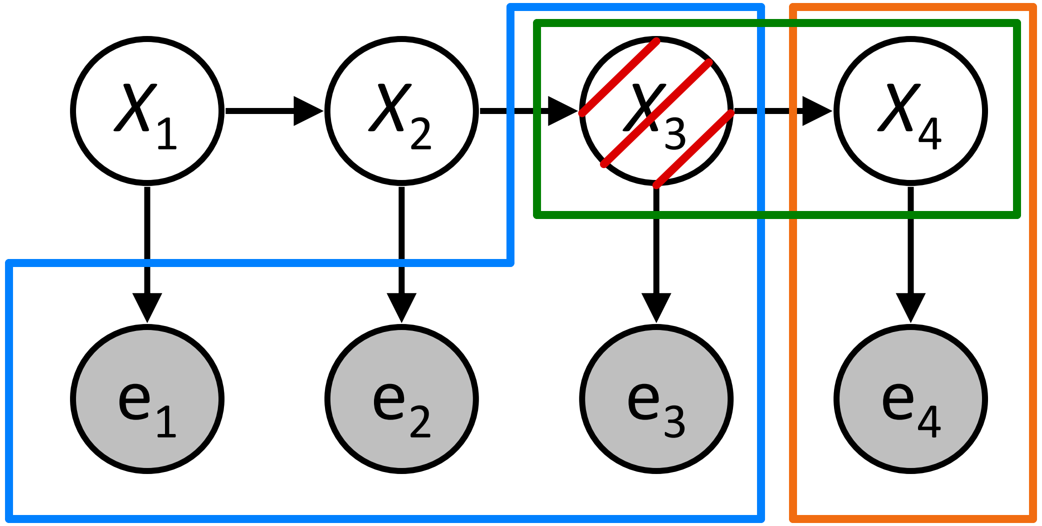
Matching math with Bayes net

$$P(X_t \mid e_{1:t}) = P(X_t \mid e_t, e_{1:t-1})$$

$$= \alpha\, P(X_t, e_t \mid e_{1:t-1})$$

$$= \alpha \sum_{x_{t-1}} P(x_{t-1}, X_t, e_t \mid e_{1:t-1})$$

$$= \alpha \sum_{x_{t-1}} P(x_{t-1} \mid e_{1:t-1})\, P(X_t \mid x_{t-1})\, P(e_t \mid X_t)$$

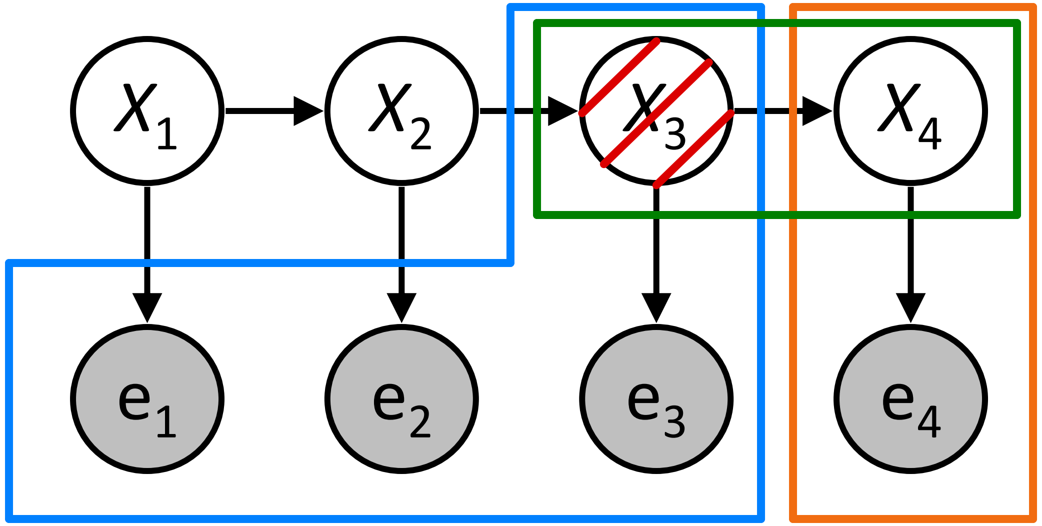


Bayes net conditional independence

Filtering **Algorithm**

Query: What is the current state, given all of the current and past evidence?

