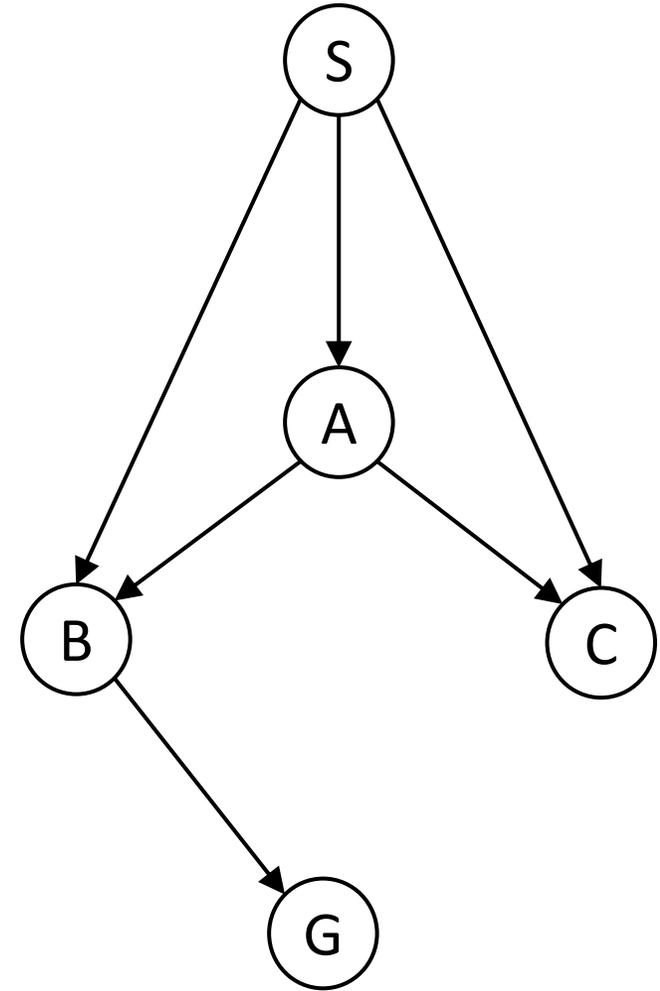


# Warm-up: DFS Graph Search

Why is the answer  $S \rightarrow B \rightarrow G$ , not  $S \rightarrow A \rightarrow B \rightarrow G$ ?

After all, we were doing DFS and breaking ties alphabetically.



# Announcements

## Assignments:

- HW1 (online) due *tonight* (Sep 5), 10pm
  - Can use late day (1 per written/online assignment, 2 per programming, 6 total for semester)
- HW2 (written) due 9/12, 10pm
- P1 due 9/18, 10pm

# Plan

## Last time

- Tree search vs graph search
- BFS, DFS, Uniform cost search

## Today

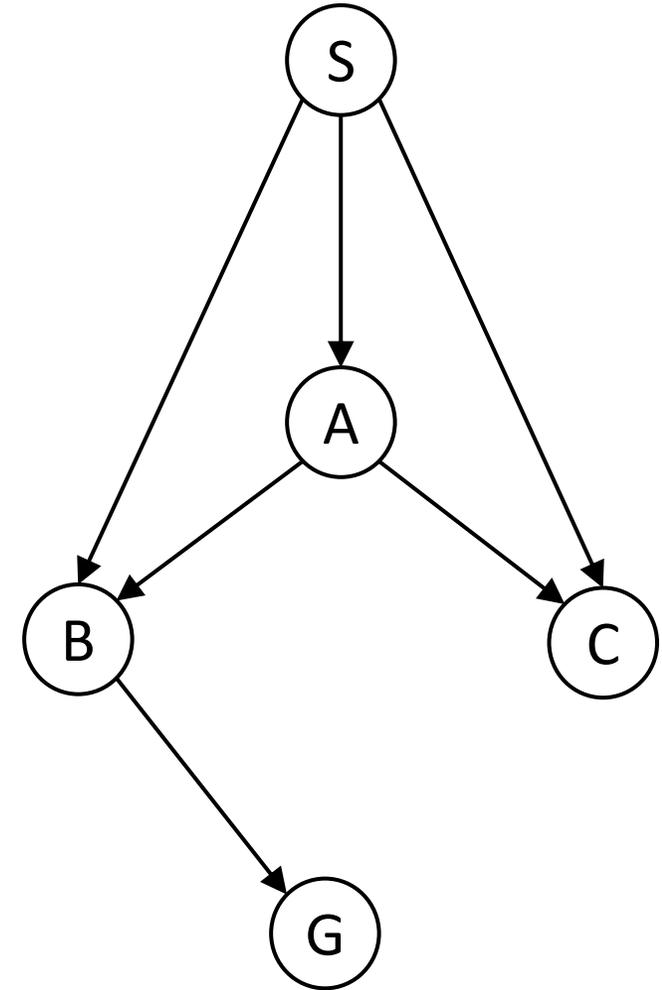
- Heuristics
- Greedy search
- A\* search
  - Optimality
- More on heuristics

# Warm-up: DFS Graph Search

Why is the answer  $S \rightarrow B \rightarrow G$ , not  $S \rightarrow A \rightarrow B \rightarrow G$ ?

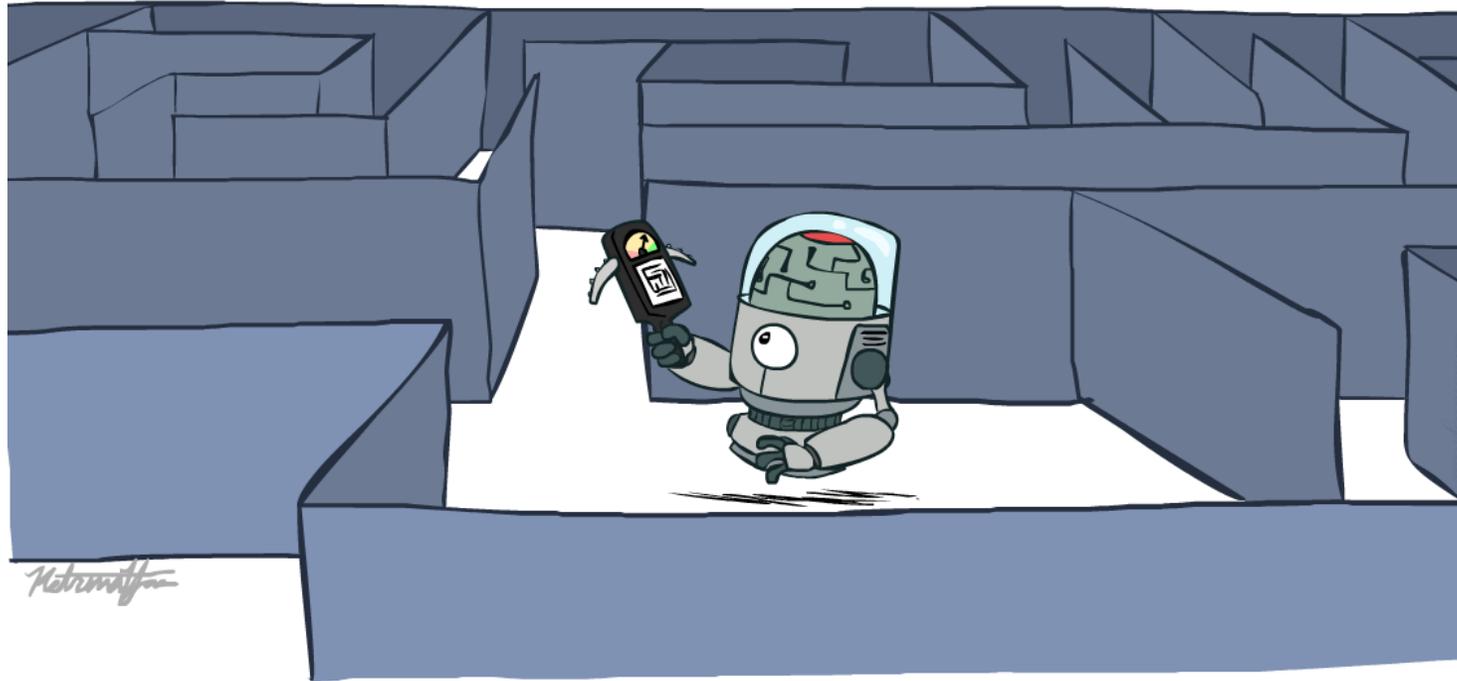
After all, we were doing DFS and breaking ties alphabetically.

Frontier	Explored
S	S
SA	<del>SA</del>
SB	<del>SB</del>
SC	
<u>SBG</u>	



# AI: Representation and Problem Solving

## Informed Search



Instructor: Vincent Conitzer and Aditi Raghunathan

Slide credits: CMU AI, <http://ai.berkeley.edu>

# Breadth-First Search (BFS) Properties

## What nodes does BFS expand?

- Processes all nodes above shallowest solution
- Let depth of shallowest solution be  $s$
- Search takes time  $O(b^s)$

## How much space does the frontier take?

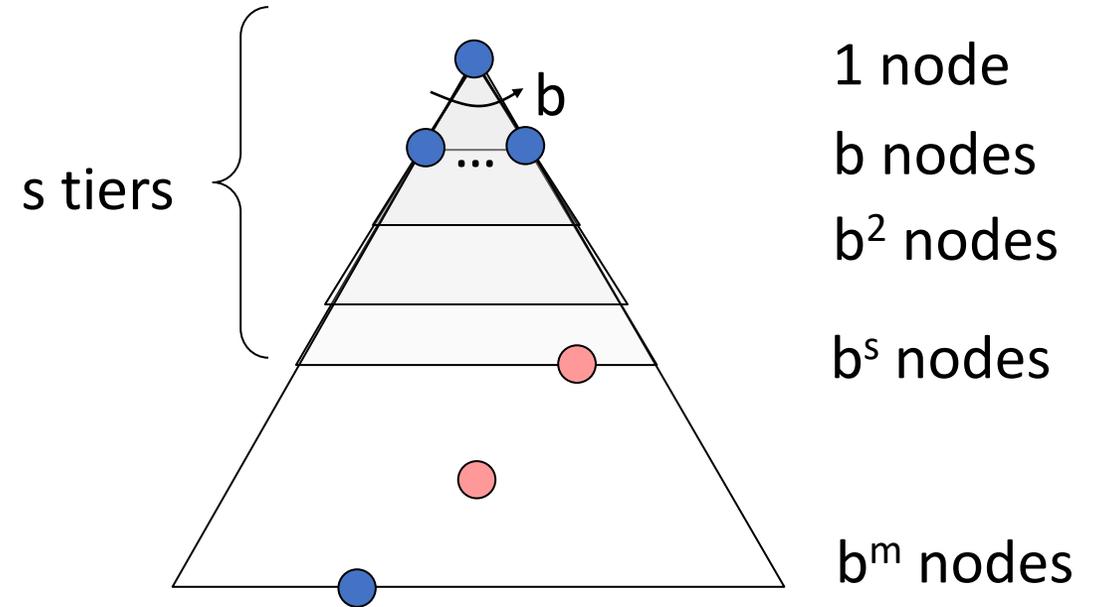
- Has roughly the last tier, so  $O(b^s)$

## Is it complete?

- $s$  must be finite if a solution exists, so yes!

## Is it optimal?

- Only if costs are all the same (more on costs later)



# Uniform Cost Search (UCS) Properties

## What nodes does UCS expand?

- Processes all nodes with cost less than cheapest solution
- If that solution costs  $C^*$  and step costs are at least  $\epsilon$ , then the “effective depth” is roughly  $C^*/\epsilon$
- Takes time  $O(b^{C^*/\epsilon})$  (exponential in effective depth)

## How much space does the frontier take?

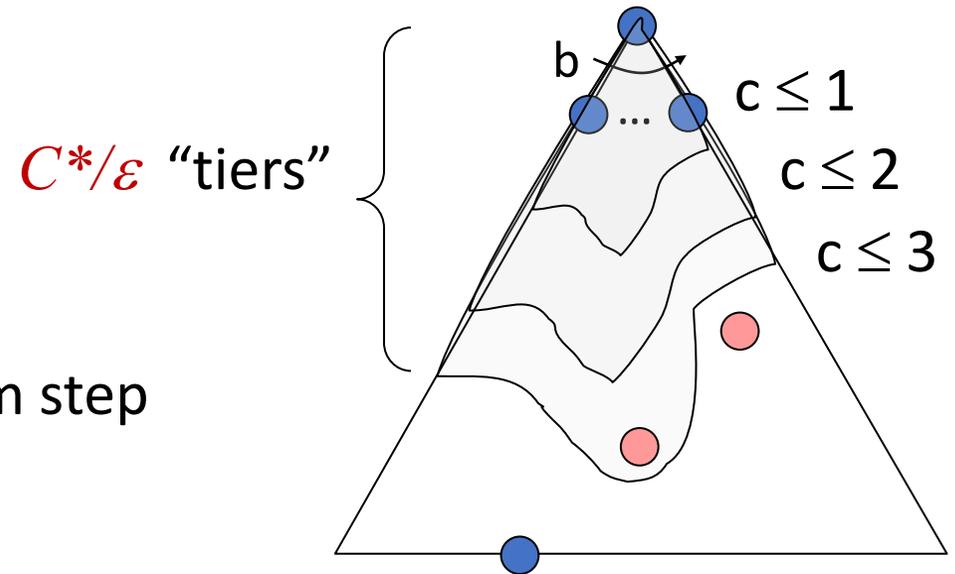
- Has roughly the last tier, so  $O(b^{C^*/\epsilon})$

## Is it complete?

- Assuming best solution has a finite cost and minimum step cost is positive, yes!

