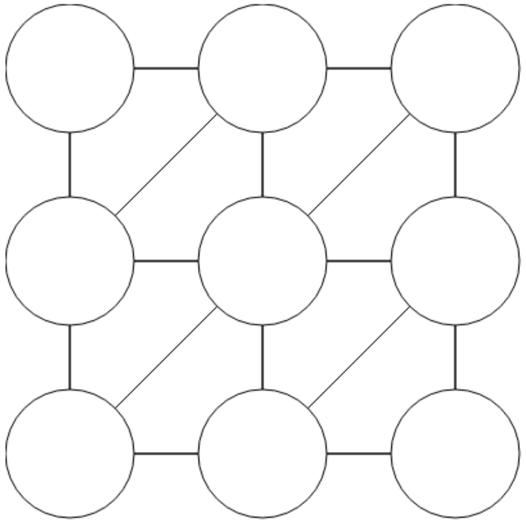
Warm-up as You Walk In (also see activity sheet on website)

Assign Red, Green, or Blue to each node
Neighbors must be different



Sudoku

1			
	2	1	
		3	
			4

1) What is your brain doing to solve these?
2) How would you solve these with search (BFS, DFS, etc.)?

Announcements

Assignments:

- HW2 (written)
  - Due tonight (9/12), 10 pm
- HW3 (online)
  - Out tonight (9/12), due 9/19 at 10 pm
- P1: Search and Games
  - Due Monday (9/18), 10 pm (NOTE THE CLOSE DEADLINES)
  - Recommended to work in pairs
  - Submit to Gradescope early and as often as you like
  - Don’t submit separately; Enter your partner’s name when submitting

Plan

Last Time

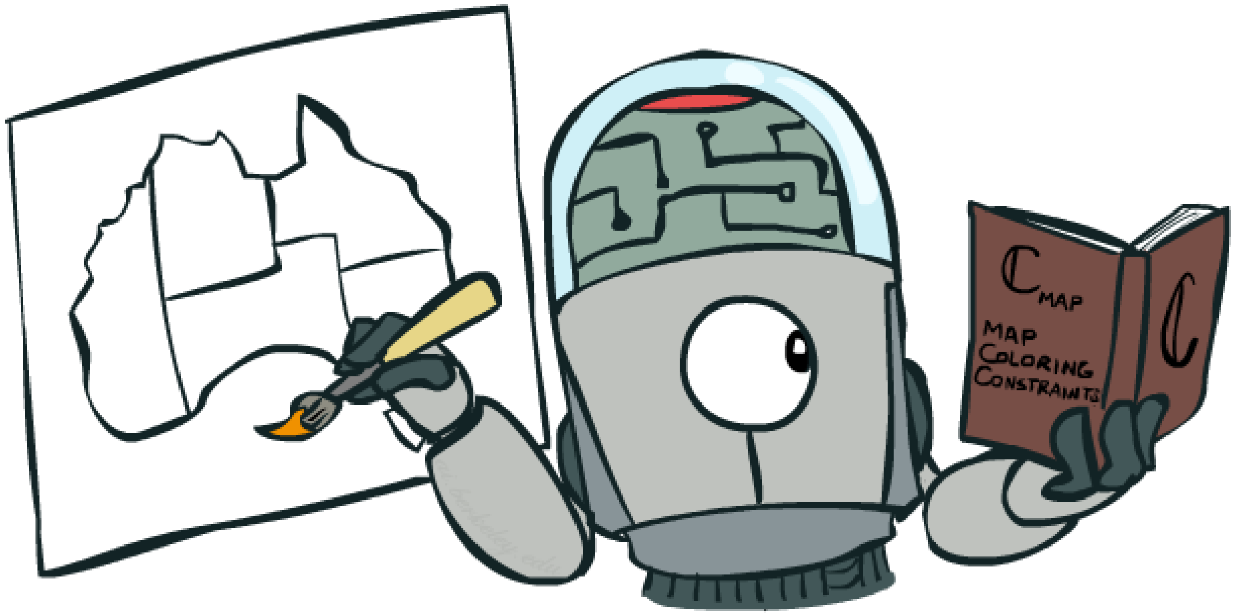
- Adversarial search
  - Minimax
  - Evaluation functions
  - Pruning
  - Expectimax (actually no, didn’t finish that, we’ll quickly do this now)