Matching math with Bayes net



$$P(X_t \mid e_{1:t}) = P(X_t \mid e_t, e_{1:t-1})$$

$$= \alpha\, P(X_t, e_t \mid e_{1:t-1})$$

$$= \alpha \sum_{x_{t-1}} P(x_{t-1}, X_t, e_t \mid e_{1:t-1})$$

$$= \alpha \sum_{x_{t-1}} P(x_{t-1} \mid e_{1:t-1})\, P(X_t \mid x_{t-1})\, P(e_t \mid X_t)$$

$$= \alpha\, P(e_t \mid x_t) \sum_{x_{t-1}} P(x_t \mid x_{t-1})\, P(x_{t-1} \mid e_{1:t-1})$$

Pulling $P(e_t|X_t)$ out of the summation

Filtering **Algorithm**

Query: What is the current state, given all of the current and past evidence?

Matching math with Bayes net

$$P(X_t \mid e_{1:t}) = P(X_t \mid e_t, e_{1:t-1})$$

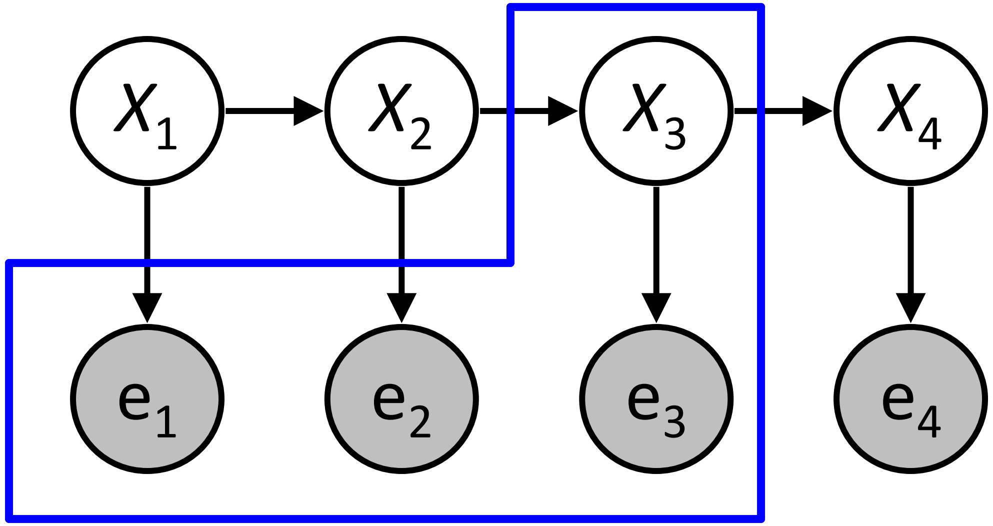
$$= \alpha\, P(X_t, e_t \mid e_{1:t-1})$$

$$= \alpha \sum_{x_{t-1}} P(x_{t-1}, X_t, e_t \mid e_{1:t-1})$$

$$= \alpha \sum_{x_{t-1}} P(x_{t-1} \mid e_{1:t-1})\, P(X_t \mid x_{t-1})\, P(e_t \mid X_t)$$

$$= \alpha\, P(e_t \mid x_t) \sum_{x_{t-1}} P(x_t \mid x_{t-1})\, P(x_{t-1} \mid e_{1:t-1})$$