## Is it optimal?

- Yes! (Proof via A\*)



# Uniform Cost Issues

## Strategy:

- Explore (expand) the lowest path cost on frontier

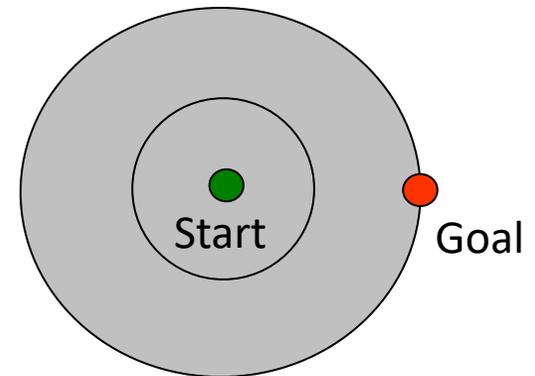
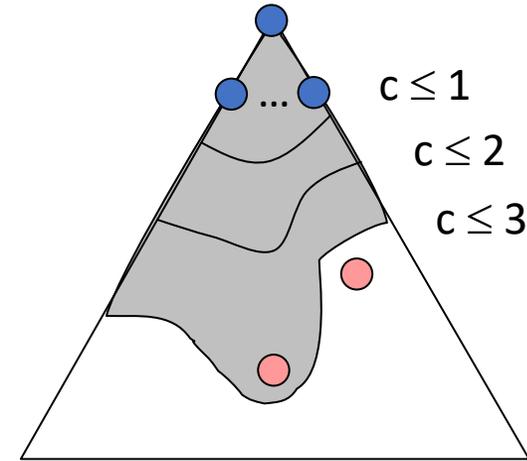
## The good:

- UCS is complete and optimal!

## The bad:

- Explores options in every “direction”
- No information about goal location

We'll fix that today!



**function** GRAPH-SEARCH(**problem**) **returns** a solution, or failure

initialize the **explored set** to be empty

initialize the **frontier** as a **priority queue** using some metric as the **priority**

add initial state of **problem** to **frontier** with **initial metric = 0**

**loop do**

**if** the **frontier** is empty **then**

**return** failure

choose a **node** and remove it from the **frontier**

**if** the **node** contains a goal state **then**

**return** the corresponding solution

add the **node** state to the **explored set**

for each resulting **child** from node

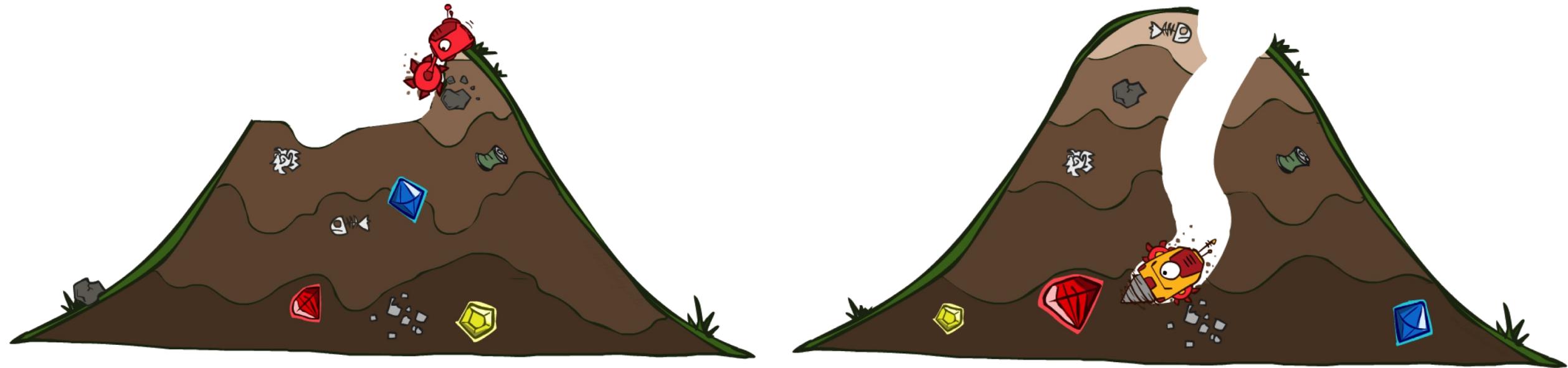
**if** the **child** state is not already in the **frontier** or **explored set** **then**

add **child** to the **frontier**

**else if** the **child** is already in the **frontier** with worse metric **then**

replace that **frontier** node with **child**

# Uninformed vs Informed Search



# Today

## Informed Search

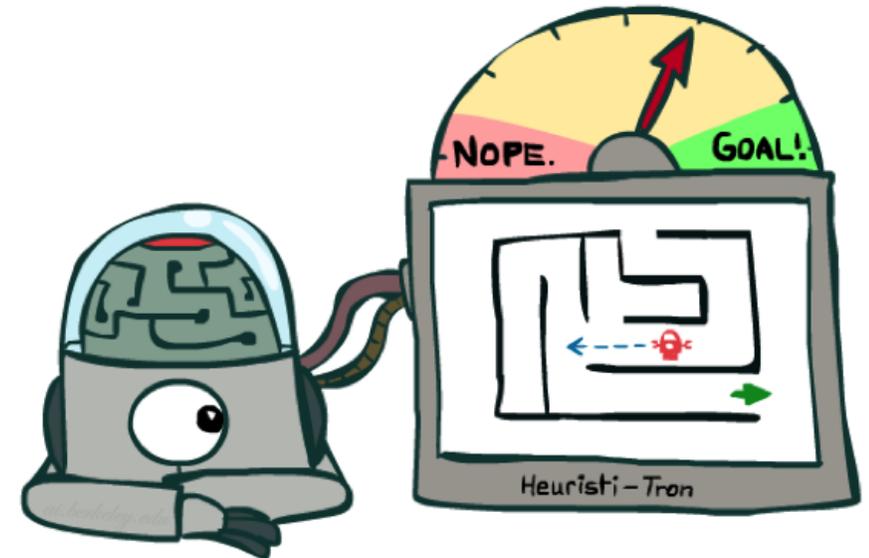
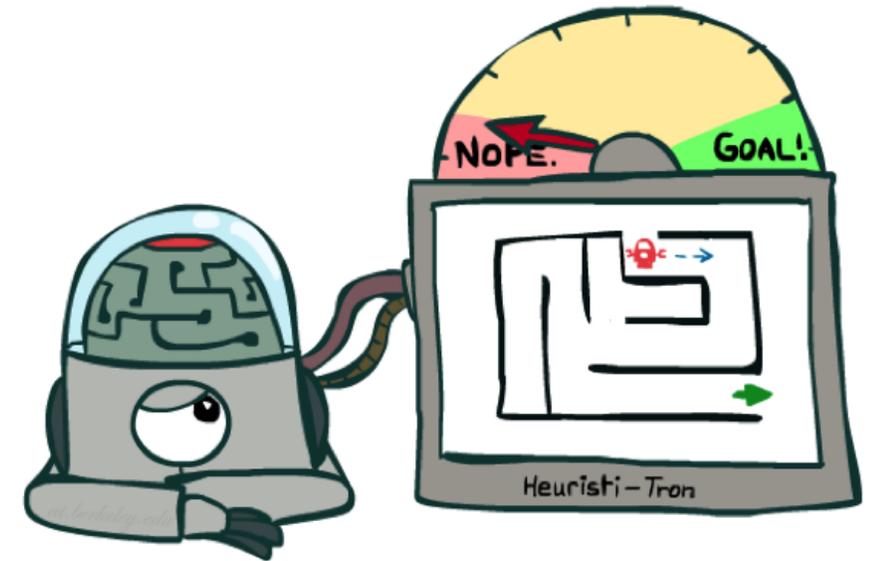
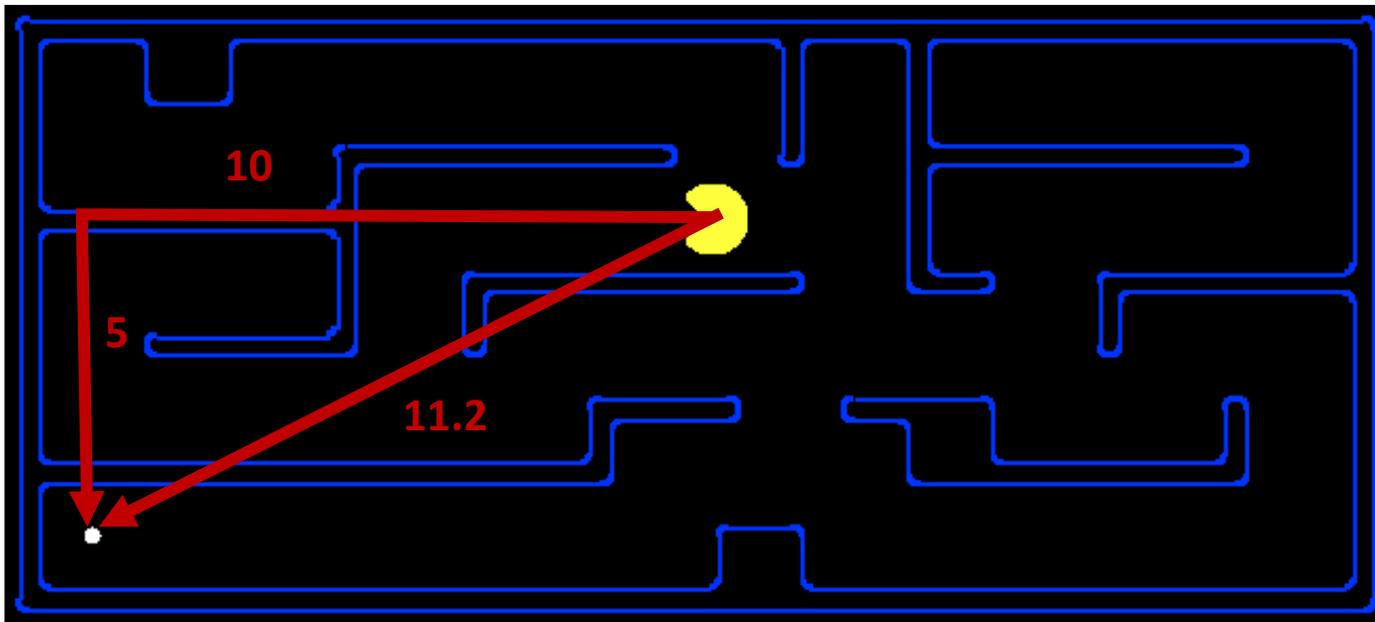
- Heuristics
- Greedy Search
- A\* Search



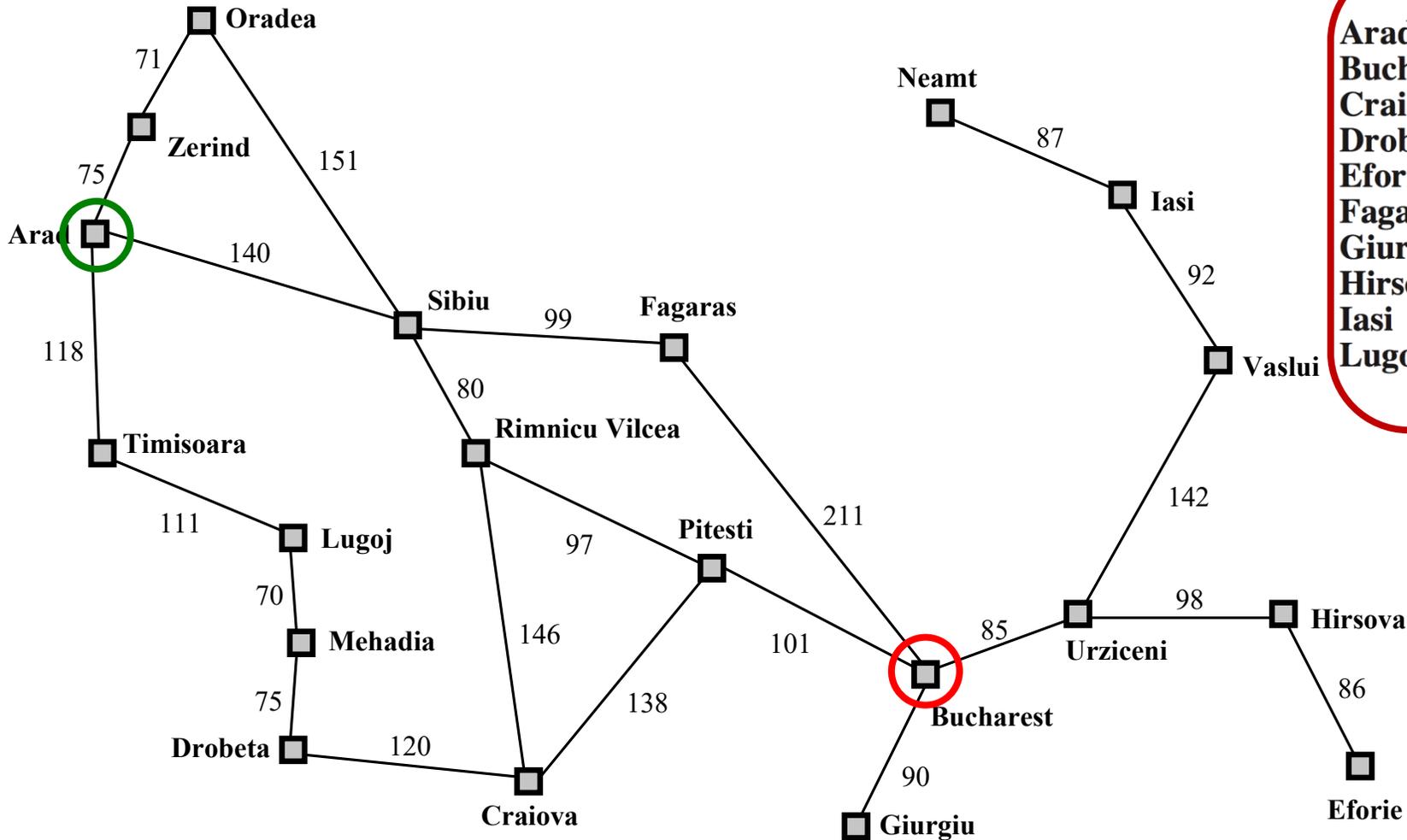
# Search Heuristics

## A heuristic is:

- A function that *estimates* how close a state is to a goal
- Designed for a particular search problem
- Examples: Manhattan distance, Euclidean distance for pathing