Today

- Constraint Satisfaction Problems

AI: Representation and Problem Solving

Constraint Satisfaction Problems (CSPs)

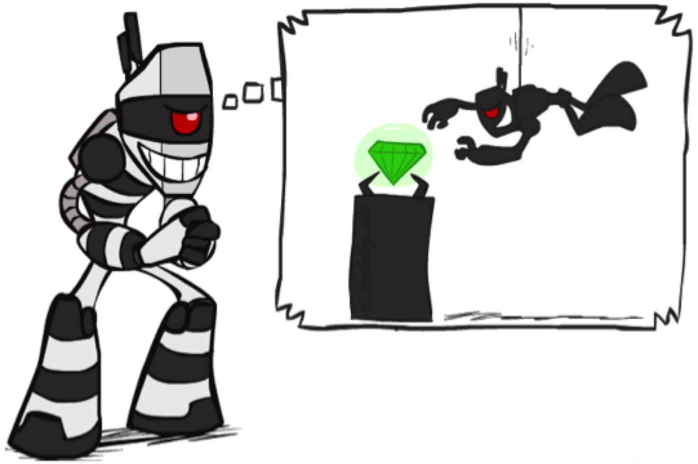


Instructor: Vincent Conitzer and Aditi Raghunathan

Slide credits: CMU AI, http://ai.berkeley.edu

What is Search For?

- Planning: sequences of actions
  - The path to the goal is the important thing
  - Paths have various costs, depths
  - Heuristics give problem-specific guidance



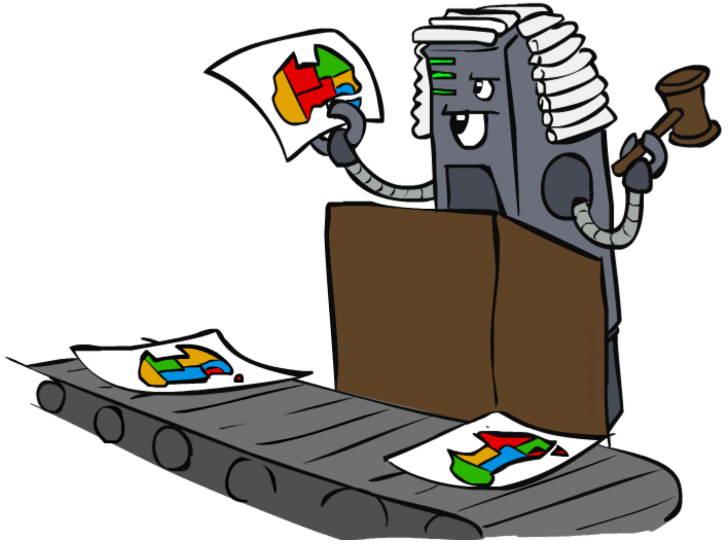
- Identification: assignments to variables
  - The goal itself is important, not the path
  - All paths at the same depth (for some formulations)

Are the warm-up assignments (i.e., sudoku) planning or identification problems?

Constraint Satisfaction Problems

CSP is a special class of search problems

- Mostly identification problems
- Have specialized algorithms for them

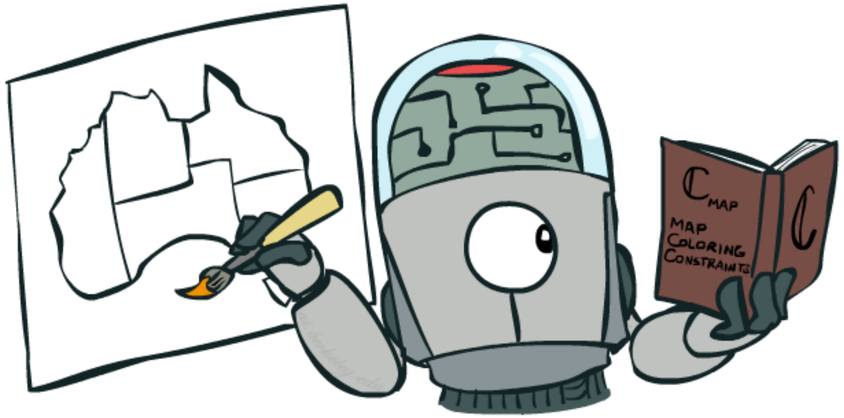


Standard search problems:

- State is an arbitrary data structure
- Goal test can be any function over states

Constraint satisfaction problems (CSPs):

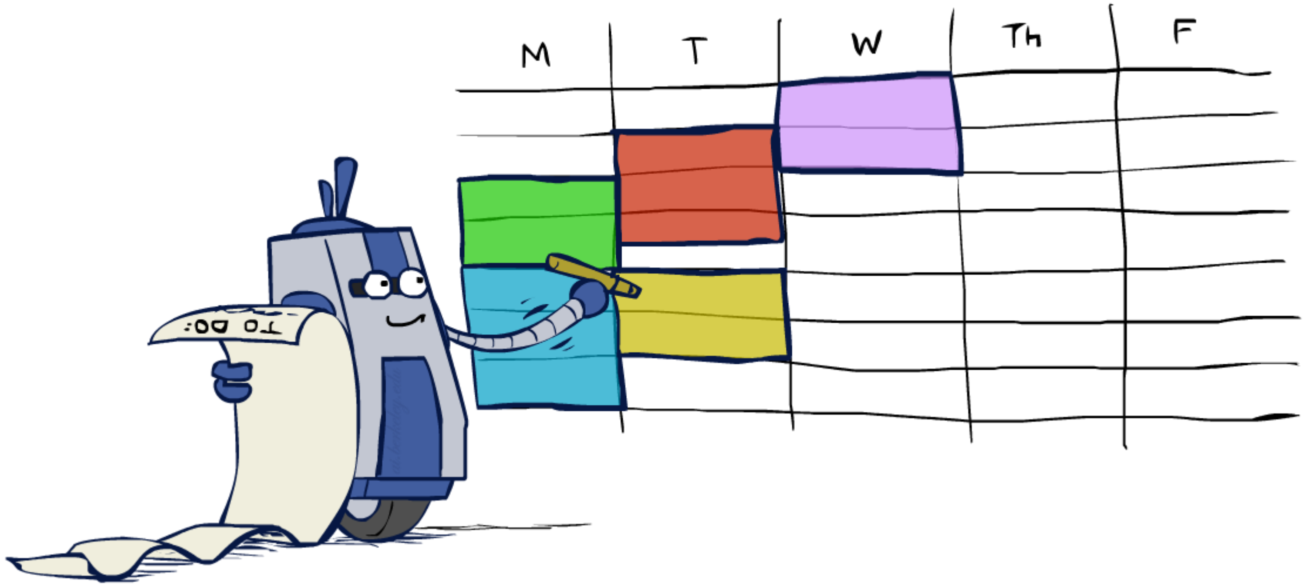
- State is defined by variables X_i with values from a domain D (sometimes D depends on i)
- Goal test is a set of constraints specifying allowable combinations of values for subsets of variables



Why study CSPs?

Many real-world problems can be formulated as CSPs

- Assignment problems: e.g., who teaches what class
- Timetabling problems: e.g., which class is offered when and where?
- Hardware configuration
- Transportation scheduling
- Factory scheduling
- Circuit layout
- Fault diagnosis
- … lots more!