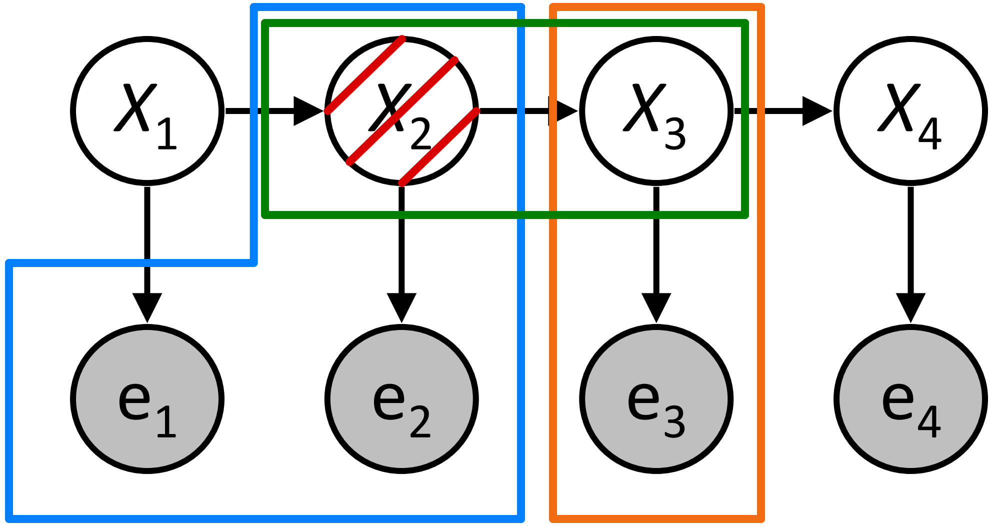
Recursion!



Filtering **Algorithm**

Query: What is the current state, given all of the current and past evidence?

Matching math with Bayes net



Recursion!

$$P(X_t \mid e_{1:t}) = P(X_t \mid e_t, e_{1:t-1})$$

$$= \alpha\, P(X_t, e_t \mid e_{1:t-1})$$

$$= \alpha \sum_{x_{t-1}} P(x_{t-1}, X_t, e_t \mid e_{1:t-1})$$

$$= \alpha \sum_{x_{t-1}} P(x_{t-1} \mid e_{1:t-1})\, P(X_t \mid x_{t-1})\, P(e_t \mid X_t)$$

$$= \alpha\, P(e_t \mid x_t) \sum_{x_{t-1}} P(x_t \mid x_{t-1})\, P(x_{t-1} \mid e_{1:t-1})$$

Poll 2

$$P(X_{t+1}|e_{1:t+1}) = \underbrace{\alpha}_{\text{Normalize}} \underbrace{P(e_{t+1}|X_{t+1})}_{\text{Update}} \underbrace{\sum_{x_t} P(X_{t+1}| x_t)\, P(x_t \mid e_{1:t})}_{\text{Predict}}$$

What is the runtime of the forward algorithm in terms of the number of states |X| and time t? Assume all 3 CPTs are given.

A) $O(|X|^2 * t)$ ←

B) $O(|X| * t)$

[✓ C) $O(|X|^2)$]

D) $O(|X|)$

Filtering **Algorithm**

$$P(X_{t+1} \mid e_{1:t+1}) = \underbrace{\alpha}_{\text{Normalize}} \; \underbrace{P(e_{t+1} \mid X_{t+1})}_{\text{Update}} \; \underbrace{\sum_{x_t} P(X_{t+1} \mid x_t)\, P(x_t \mid e_{1:t})}_{\text{Predict}}$$

$\boldsymbol{f}_{1:t+1} = \textsc{Forward}(\boldsymbol{f}_{1:t}, e_{t+1})$

Cost per time step: $O(|X|^2)$ where $|X|$ is the number of states

Time and space costs are ***constant***, independent of t

$O(|X|^2)$ is infeasible for models with many state variables

We get to invent really cool approximate filtering algorithms

In Class Activity: Weather HMM

An HMM is defined by:

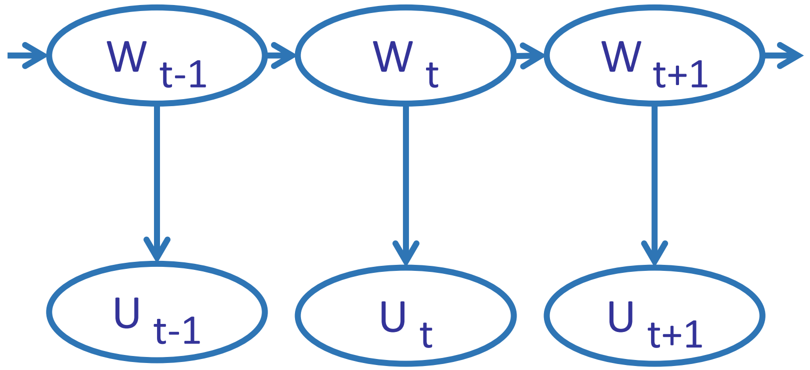
- Initial distribution: $P(X_1)$
- Transition model: $P(X_t \mid X_{t-1}) = P(W_t \mid W_{t-1})$
- Sensor model: $P(E_t \mid X_t) = P(U_t \mid W_t)$