# Example: Euclidean distance to Bucharest

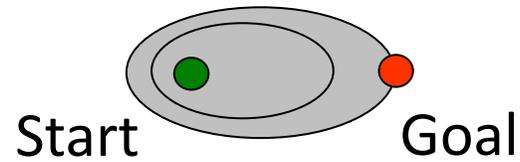


<b>Arad</b>	366	<b>Mehadia</b>	241
<b>Bucharest</b>	0	<b>Neamt</b>	234
<b>Craiova</b>	160	<b>Oradea</b>	380
<b>Drobeta</b>	242	<b>Pitesti</b>	100
<b>Eforie</b>	161	<b>Rimnicu Vilcea</b>	193
<b>Fagaras</b>	176	<b>Sibiu</b>	253
<b>Giurgiu</b>	77	<b>Timisoara</b>	329
<b>Hirsova</b>	151	<b>Urziceni</b>	80
<b>Iasi</b>	226	<b>Vaslui</b>	199
<b>Lugoj</b>	244	<b>Zerind</b>	374

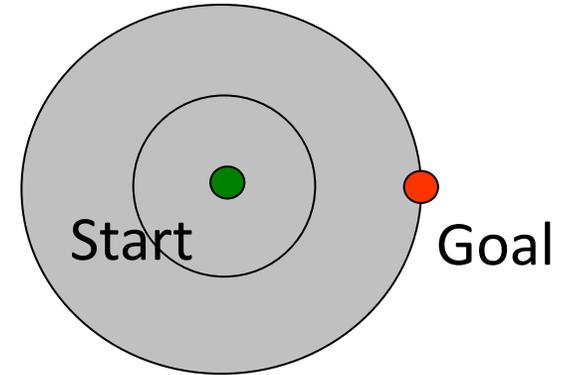
$h(\text{state}) \rightarrow \text{value}$

# Effect of heuristics

Guide search *towards the goal* instead of *all over the place*



Informed



Uninformed

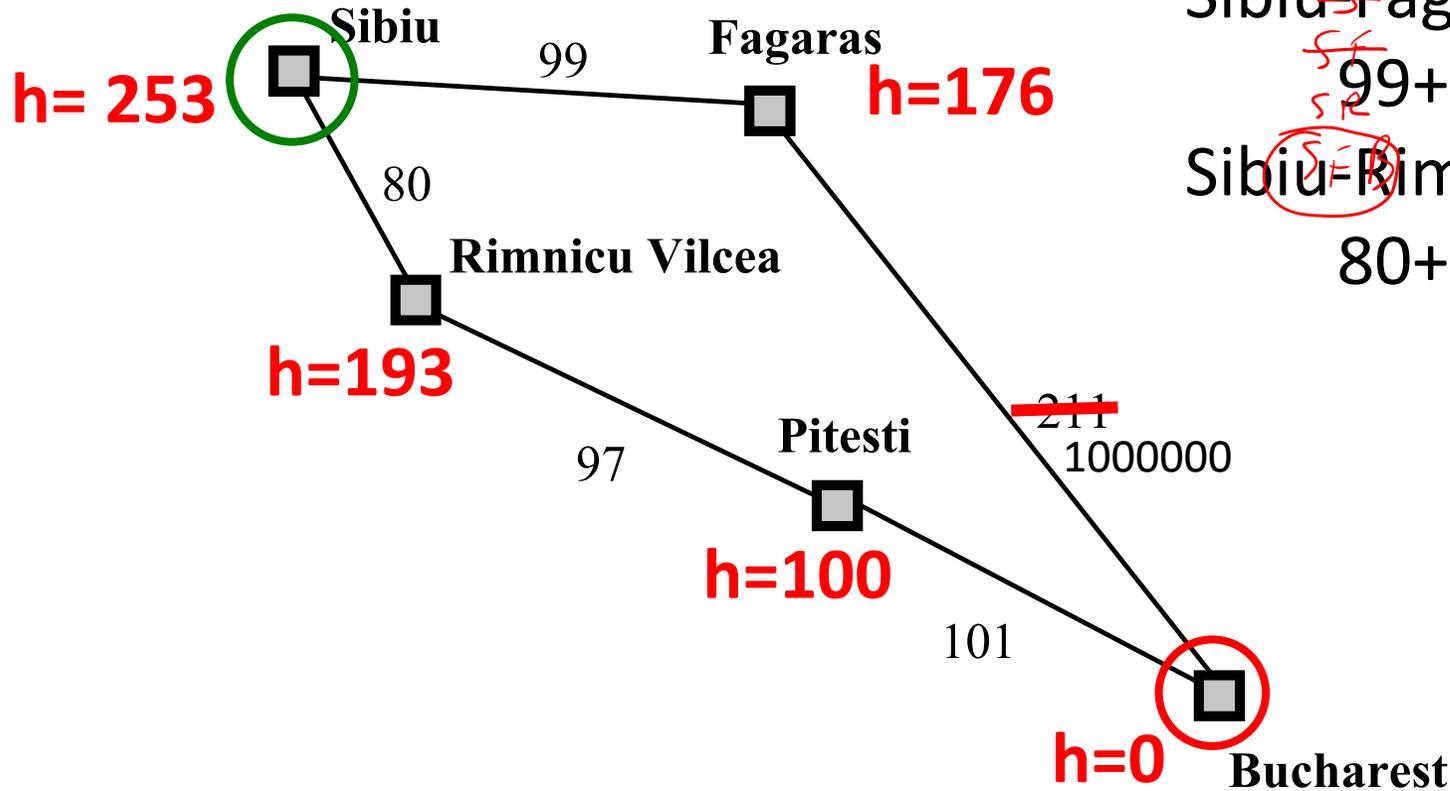
# Greedy Search



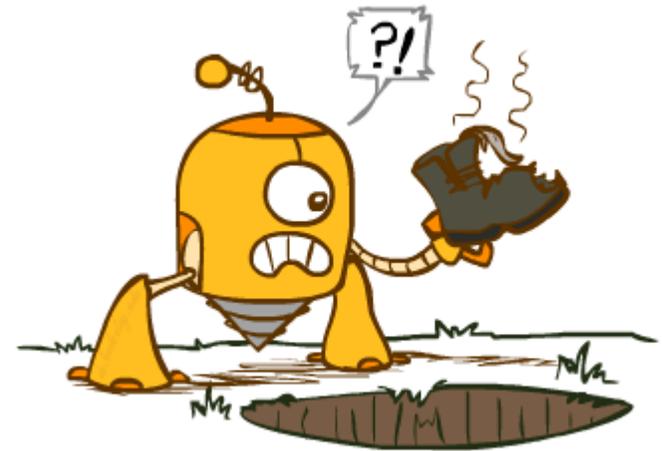
# Greedy Search

Expand the node that seems closest...(order frontier by h)

What can possibly go wrong?



*Frontier*  
Sibiu-Fagaras-Bucharest =  
~~SF~~  
~~SR~~  
99+211 = 310  
Sibiu-Rimnicu Vilcea-Pitesti-Bucharest =  
~~SF~~  
~~SR~~  
80+97+101 = 278

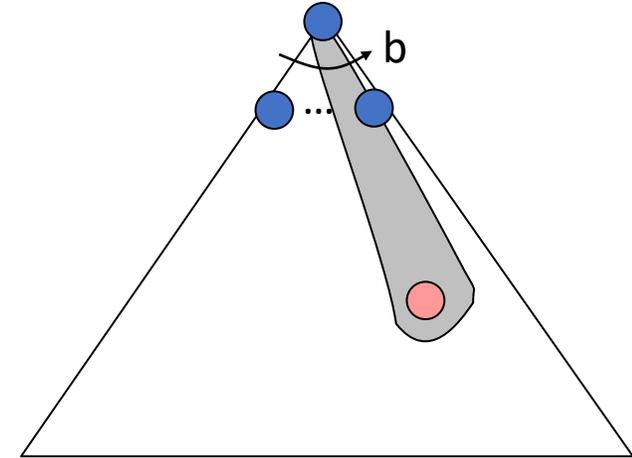


# Greedy Search

Strategy: expand a node that *seems* closest to a goal state, according to  $h$

Problem 1: it chooses a node even if it's at the end of a very long and winding road

Problem 2: it takes  $h$  literally even if it's completely wrong



# A\* Search



# A\* Search



UCS



Greedy

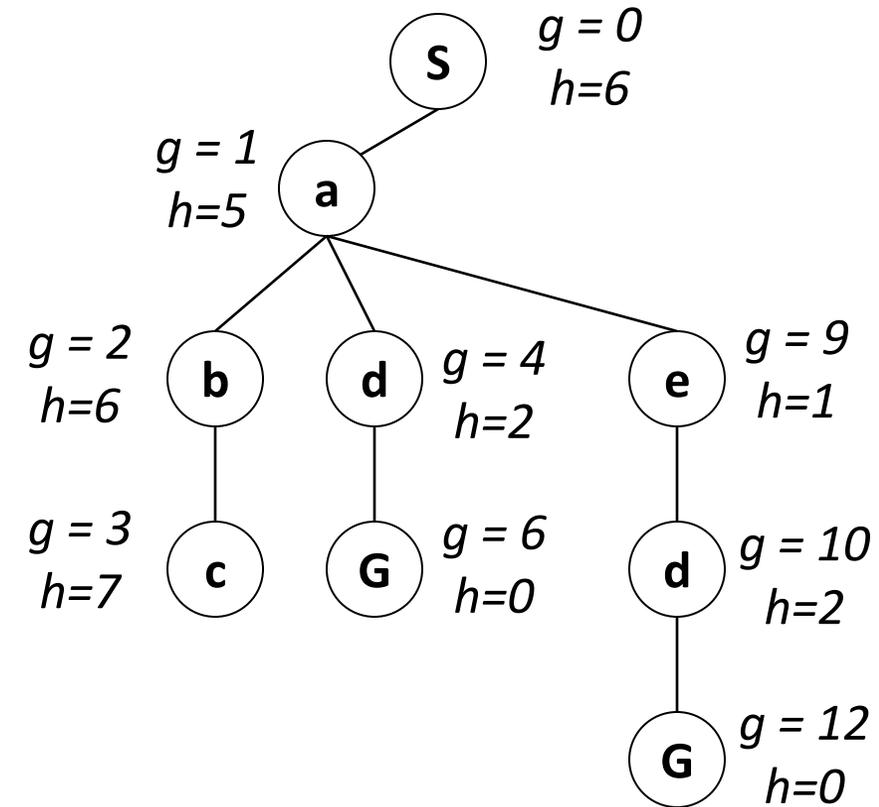
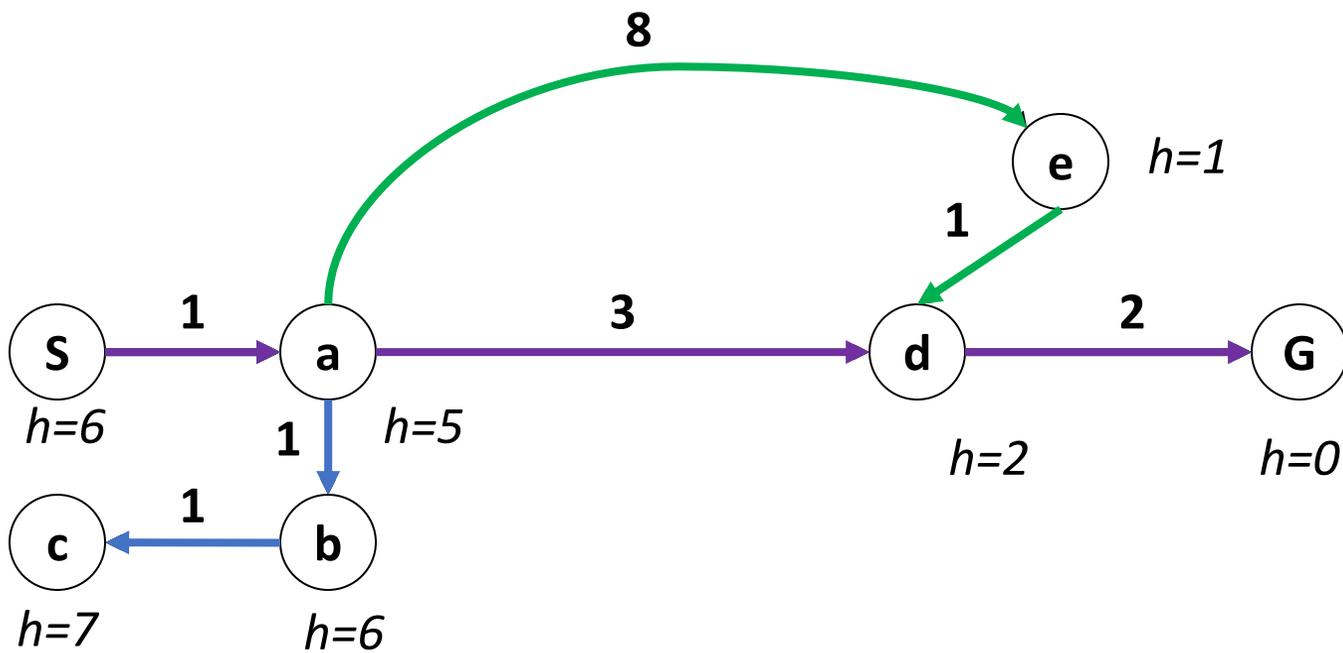


A\*

# Combining UCS and Greedy

Uniform-cost orders by path cost, or *backward cost*  $g(n)$

Greedy orders by goal proximity, or *forward cost*  $h(n)$



A\* Search orders by the sum:  $f(n) = g(n) + h(n)$

function UNIFORM-COST-SEARCH(problem) returns a solution, or failure

initialize the explored set to be empty

initialize the frontier as a priority queue using  $g(n)$  as the priority

add initial state of problem to frontier with priority  $g(S) = 0$

loop do

if the frontier is empty then

return failure

choose a node and remove it from the frontier

if the node contains a goal state then

return the corresponding solution

add the node state to the explored set

for each resulting child from node

if the child state is not already in the frontier or explored set then

add child to the frontier

else if the child is already in the frontier with higher  $g(n)$  then

replace that frontier node with child

function A-STAR-SEARCH(problem) returns a solution, or failure

initialize the explored set to be empty

initialize the frontier as a priority queue using  $f(n) = g(n) + h(n)$  as the priority

add initial state of problem to frontier with priority  $f(S) = 0 + h(S)$

loop do

if the frontier is empty then

return failure

choose a node and remove it from the frontier

if the node contains a goal state then

return the corresponding solution

add the node state to the explored set

for each resulting child from node

if the child state is not already in the frontier or explored set then

add child to the frontier

else if the child is already in the frontier with higher  $f(n)$  then

replace that frontier node with child

# A\* Search Algorithms

## A\* Tree Search

- Same tree search algorithm but with a **frontier** that is a priority queue using priority  $f(n) = g(n) + h(n)$