- Sometimes involve real-valued variables…

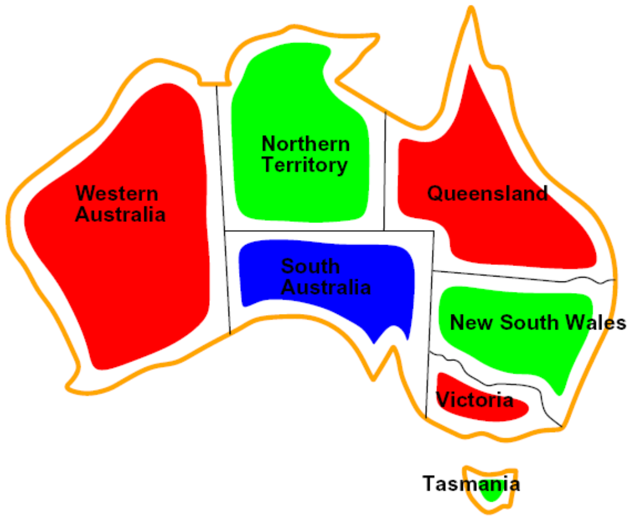
Varieties of CSPs and Constraints

Example: Map Coloring

- Variables: $\mathrm{WA,\ NT,\ Q,\ NSW,\ V,\ SA,\ T}$
- Domains: $D = \{\text{red}, \text{green}, \text{blue}\}$
- Constraints: adjacent regions must have different colors

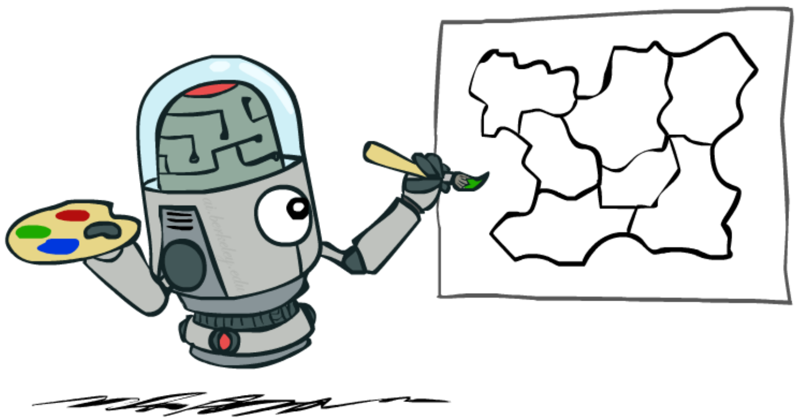
  Implicit: $\mathrm{WA} \neq \mathrm{NT}$

  Explicit: $(\mathrm{WA}, \mathrm{NT}) \in \{(\text{red}, \text{green}), (\text{red}, \text{blue}), \ldots\}$