W_{t-1}	$P(W_t \mid W_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Given $P(X_1)$ = {sun:0.5, rain:0.5}

Compute $P(X_4=\text{sun} \mid e_4 = e_3 = e_2 = e_1 = \text{True})$



In Class Activity: Weather HMM

An HMM is defined by:

- Initial distribution: $P(X_1)$
- Transition model: $P(X_t \mid X_{t-1})$
- Sensor model: $P(E_t \mid X_t)$

W_{t-1}	$P(W_t \mid W_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Compute $P(X_4\text{=sun} \mid e_4\text{=}\ e_3\text{=}\ e_2\text{=}\ e_1\text{=True})$ and $P(X_1)$ = {sun:0.5, rain:0.5}

$P(X_1, e_1) = P(e_1|X_1)P(X_1)$ #OBSERVE (chain rule)

$P(X_1|e_1) = \alpha P(X_1, e_1) \rightarrow \alpha = 1/\sum_{x_1} P(e_1|x_1)P(x_1)$ #Don't forget to NORMALIZE

$P(X_2|e_1) = \sum_{x \in X_1} P(X_2|x)P(x|e_1)$ #PREDICT

In Class Activity: Weather HMM

An HMM is defined by:

- Initial distribution: $P(X_1)$
- Transition model: $P(X_t \mid X_{t-1})$
- Sensor model: $P(E_t \mid X_t)$

W_{t-1}	**$P(W_t \mid W_{t-1})$**	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	**$P(U_t \mid W_t)$**	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Compute $P(X_4\text{=sun} \mid e_4= e_3= e_2= e_1\text{=True})$ and $P(X_1) = \{\text{sun}:0.5, \text{rain}:0.5\}$

$P(X_2|e_1) = \sum_{x\in X_1} P(X_2|x)P(x|e_1)$ #PREDICT

$$P(X_2|e_1,e_2) = \alpha P(X_2,e_2|e_1) = \alpha P(e_2|X_2)P(X_2|e_1);\ \alpha = 1/\sum_{x\in X_2} P(e_2|x)P(x|e_1)$$

$P(X_3|e_1,e_2) = \sum_{x_2\in X_2} P(X_3|x_2)P(x_2|e_1,e_2)$ #PREDICT

In Class Activity: Weather HMM

An HMM is defined by:

- Initial distribution: $P(X_1)$
- Transition model: $P(X_t \mid X_{t-1})$
- Sensor model: $P(E_t \mid X_t)$

W_{t-1}	**$P(W_t \mid W_{t-1})$**	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	**$P(U_t \mid W_t)$**	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Compute $P(X_4\text{=sun} \mid e_4\text{= }e_3\text{= }e_2\text{= }e_1\text{=True})$ and $P(X_1)$ = {sun:0.5, rain:0.5}

$$P(X_3|e_1,e_2) = \sum\nolimits_{x_2 \in X_2} P(X_3|x_2)P(x_2|e_1,e_2)$$ #PREDICT

$$P(X_3|e_1,e_2,e_3) = \alpha P(X_3,e_3|e_1,e_2) = \alpha P(e_3|X_3)P(X_3|e_1,e_2);$$

$$\alpha = 1/\sum_{x \in X_3} P(e_3|x)P(x|e_1,e_2)$$

$$P(X_4|e_1,e_2,e_3) = \sum\nolimits_{x \in X_3} P(X_4|x)P(x|e_1,e_2,e_3)$$ #PREDICT

In Class Activity: Weather HMM

An HMM is defined by:

- Initial distribution: $P(X_1)$
- Transition model: $P(X_t \mid X_{t-1})$
- Sensor model: $P(E_t \mid X_t)$