# A\* Search Algorithms

## A\* Tree Search

- Same tree search algorithm but with a **frontier** that is a priority queue using priority  $f(n) = g(n) + h(n)$

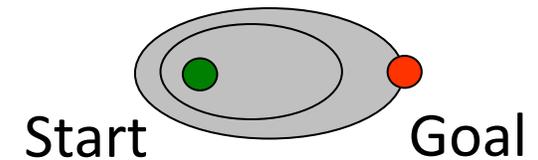
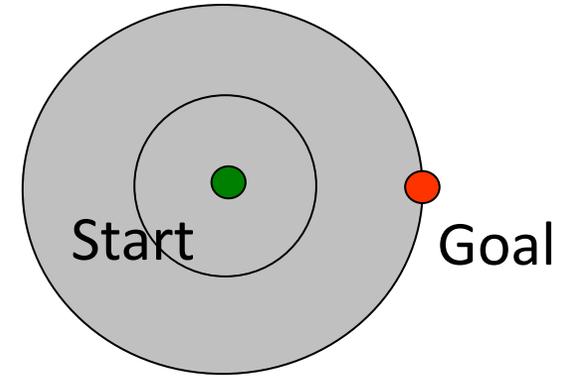
## A\* Graph Search

- Same as **UCS** graph search algorithm but with a **frontier** that is a priority queue using priority  $f(n) = g(n) + h(n)$

# UCS vs A\* Contours

Uniform-cost expands equally in all “directions”

A\* expands mainly toward the goal, but does hedge its bets to ensure optimality



# Comparison



Greedy

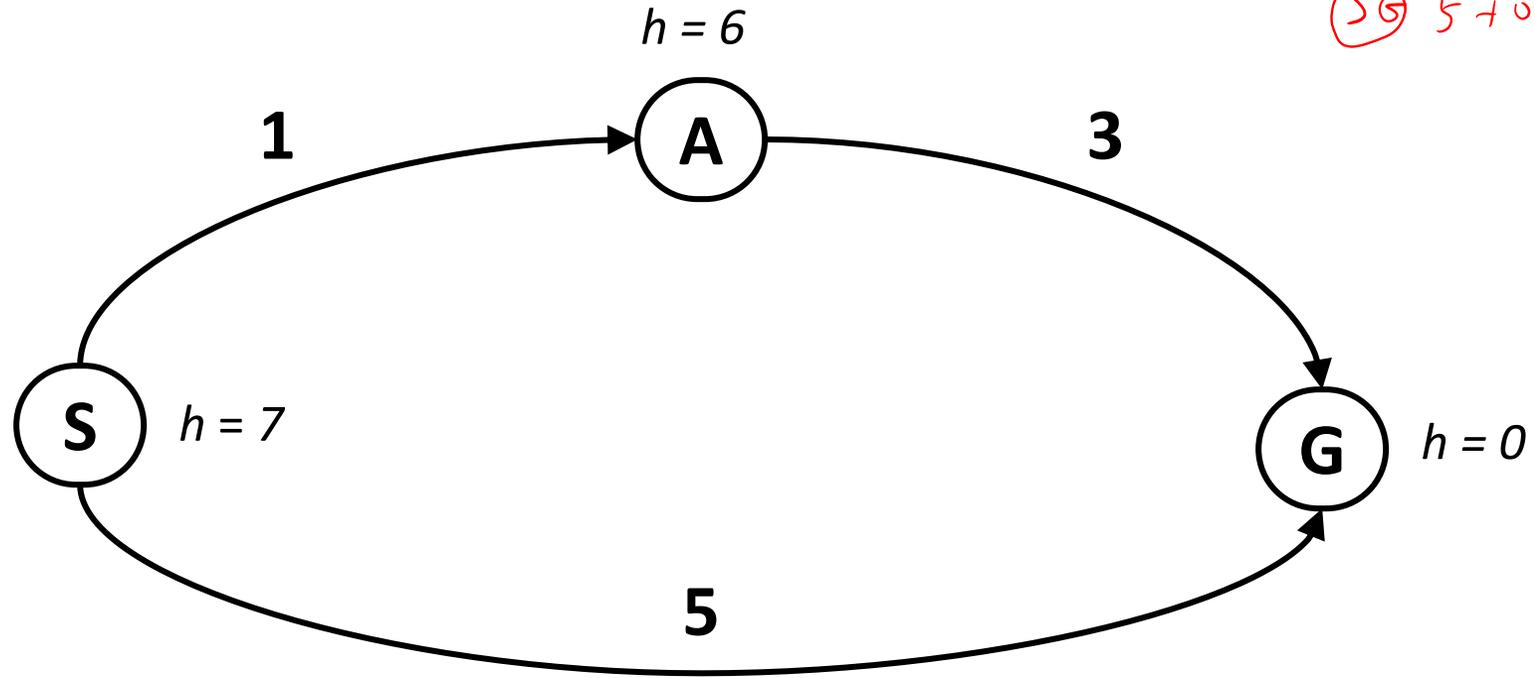


Uniform Cost



A\*

# Is A\* Optimal?



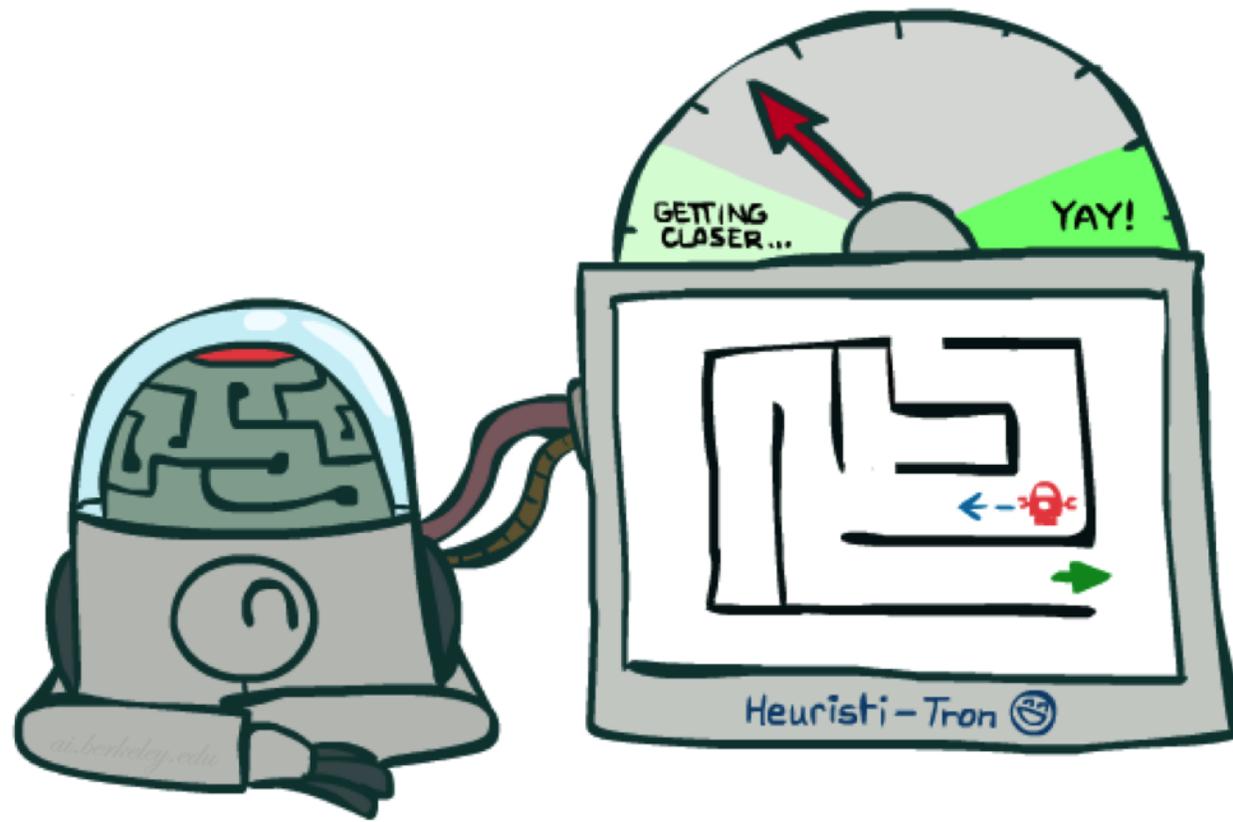
$$\begin{array}{r} f \\ \hline f \\ SA \ 1+6=7 \\ \textcircled{SG} \ 5+0=5 \end{array}$$

What went wrong?

**Actual** bad goal cost < **estimated** good goal cost

We need estimates to be less than actual costs!

# Admissible Heuristics



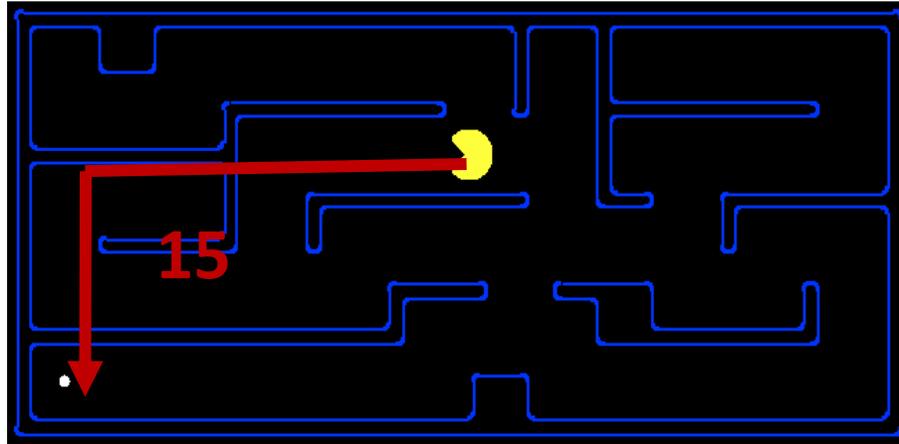
# Admissible Heuristics

A heuristic  $h$  is *admissible* (optimistic) if:

$$0 \leq h(n) \leq h^*(n)$$

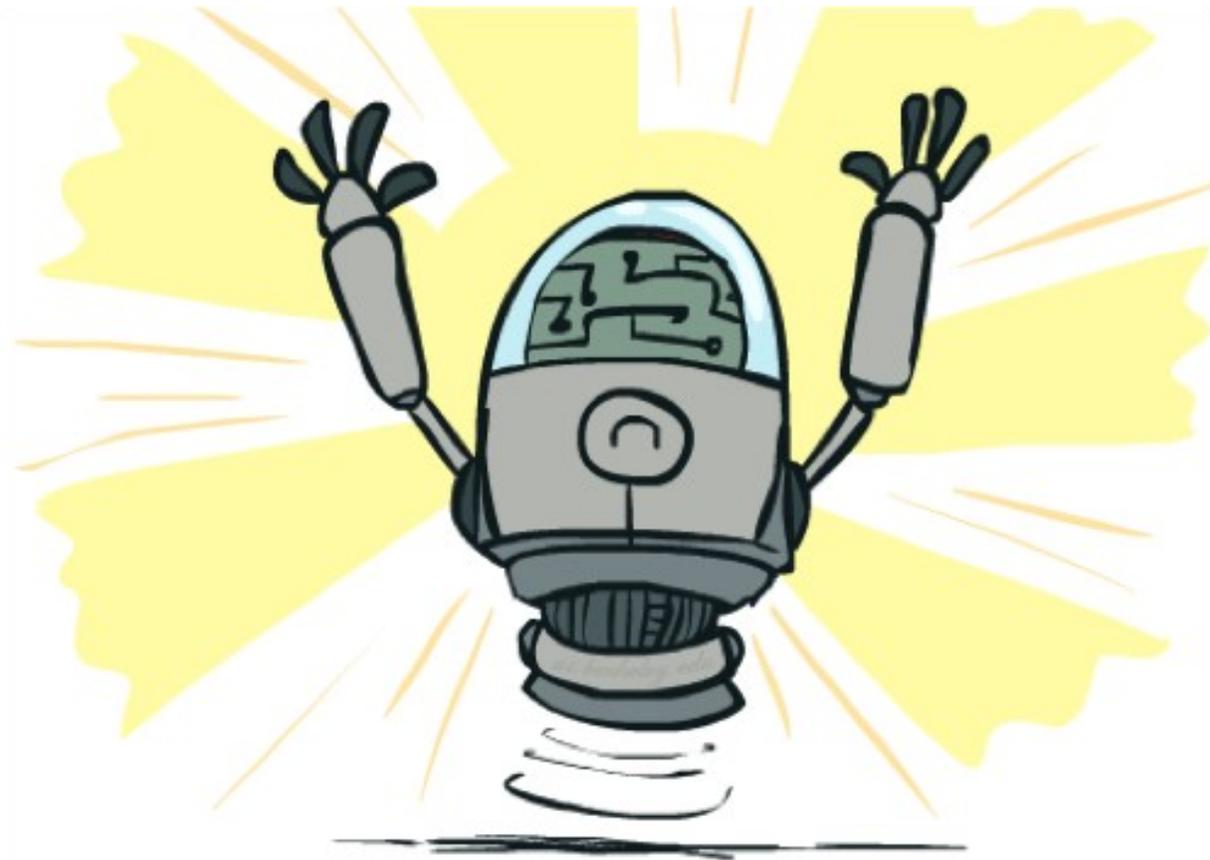
where  $h^*(n)$  is the true cost to a nearest goal

Example:



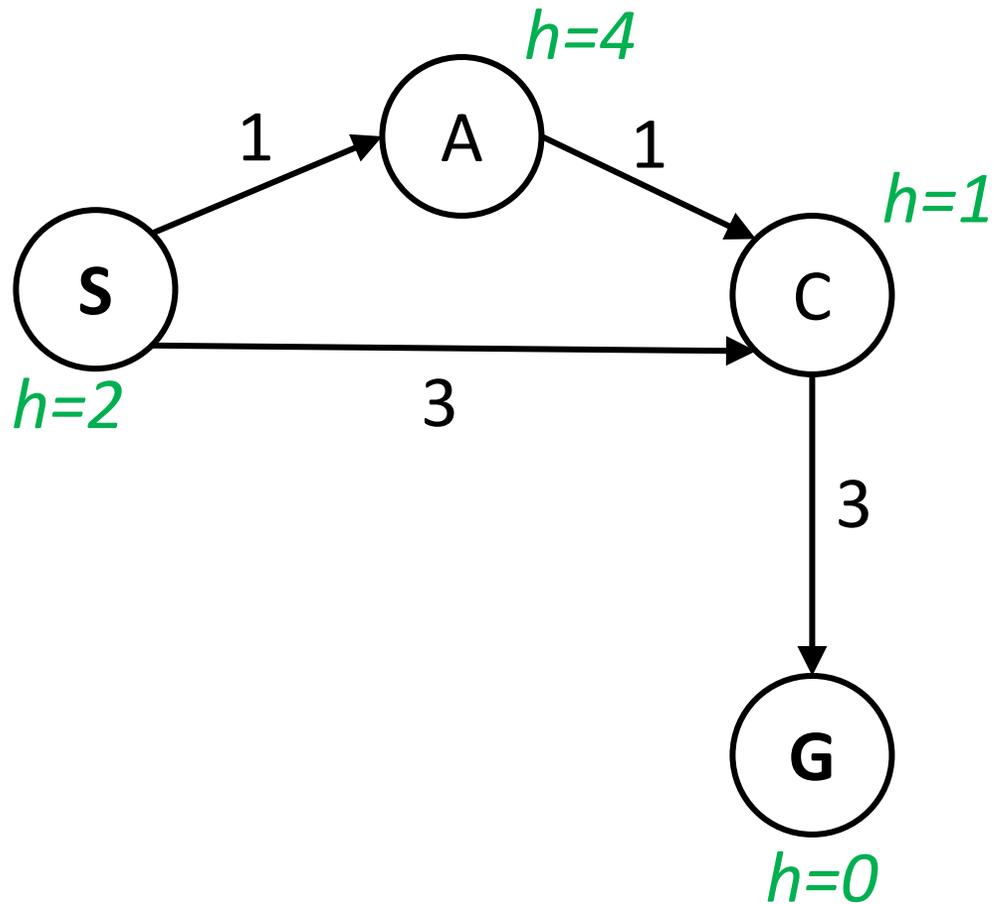
Coming up with admissible heuristics is most of what's involved in using  $A^*$  in practice.

# Optimality of A\* Tree Search



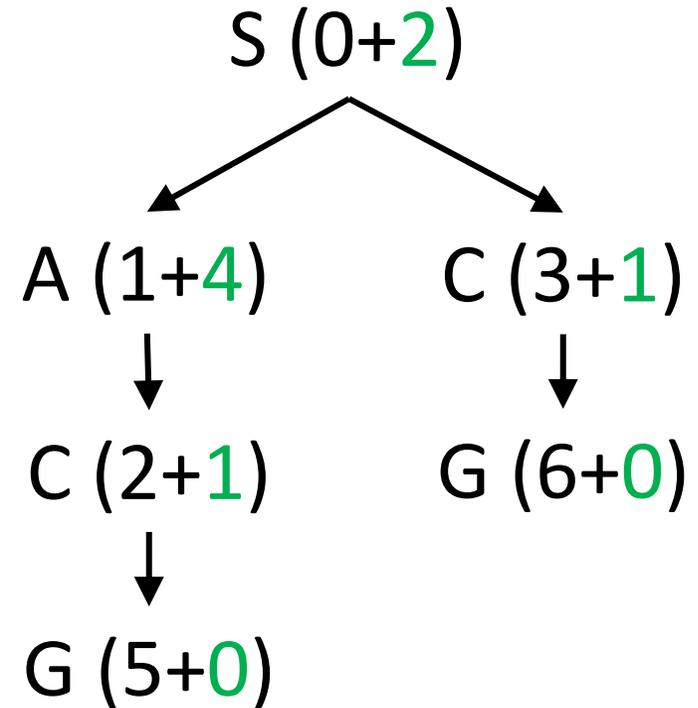
# A\* Tree Search

State space graph



F  
~~S 0+2~~  
~~SA 1+4~~  
~~SC 3+1~~  
SCG 6+0  
~~SAC 2+1~~  
SACG 5+0

Search tree



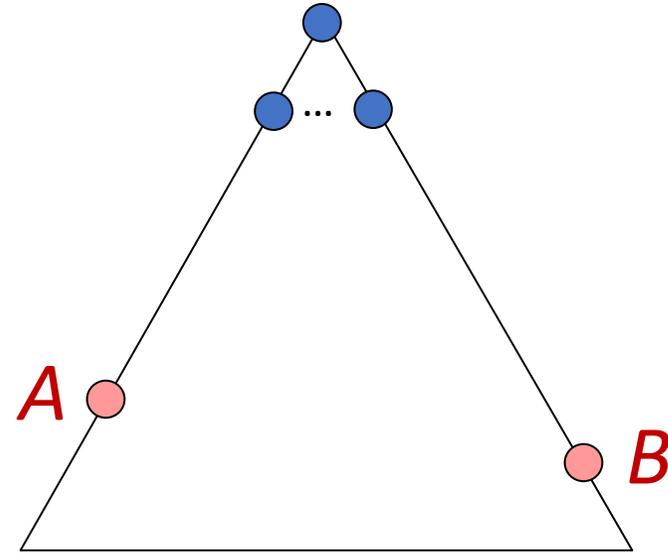
# Optimality of A\* Tree Search

Assume:

$A$  is an optimal goal node

$B$  is a suboptimal goal node

$h$  is admissible



Claim:

$A$  will be chosen for exploration (popped off the frontier) before  $B$

# Optimality of A\* Tree Search: Blocking

$$f(x) = g(x) + h(x)$$
$$h(x) \leq h^*(x)$$

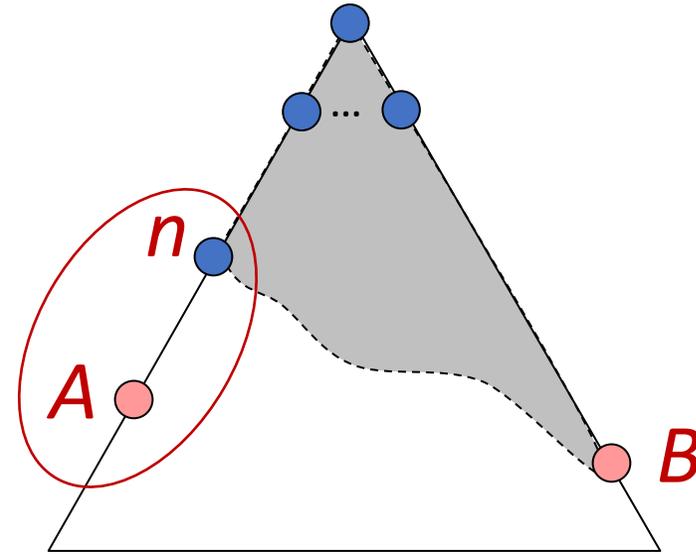
Proof:

Imagine  $B$  is on the frontier

Some ancestor  $n$  of  $A$  is on the frontier, too  
(Maybe the start state; maybe  $A$  itself!)

Claim:  $n$  will be explored before  $B$

1.  $f(n)$  is less than or equal to  $f(A)$



$$f(n) = g(n) + h(n)$$

$$f(n) \leq g(A)$$

$$g(A) = f(A)$$

Definition of  $f$ -cost

Admissibility of  $h$

$h = 0$  at a goal

# Optimality of A\* Tree Search: Blocking

$$f(x) = g(x) + h(x)$$
$$h(x) \leq h^*(x)$$

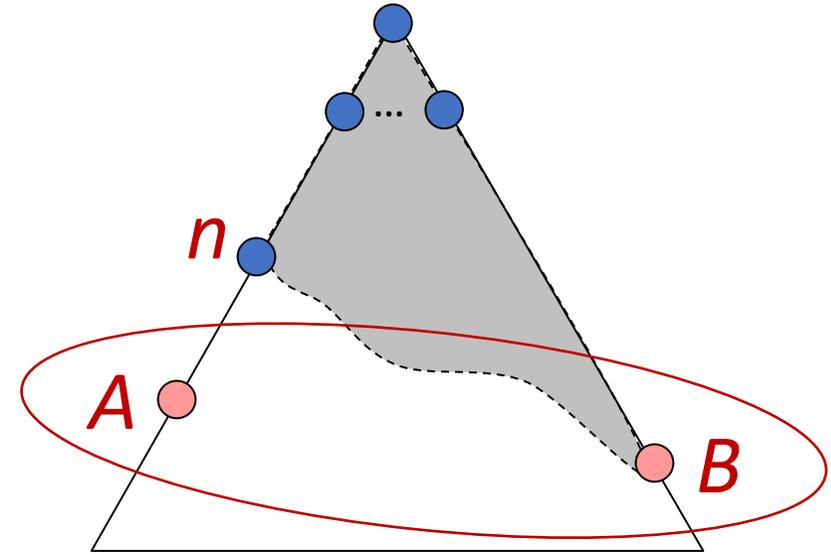
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Claim:  $n$  will be explored before  $B$

1.  $f(n)$  is less than or equal to  $f(A)$
2.  $f(A)$  is less than  $f(B)$



$$g(A) < g(B)$$
$$f(A) < f(B)$$

Suboptimality of  $B$   
 $h = 0$  at a goal

# Optimality of A\* Tree Search: Blocking

$$f(x) = g(x) + h(x)$$
$$h(x) \leq h^*(x)$$

Proof:

Imagine  $B$  is on the frontier

Some ancestor  $n$  of  $A$  is on the frontier, too  
(Maybe the start state; maybe  $A$  itself!)

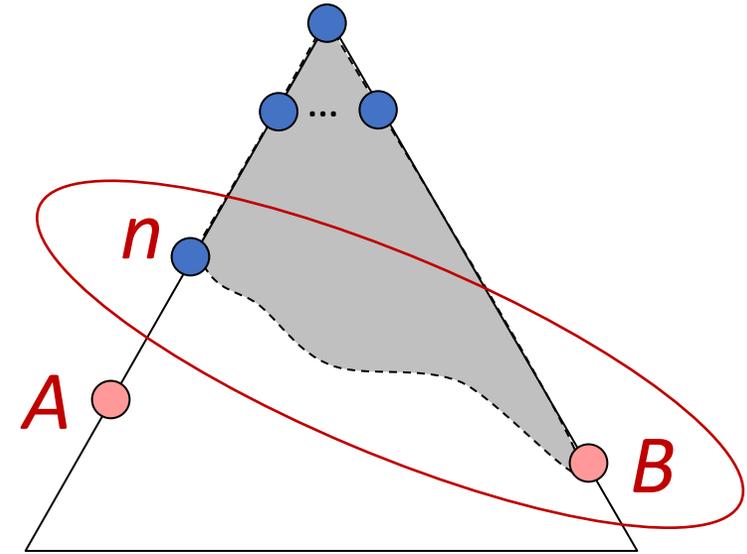
Claim:  $n$  will be explored before  $B$

1.  $f(n)$  is less than or equal to  $f(A)$
2.  $f(A)$  is less than  $f(B)$
3.  $n$  is explored before  $B$