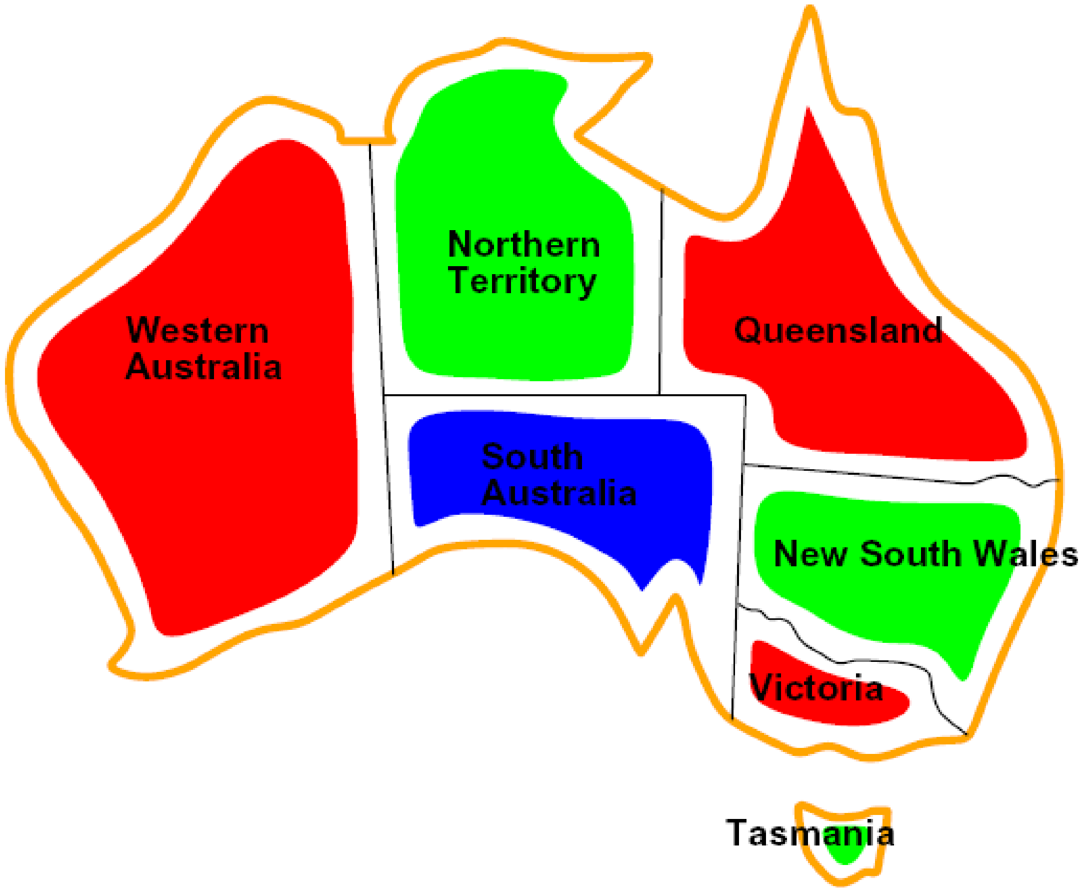


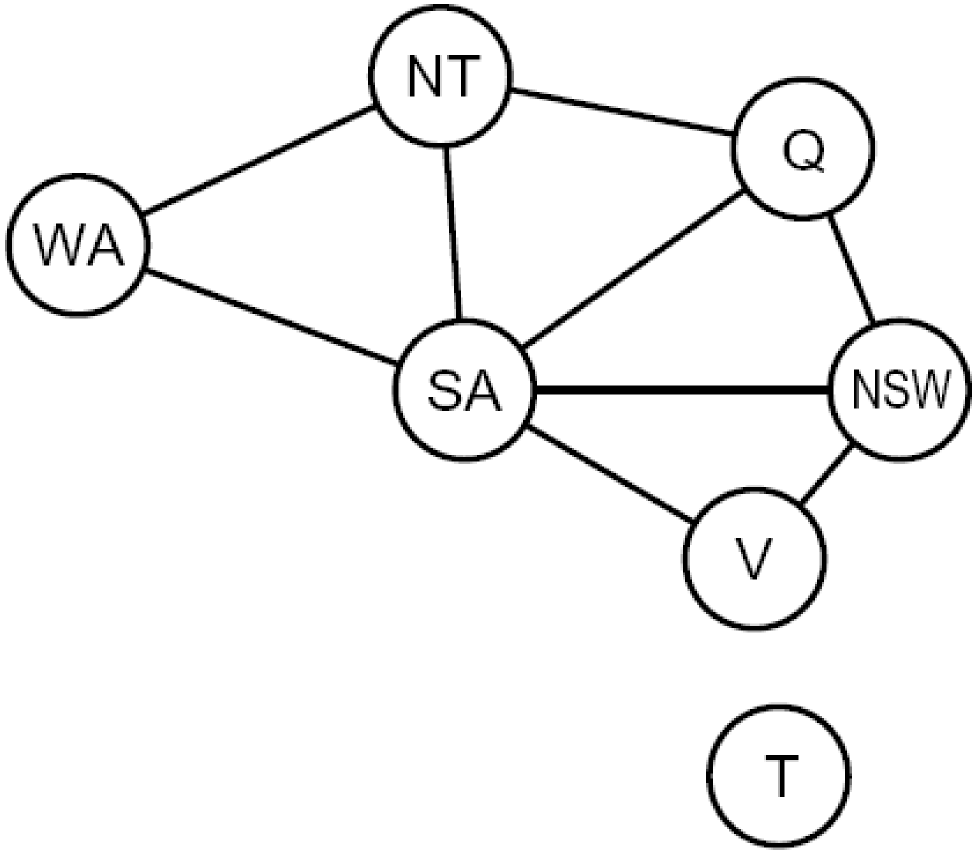
- Solutions are assignments satisfying all constraints, e.g.:

  $\{\mathrm{WA{=}red,\ NT{=}green,\ Q{=}red,\ NSW{=}green,\ V{=}red,\ SA{=}blue,\ T{=}green}\}$