W_{t-1}	$P(W_t \mid W_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Compute P(X_4=sun | e_4= e_3= e_2= e_1=True) and P(X_1) = {sun:0.5, rain:0.5}

$P(X_4|e_1,e_2,e_3) = \sum_{x\in X_3} P(X_4|x)P(x|e_1,e_2,e_3)$ #PREDICT

$$P(X_4|e_1,e_2,e_3,e_4) = \alpha P(X_4,e_4|e_1,e_2,e_3) = \alpha P(e_4|X_4)P(X_4|e_1,e_2,e_3);$$
$$\alpha = 1/\sum_{x\in X_4} P(e_4|x)P(x|e_1,e_2,e_3)$$

In Class Activity: Weather HMM

An HMM is defined by:

- Initial distribution: $P(X_1)$
- Transition model: $P(X_t \mid X_{t-1})$
- Sensor model: $P(E_t \mid X_t)$

W_{t-1}	**$P(W_t \mid W_{t-1})$**	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	**$P(U_t \mid W_t)$**	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Compute P(X_4=sun | e_4= e_3= e_2= e_1=True) and P(X_1) = {sun:0.5, rain:0.5}

$P(X_1, e_1) = P(e_1|X_1)P(X_1)$ #OBSERVE (chain rule)

$P(e_1 = True|X_1 = sun)P(X_1 = sun) = .2 * .5 = \ .1$

$P(e_1 = True|X_1 = rain)P(X_1 = rain) = \ .9 * .5 = \ .45$

$P(X_1|e_1) = \frac{P(X_1,e_1)}{P(e_1)} = \ P(e_1|X_1)P(X_1)/\sum_{x \in X_1} P(e_1|x)P(x)$ #NORMALIZE USING BAYES RULE

$$P(X_1 = sun|e_1 = True) = \frac{.1}{.1 + .45} = \ .18$$

$$P(X_1 = rain|e_1 = True) = \frac{.45}{.1 + .45} = \ .82$$

In Class Activity: Weather HMM

An HMM is defined by:

- Initial distribution: $P(X_1)$
- Transition model: $P(X_t \mid X_{t-1})$
- Sensor model: $P(E_t \mid X_t)$

W_{t-1}	$P(W_t \mid W_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Compute $P(X_4\text{=sun} \mid e_4\text{= }e_3\text{= }e_2\text{= }e_1\text{=True})$ and $P(X_1)$ = {sun:0.5, rain:0.5}

$P(X_2|e_1) = \sum_{x \in X_1} P(X_2|x)P(x|e_1)$ #PREDICT

$$P(X_2 = sun|e_1 = True) = \sum_{x \in X_1} P(X_2 = sun|x)P(x|e_1 = True) = .9 * .18 + .3 * .82 = .41$$

$$P(X_2 = rain|e_1 = True) = \sum_{x \in X_1} P(X_2 = rain|x)P(x|e_1 = True) = .1 * .18 + .7 * .82 = .59$$

In Class Activity: Weather HMM

An HMM is defined by:

- Initial distribution: $P(X_1)$
- Transition model: $P(X_t \mid X_{t-1})$
- Sensor model: $P(E_t \mid X_t)$

W_{t-1}	$P(W_t \mid W_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Compute $P(X_4\text{=sun} \mid e_4\text{= }e_3\text{= }e_2\text{= }e_1\text{=True})$ and $P(X_1)$ = {sun:0.5, rain:0.5}

$$P(X_2|e_1,e_2) = \alpha P(X_2,e_2|e_1) = \alpha P(e_2|X_2)P(X_2|e_1);\ \alpha = 1/\sum_{x \in X_2} P(e_2|x)P(x|e_1)$$

$$P(X_2 = sun|e_1, e_2 = True) = \alpha P(e_2|X_2 = sun)P(X_2 = sun|e_1) = \alpha(.2)(.41) = \ .13$$

$$P(X_2 = rain|e_1, e_2 = True) = \alpha P(e_2|X_2 = rain)P(X_2 = rain|e_1) = \alpha(.9)(.59) = \ .87$$