All ancestors of  $A$  are explored before  $B$

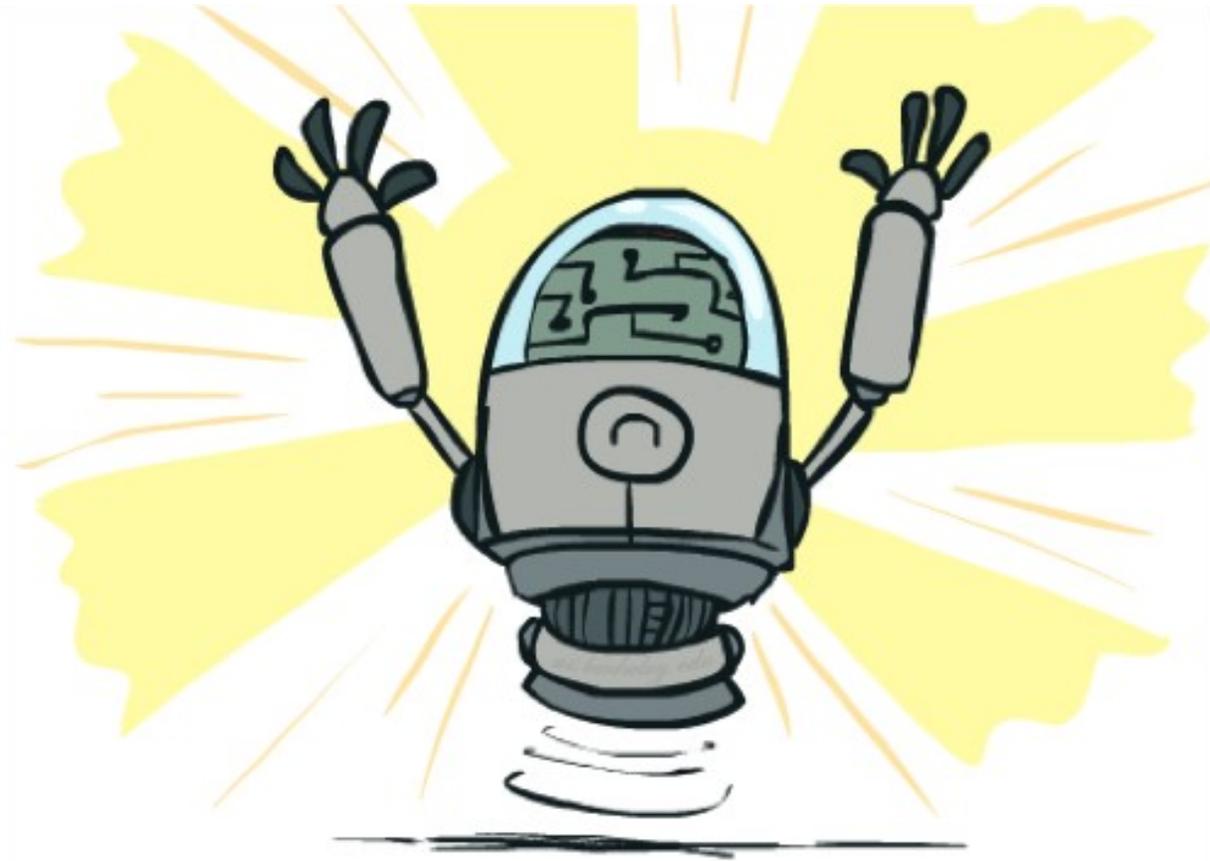
$A$  is explored before  $B$

**A\* search is optimal**



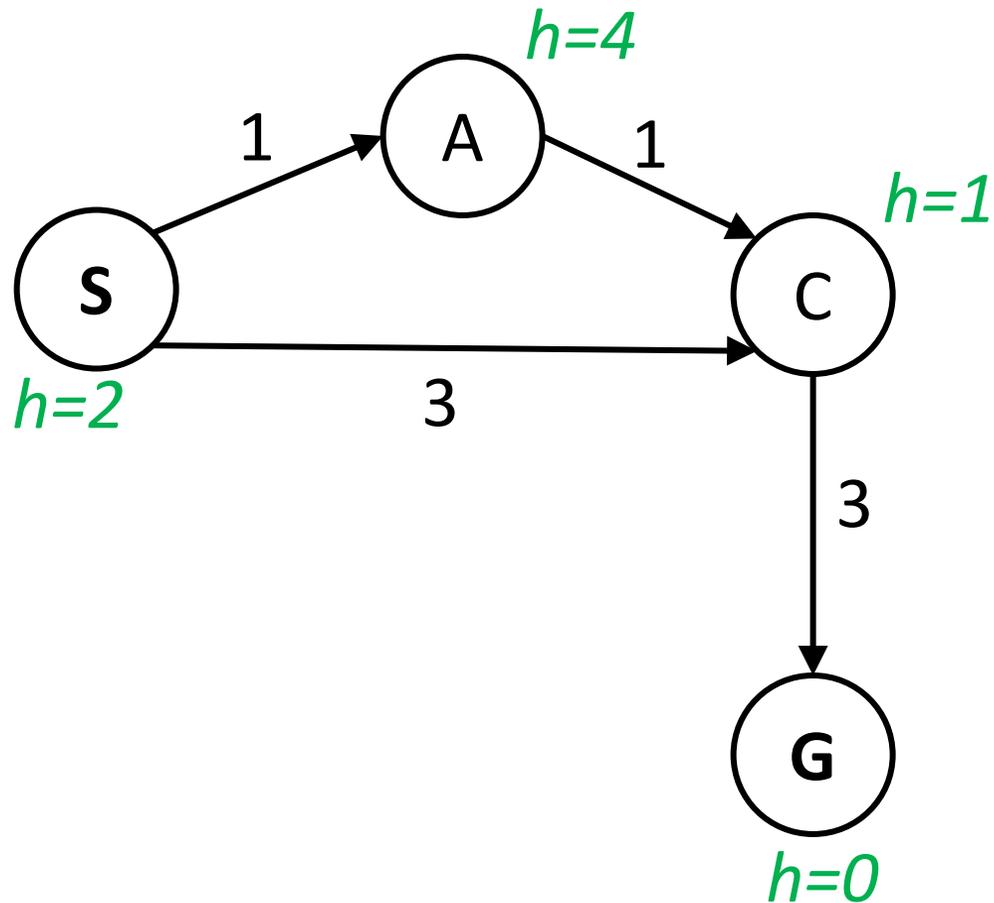
$$f(n) \leq f(A) < f(B)$$

# Optimality of A\* Graph Search



# Poll 1: A\* Graph Search

What nodes does A\* graph search consider during its search?

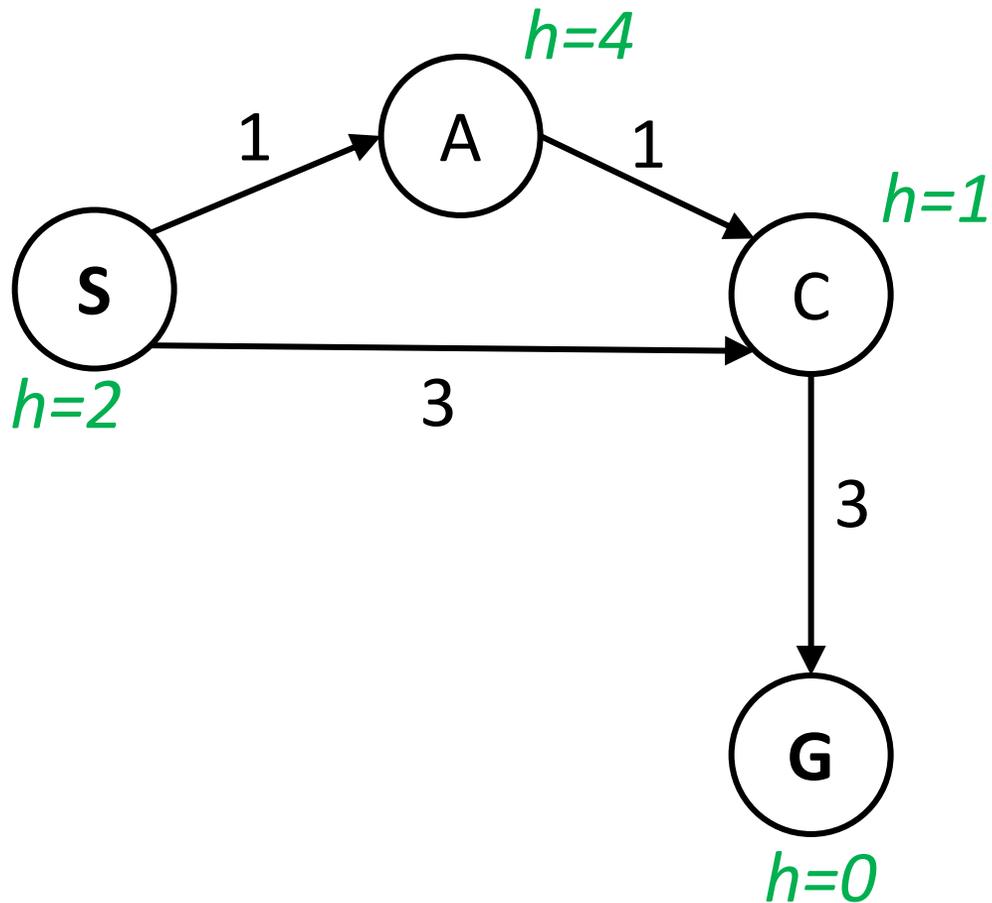


F E  
S+2 S  
S+4 C  
~~S+1~~  
SCG 6+0

- A) ~~S~~, ~~S-A~~, ~~S-C~~, S-C-G
- B) ~~S~~, ~~S-A~~, S-C, ~~S-A-C~~, S-C-G
- C) ~~S~~, ~~S-A~~, ~~S-A-C~~, S-A-C-G
- D) ~~S~~, ~~S-A~~, ~~S-C~~, ~~S-A-C~~, S-A-C-G

# Poll 1: A\* Graph Search

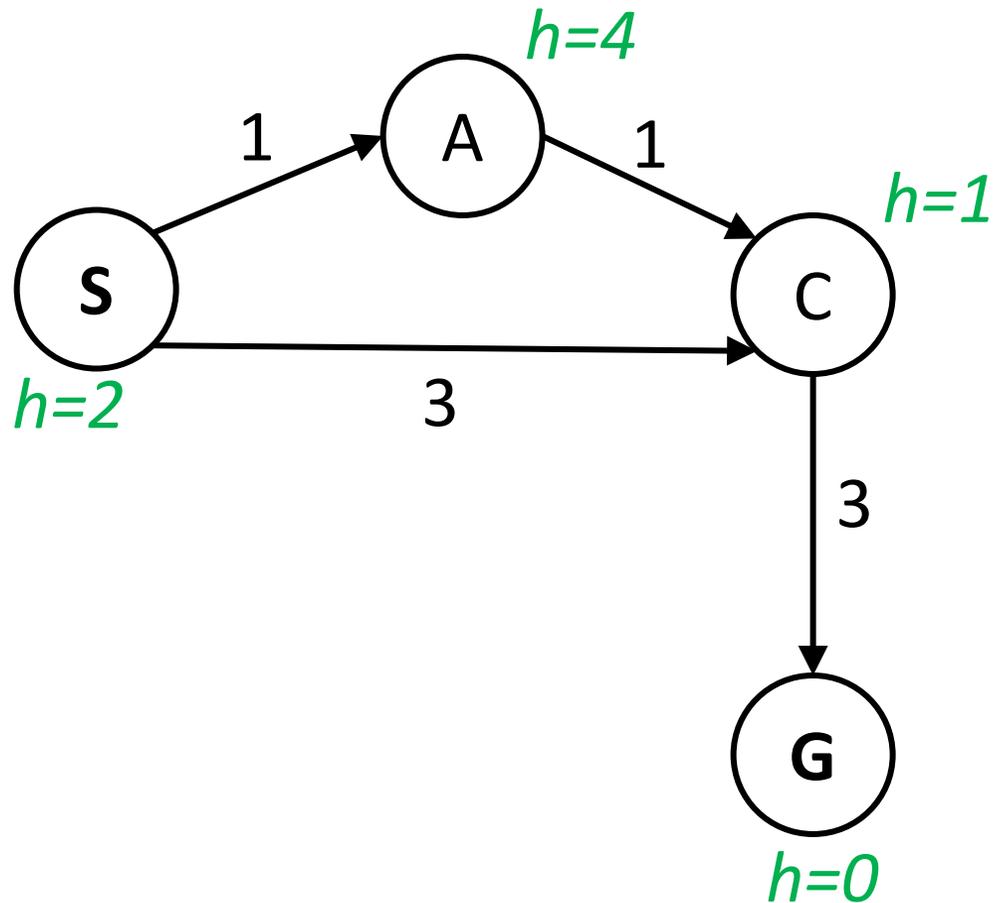
Which paths does A\* graph search consider during its search?



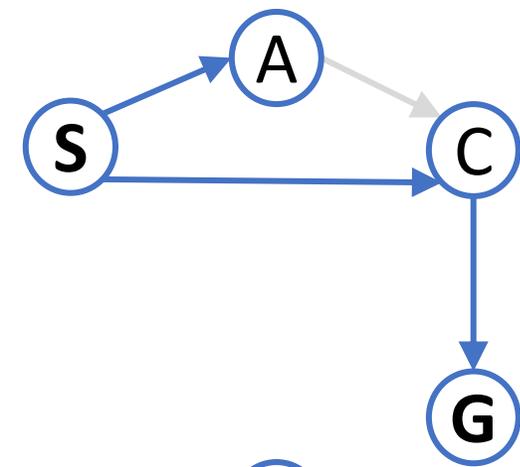
- A) ~~S~~, ~~S-A~~, ~~S-C~~, S-C-G
- B) ~~S~~, ~~S-A~~, S-C, ~~S-A-C~~, S-C-G
- C) ~~S~~, ~~S-A~~, ~~S-A-C~~, S-A-C-G
- D) ~~S~~, ~~S-A~~, ~~S-C~~, ~~S-A-C~~, S-A-C-G

# A\* Graph Search

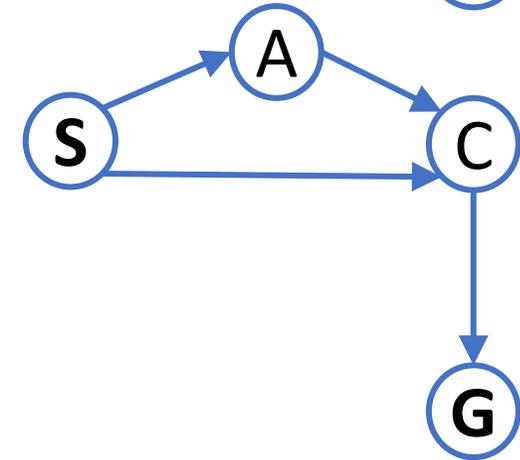
What does the resulting graph search tree look like?