Constraint Graphs



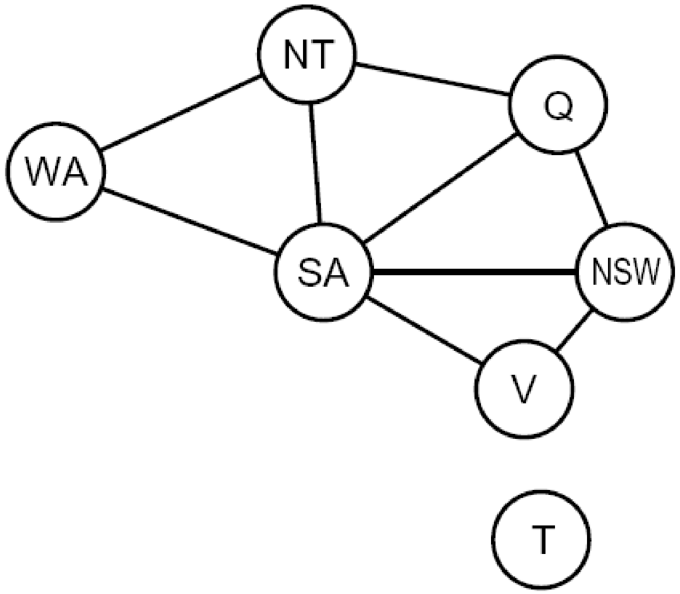


Constraint Graphs

- Binary CSP: each constraint relates (at most) two variables

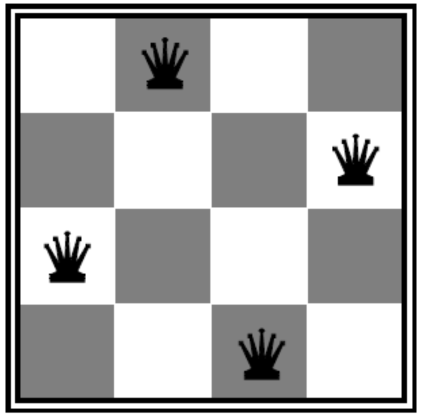
- Binary constraint graph: nodes are variables, arcs show constraints

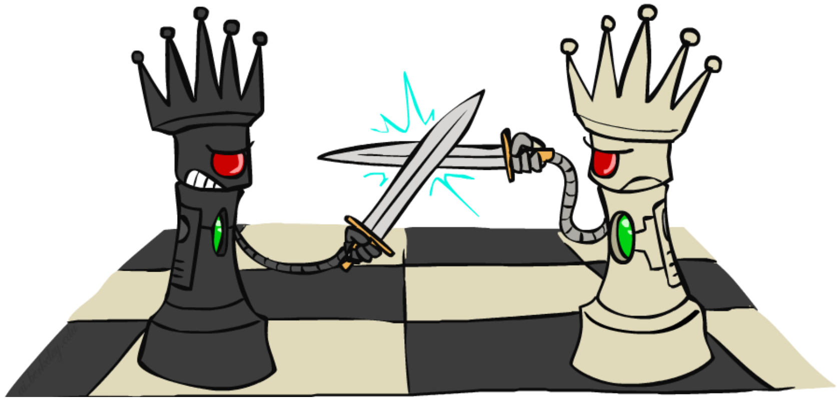
- General-purpose CSP algorithms use the graph structure to speed up search. E.g., Tasmania is an independent subproblem!



Example: N-Queens

- Formulation 1:
  - Variables: X_{ij}
  - Domains: $\{0, 1\}$
  - Constraints



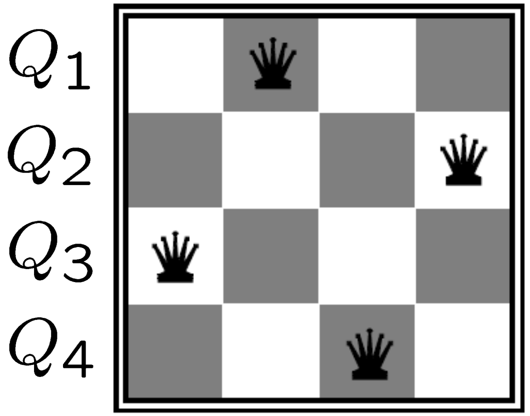


$$\forall i, j, k \;\; (X_{ij}, X_{ik}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i, j, k \;\; (X_{ij}, X_{kj}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i, j, k \;\; (X_{ij}, X_{i+k,j+k}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i, j, k \;\; (X_{ij}, X_{i+k,j-k}) \in \{(0,0),(0,1),(1,0)\}$$

$$\sum_{i,j} X_{ij} = N$$

Example: N-Queens

- Formulation 2:
  - Variables: Q_k
  - Domains: $\{1, 2, 3, \ldots N\}$
  - Constraints:



Implicit: $\forall i, j \;\; \text{non-threatening}(Q_i, Q_j)$

Explicit: $(Q_1, Q_2) \in \{(1,3), (1,4), \ldots\}$

$\cdots$

Example: Cryptarithmetic

- Variables:

$$F \ T \ U \ W \ R \ O \ X_1 \ X_2 \ X_3$$

- Domains:

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

- Constraints:

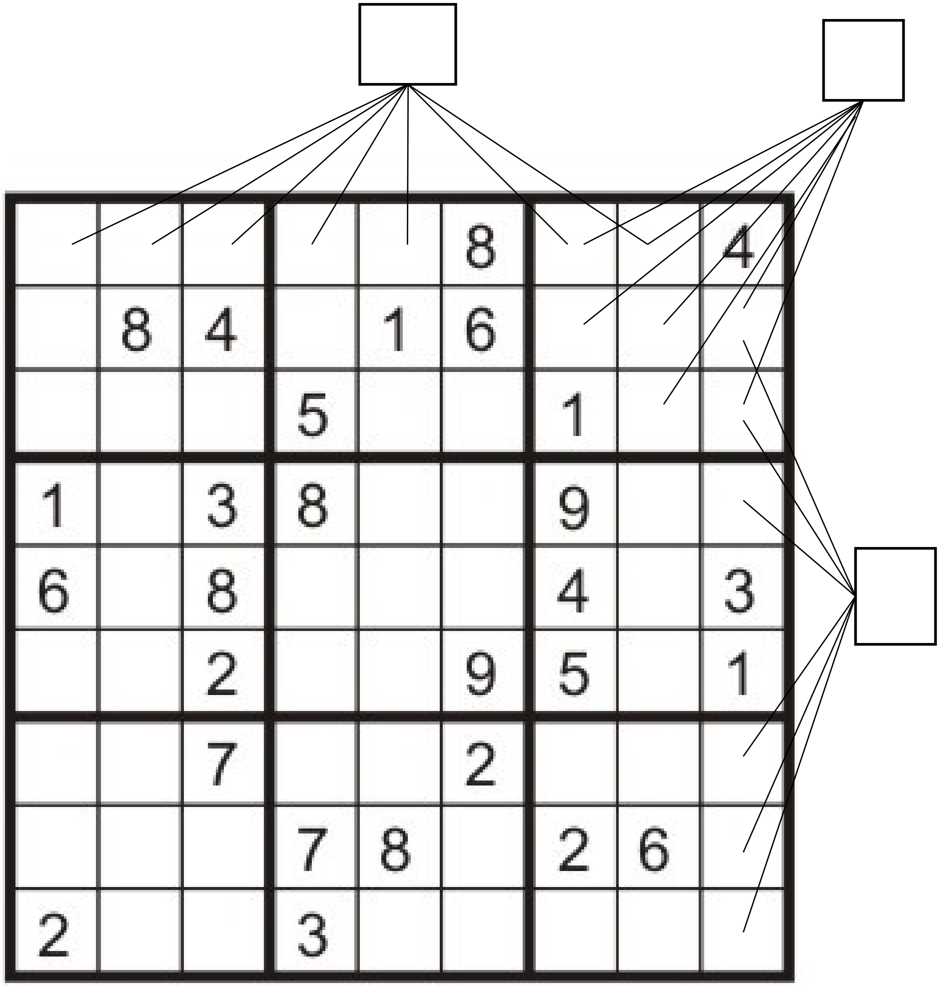
$$\text{alldiff}(F, T, U, W, R, O)$$

$$O + O = R + 10 \cdot X_1$$

$$\cdots$$

$$\begin{array}{cccc} & T & W & O \\ + & T & W & O \\ \hline F & O & U & R \end{array}$$