$P(X_3|e_1,e_2) = \sum_{x \in X_2} P(X_3|x)P(x|e_1,e_2)$ #PREDICT

$P(X_3 = sun|e_1,e_2) = P(X_3 = sun|x = sun)P(x = sun|e_1,e_2) + P(X_3|x = rain)P(x = rain|e_1,e_2)$ = 0.38

$P(X_3 = rain|e_1,e_2) = P(X_3 = rain|x = sun)P(x = sun|e_1,e_2) + P(X_3|x = rain)P(x = rain|e_1,e_2)$ = 0.62

In Class Activity: Weather HMM

An HMM is defined by:

- Initial distribution: $P(X_1)$
- Transition model: $P(X_t \mid X_{t-1})$
- Sensor model: $P(E_t \mid X_t)$

W_{t-1}	$P(W_t \mid W_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Compute $P(X_4\text{=sun} \mid e_4= e_3= e_2= e_1\text{=True})$ and $P(X_1)$ = {sun:0.5, rain:0.5}

$$P(X_3|e_1, e_2, e_3) = \alpha P(X_3, e_3|e_1, e_2) = \alpha P(e_3|X_3) P(X_3|e_1, e_2);$$

$$\alpha = 1/\sum_{x \in X_3} P(e_3|x) P(x|e_1, e_2)$$

$$P(X_3 = sun|e_1, e_2, e_3) = \alpha P(e_3 = True|X_3 = sun) P(X_3 = sun|e_1, e_2) = \alpha(.2)(.38) = .12$$

$$P(X_3 = rain|e_1, e_2, e_3) = \alpha P(e_3 = True|X_3 = rain) P(X_3 = rain|e_1, e_2) = \alpha(.9)(.62) = .88$$

In Class Activity: Weather HMM

An HMM is defined by:

- Initial distribution: $P(X_1)$
- Transition model: $P(X_t \mid X_{t-1})$
- Sensor model: $P(E_t \mid X_t)$

W_{t-1}	**$P(W_t \mid W_{t-1})$**	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	**$P(U_t \mid W_t)$**	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Compute $P(X_4\text{=sun} \mid e_4\text{= }e_3\text{= }e_2\text{= }e_1\text{=True})$ and $P(X_1)$ = {sun:0.5, rain:0.5}

$$P(X_4|e_1,e_2,e_3) = \sum_{x \in X_3} P(X_4|x)P(x|e_1,e_2,e_3) \text{ \#PREDICT}$$

$$P(X_4 = sun|e_1,e_2,e_3) = \sum_{x \in \{sun,rain\}} P(X_4 = sun|x)P(x|e_1,e_2,e_3) = .9 * .12 + .3 * .88 = .37$$

$$P(X_4 = rain|e_1,e_2,e_3) = \sum_{x \in \{sun,rain\}} P(X_4 = rain|x)P(x|e_1,e_2,e_3) = .1 * .12 + .7 * .88 = .63$$

In Class Activity: Weather HMM

An HMM is defined by:

- Initial distribution: $P(X_1)$
- Transition model: $P(X_t \mid X_{t-1})$
- Sensor model: $P(E_t \mid X_t)$

W_{t-1}	$P(W_t \mid W_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Compute $P(X_4\text{=sun} \mid e_4\text{=} e_3\text{=} e_2\text{=} e_1\text{=True})$ and $P(X_1)$ = {sun:0.5, rain:0.5}

$$P(X_4|e_1,e_2,e_3,e_4) = \alpha P(X_4,e_4|e_1,e_2,e_3) = \alpha P(e_4|X_4)P(X_4|e_1,e_2,e_3);$$

$$\alpha = 1/\sum_{x\in X_4} P(e_4|x)P(x|e_1,e_2,e_3)$$

$$\alpha P(e_4 = True|X_4 = sun)P(X_4 = sun|e_1,e_2,e_3) = \alpha(.2*.37) = .115$$

$$\alpha P(e_4 = True|X_4 = rain)P(X_4 = rain|e_1,e_2,e_3) = \alpha(.9*.63) = .885$$

Poll 3

Suppose we are given P(X4=sun | e4= e3= e2= e1=True), along with the same CPT tables as the activity example, and we want to compute P(X5=sun | e5= e4= e3= e2= e1=True).

What is the first step we would perform?