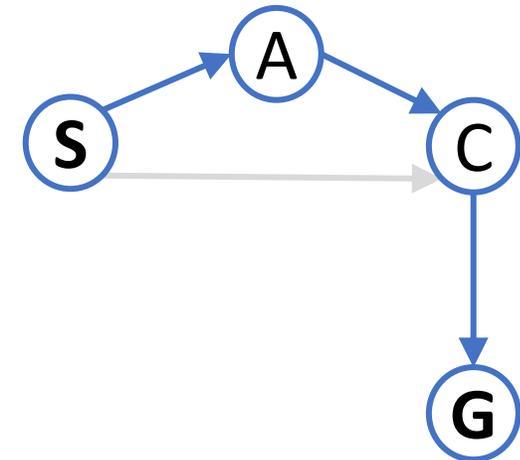
A)



B)

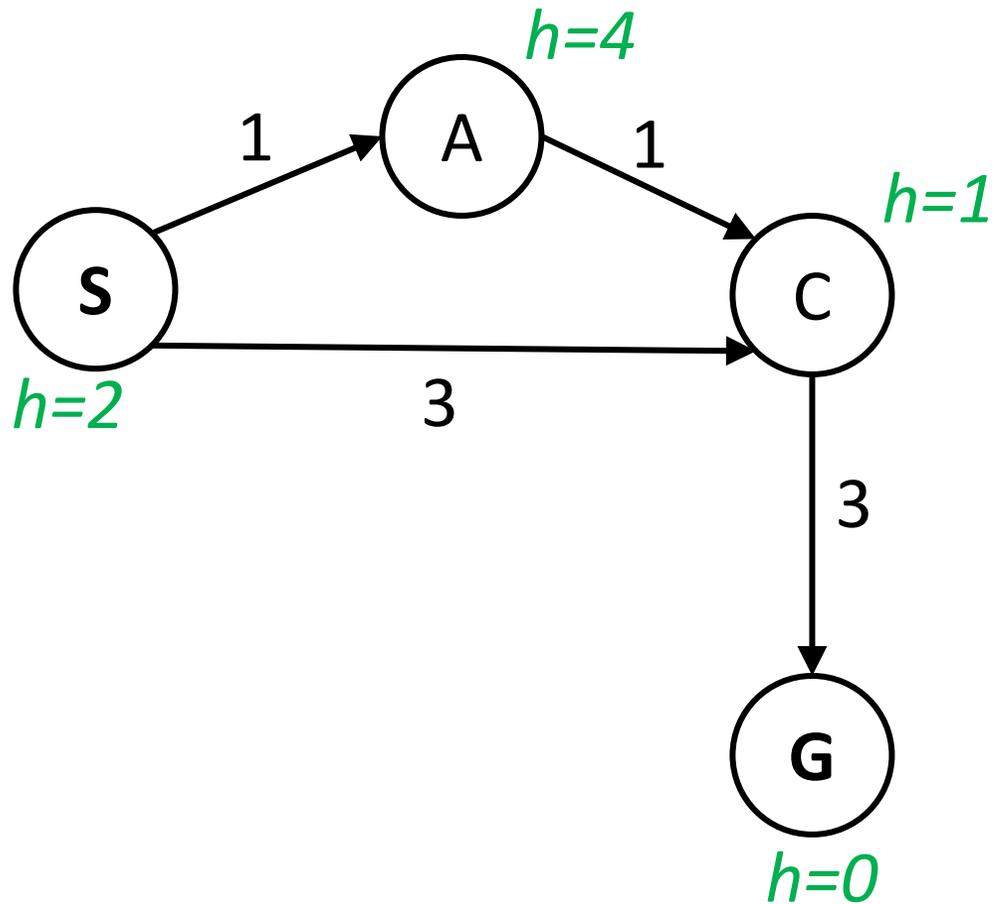


C)

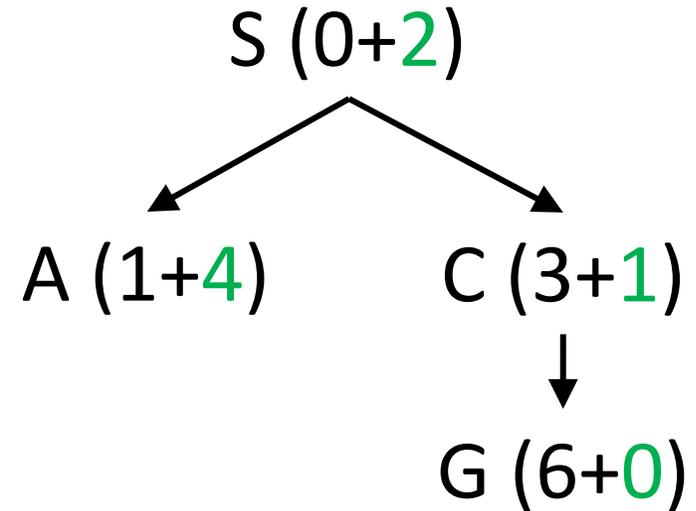


# A\* Graph Search Gone Wrong?

State space graph



Search tree



Simple check against explored set blocks C

Fancy check allows new C if cheaper than old but requires recalculating C's descendants

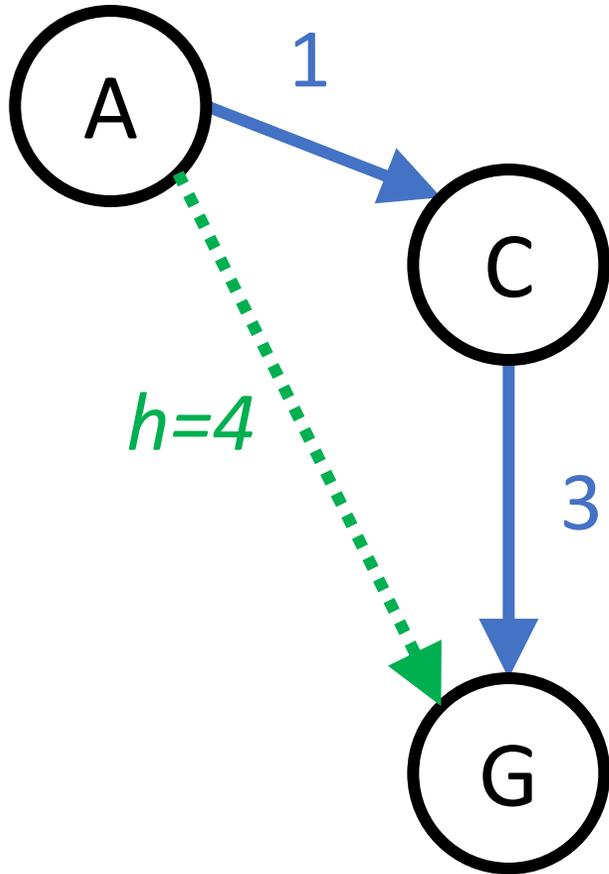
# Admissibility of Heuristics

Main idea: Estimated heuristic values  $\leq$  actual costs

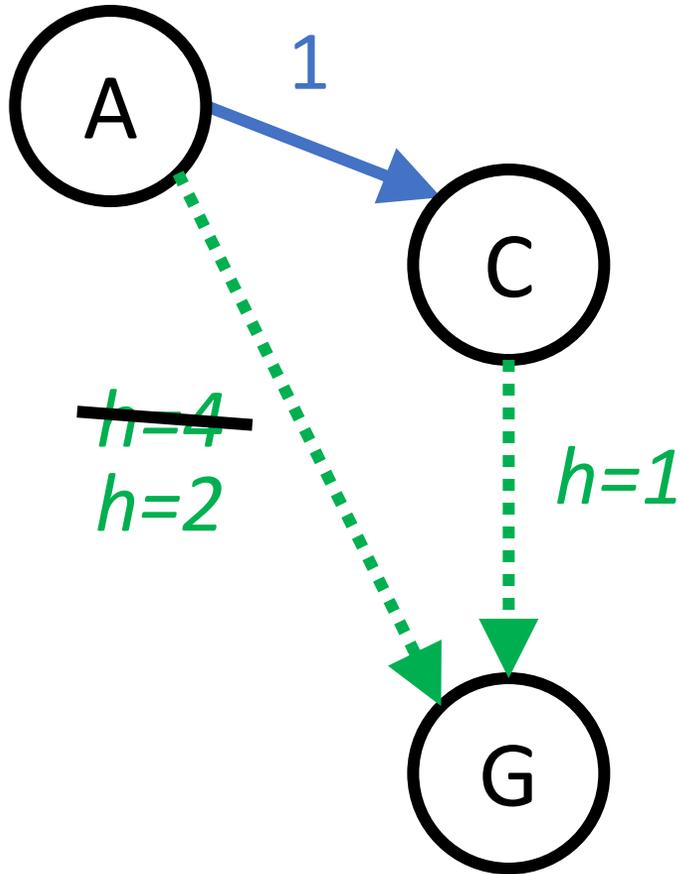
▪ Admissibility:

heuristic value  $\leq$  actual cost to goal

$$h(A) \leq \text{actual cost from A to G}$$



# Consistency of Heuristics



Main idea: Estimated heuristic costs  $\leq$  actual costs

■ Admissibility:

heuristic cost  $\leq$  actual cost to goal

$$h(A) \leq \text{actual cost from A to G}$$

■ Consistency:

“heuristic step cost”  $\leq$  actual cost for each step

$$h(A) - h(C) \leq \text{cost}(A \text{ to } C)$$

triangle inequality

$$h(A) \leq \text{cost}(A \text{ to } C) + h(C)$$

Consequences of consistency:

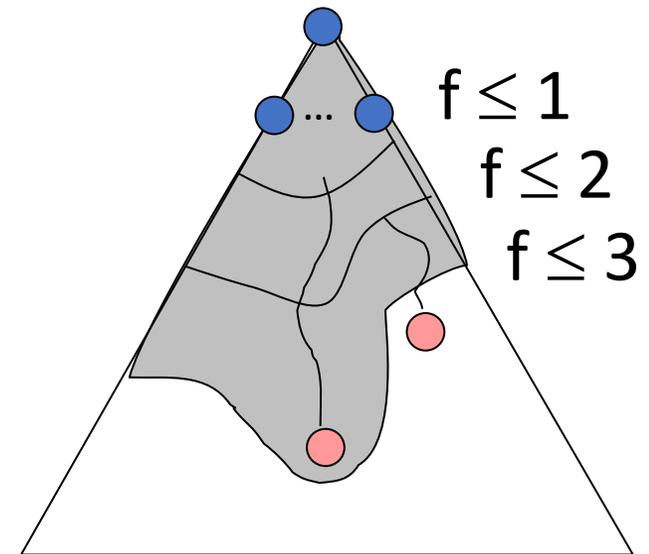
■ The f value along a path never decreases

■ A\* graph search is optimal

# Optimality of A\* Graph Search

Sketch: consider what A\* does with a consistent heuristic:

- Fact 1: In tree search, A\* expands nodes in increasing total **f** value (**f-contours**)
- Fact 2: For every state **s**, nodes that reach **s** optimally are explored before nodes that reach **s** suboptimally
- Result: A\* graph search is optimal



# Optimality

## Tree search:

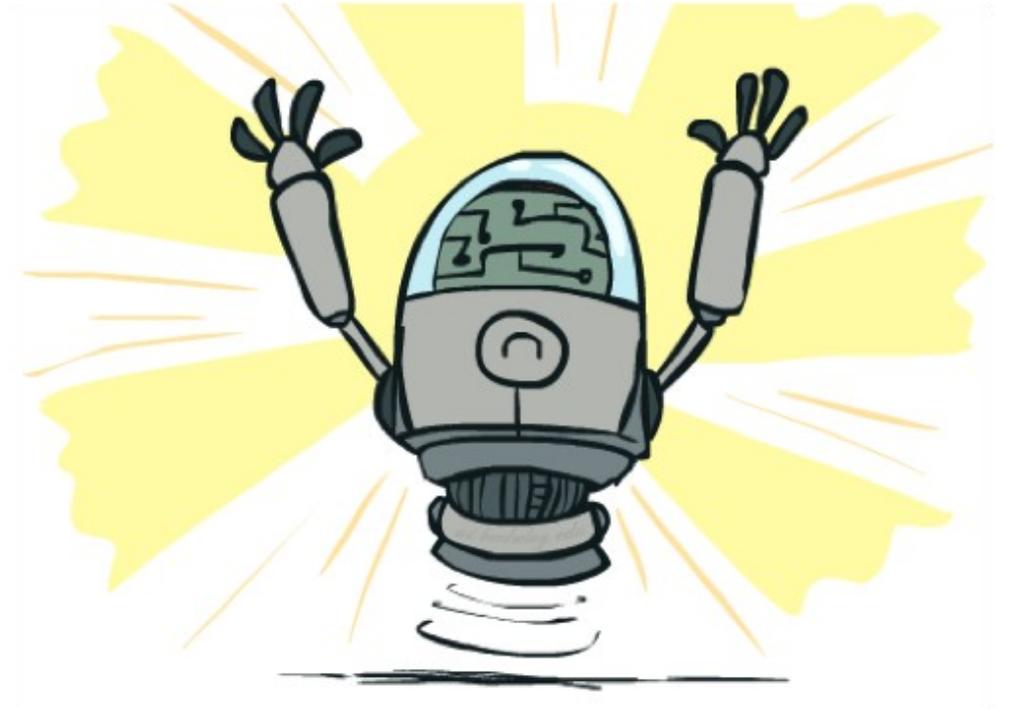
- A\* is optimal if heuristic is admissible
- UCS is a special case ( $h = 0$ )

## Graph search:

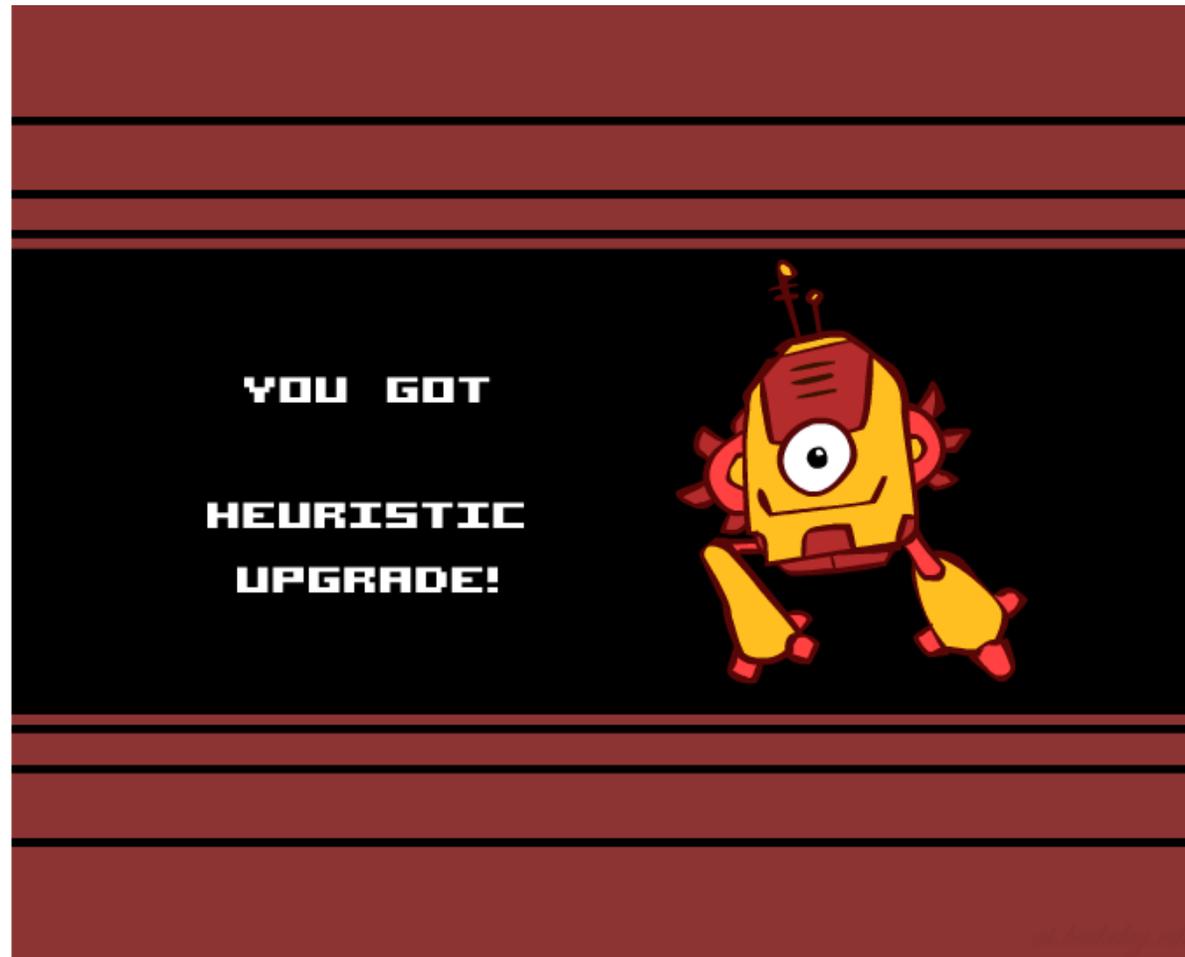
- A\* optimal if heuristic is consistent
- UCS optimal ( $h = 0$  is consistent)

## Consistency implies admissibility

In general, most natural admissible heuristics tend to be consistent, especially if from relaxed problems



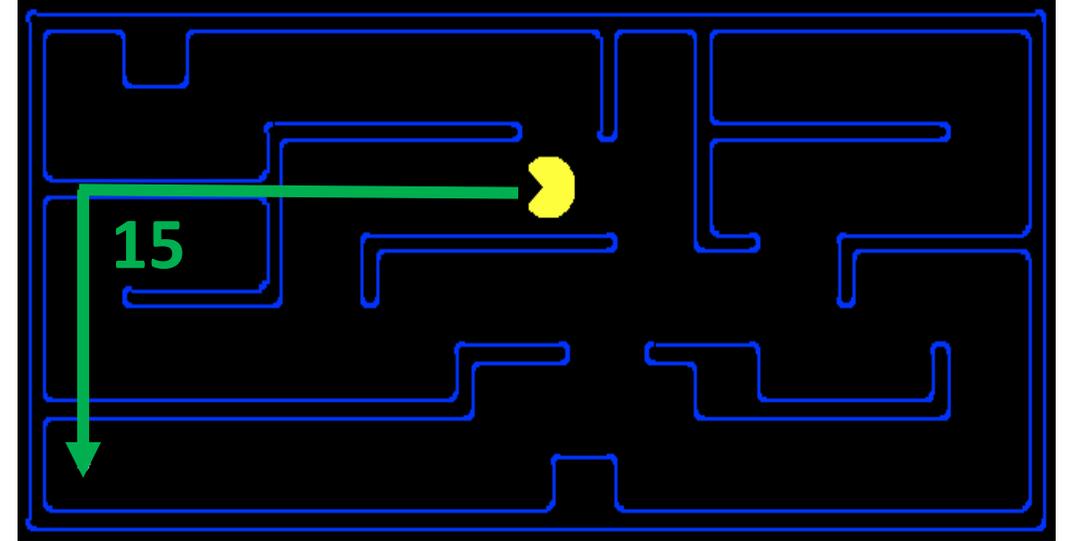
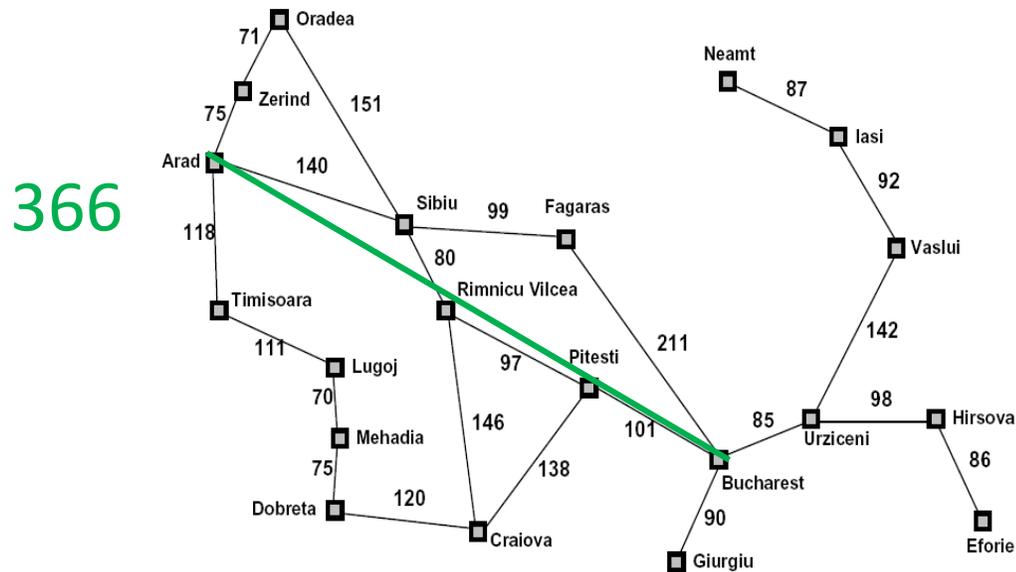
# Creating Heuristics



# Creating Admissible Heuristics

Most of the work in solving hard search problems optimally is in coming up with admissible heuristics

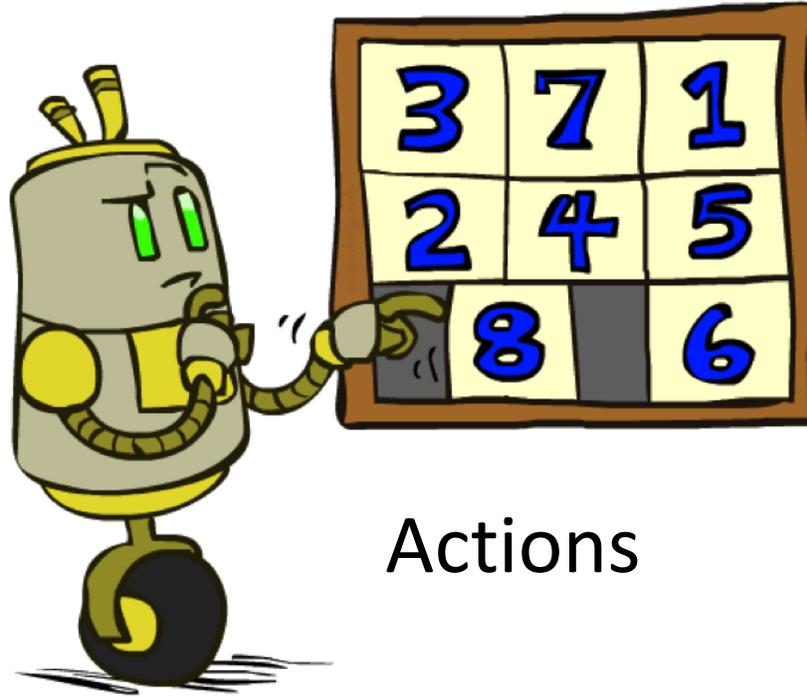
Often, admissible heuristics are solutions to *relaxed problems*, where new actions are available



# Example: 8 Puzzle

7	2	4
5		6
8	3	1

Start State



Actions

	1	2
3	4	5
6	7	8

Goal State

What are the states?

How many states?  $9!$

What are the actions?  $L R U D$

How many actions from the start state?  $4$

What should the step costs be?  $1$

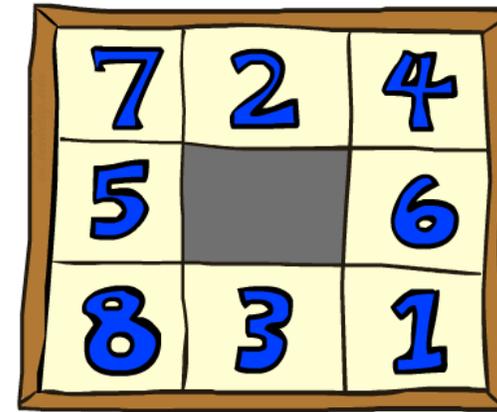
# 8 Puzzle I

Heuristic: Number of tiles misplaced

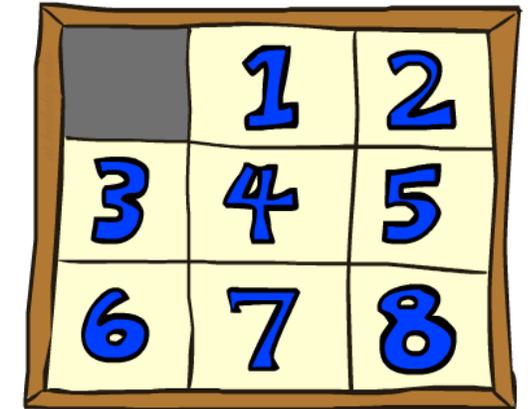
Why is it admissible?

$$h(\text{start}) = 8$$

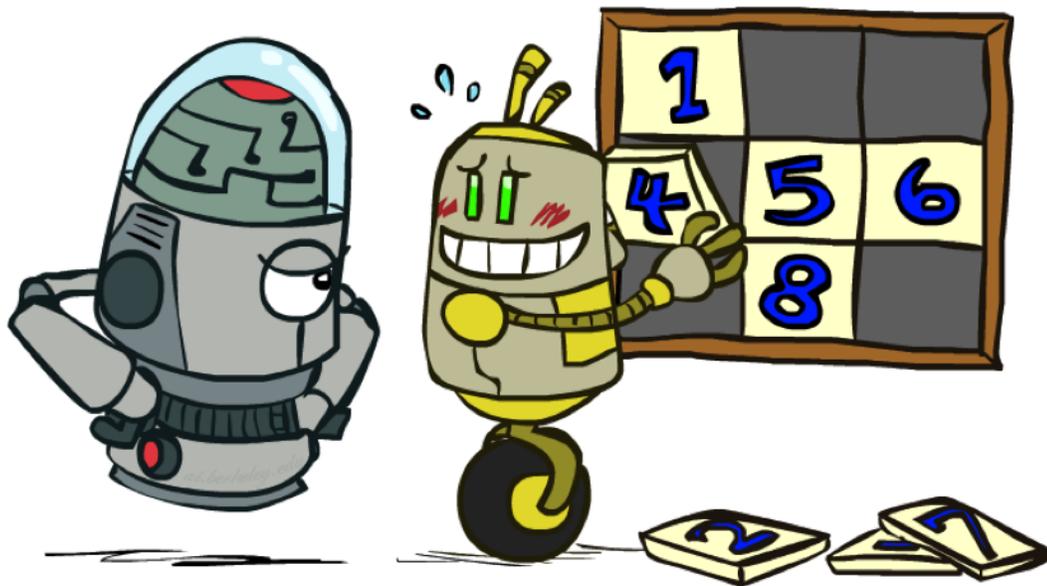
This is a *relaxed-problem* heuristic



Start State



Goal State



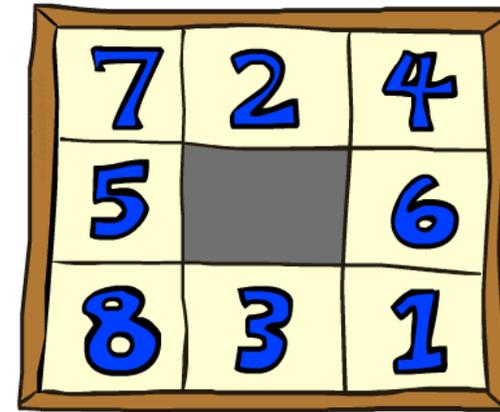
Average nodes expanded when the optimal path has...

	...4 steps	...8 steps	...12 steps
UCS	112	6,300	$3.6 \times 10^6$
A*TILES	13	39	227

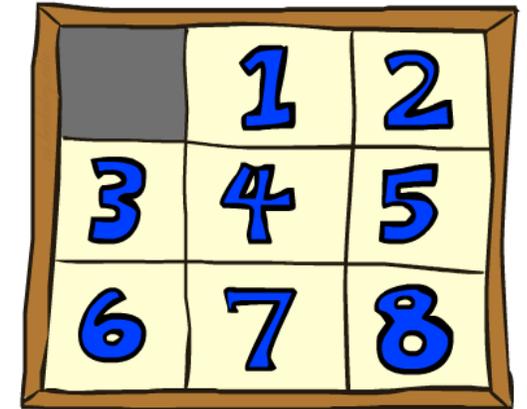
Statistics from Andrew Moore

# 8 Puzzle II

What if we had an easier 8-puzzle where any tile could slide any direction at any time, ignoring other tiles?



Start State



Goal State

Total *Manhattan* distance

Why is it admissible?

$$h(\text{start}) = 3 + 1 + 2 + \dots = 18$$

Average nodes expanded when the optimal path has...			
	...4 steps	...8 steps	...12 steps
A* <sub>TILES</sub>	13	39	227
A* <sub>MANHATTAN</sub>	12	25	73

# Combining heuristics

Dominance:  $h_a \geq h_c$  if

$$\forall n \quad h_a(n) \geq h_c(n)$$

- Roughly speaking, larger is better as long as both are admissible
- The **zero heuristic** is pretty bad (what does A\* do with  $h=0$ ?)
- The **exact heuristic** is pretty good, but usually too expensive!

What if we have two heuristics, neither dominates the other?

- Form a new heuristic by taking the max of both:

$$h(n) = \max( h_a(n), h_b(n) )$$

- Max of admissible heuristics is admissible and dominates both!

# In-Class Activity

Q1: Practice creating heuristics and running Greedy and A\* search

Q2: Walk through Amazon Robot Example

# A\*: Summary



# A\*: Summary

A\* uses both backward costs and (estimates of) forward costs

A\* is optimal with admissible / consistent heuristics

Heuristic design is key: often use relaxed problems