Predict

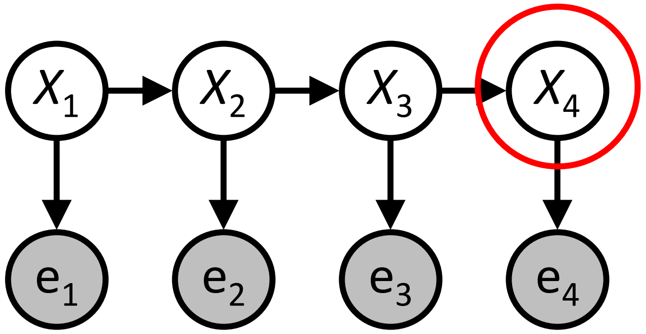
Observe

Forward

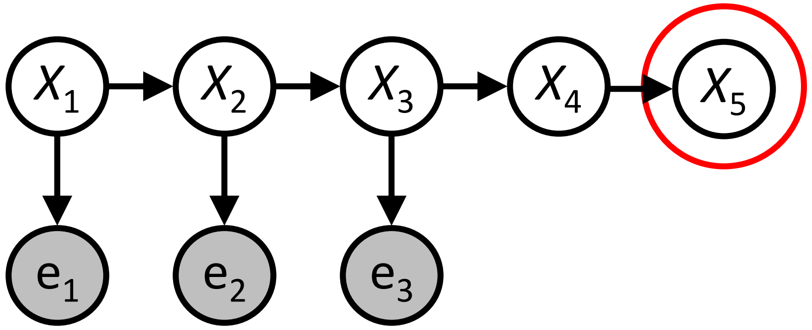
Smoothing

Other HMM Queries

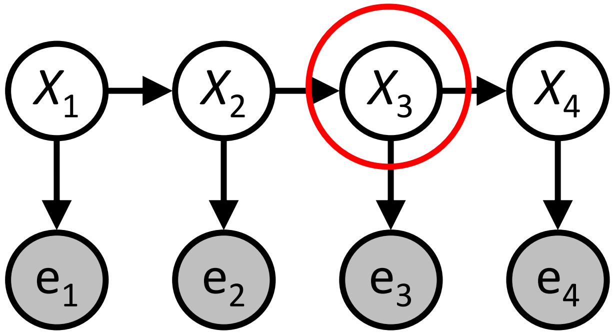
Filtering: $P(X_t | e_{1:t})$



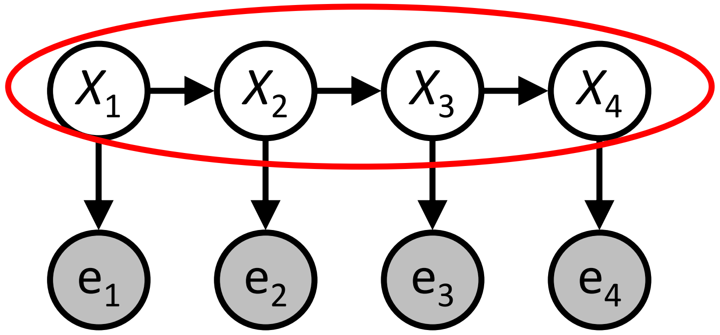
Prediction: $P(X_{t+k} | e_{1:t})$



Smoothing: $P(X_k | e_{1:t})$, $k<t$



Explanation: $P(X_{1:t} | e_{1:t})$



Inference Tasks

Filtering: $P(X_t | e_{1:t})$

- belief state—input to the decision process of a rational agent

Prediction: $P(X_{t+k} | e_{1:t})$ for $k > 0$

- evaluation of possible action sequences; like filtering without the evidence

Smoothing: $P(X_k | e_{1:t})$ for $0 \leq k < t$

- better estimate of past states, essential for learning

Most likely explanation: $\text{argmax}_{x_{1:t}} P(x_{1:t} \mid e_{1:t})$

- speech recognition, decoding with a noisy channel

Dynamic Bayes Nets (DBNs)

We want to track multiple variables over time, using multiple sources of evidence

Idea: Repeat a fixed Bayes net structure at each time

Variables from time *t* can condition on those from *t-1*