Example: Sudoku



- Variables: Each (open) square
- Domains: {1,2,…,9}
- Constraints:

  9-way alldiff for each column

  9-way alldiff for each row

  9-way alldiff for each region

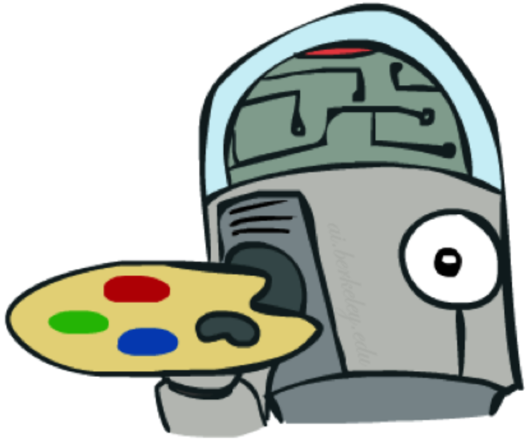
  (or can have a bunch
  of pairwise inequality
  constraints)

Varieties of CSPs

- Discrete Variables

We will cover today

  - Finite domains
    - Size d means $O(d^n)$ complete assignments
    - E.g., Boolean CSPs, including Boolean satisfiability (NP-complete)
  - Infinite domains (integers, strings, etc.)
    - E.g., job scheduling, variables are start/end times for each job
    - Linear constraints solvable, nonlinear undecidable

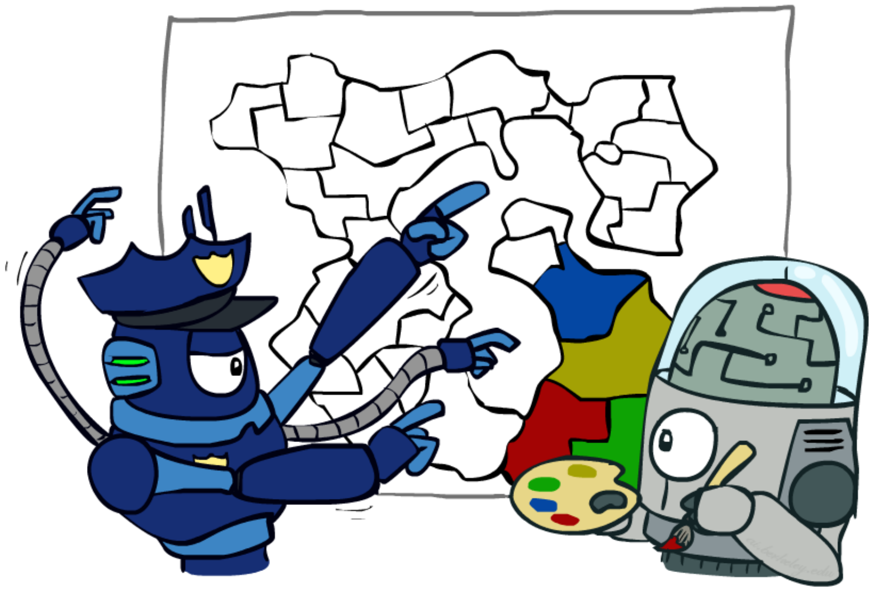


We will cover in a later lecture (linear programming)

- Continuous variables
  - E.g., start/end times for Hubble Telescope observations
  - Linear constraints solvable in polynomial time

Varieties of Constraints

- Varieties of Constraints
  - Unary constraints involve a single variable (equivalent to reducing domains), e.g.:
    $$\text{SA} \neq \text{green}$$
    Focus of today
  - Binary constraints involve pairs of variables, e.g.:
    $$\text{SA} \neq \text{WA}$$
  - Higher-order constraints involve 3 or more variables:
    e.g., cryptarithmetic column constraints
    $$O + O = R + 10 \cdot X_1$$
- Preferences (soft constraints):
  - E.g., red is better than green
  - Often representable by a cost for each variable assignment
  - Gives constrained optimization problems



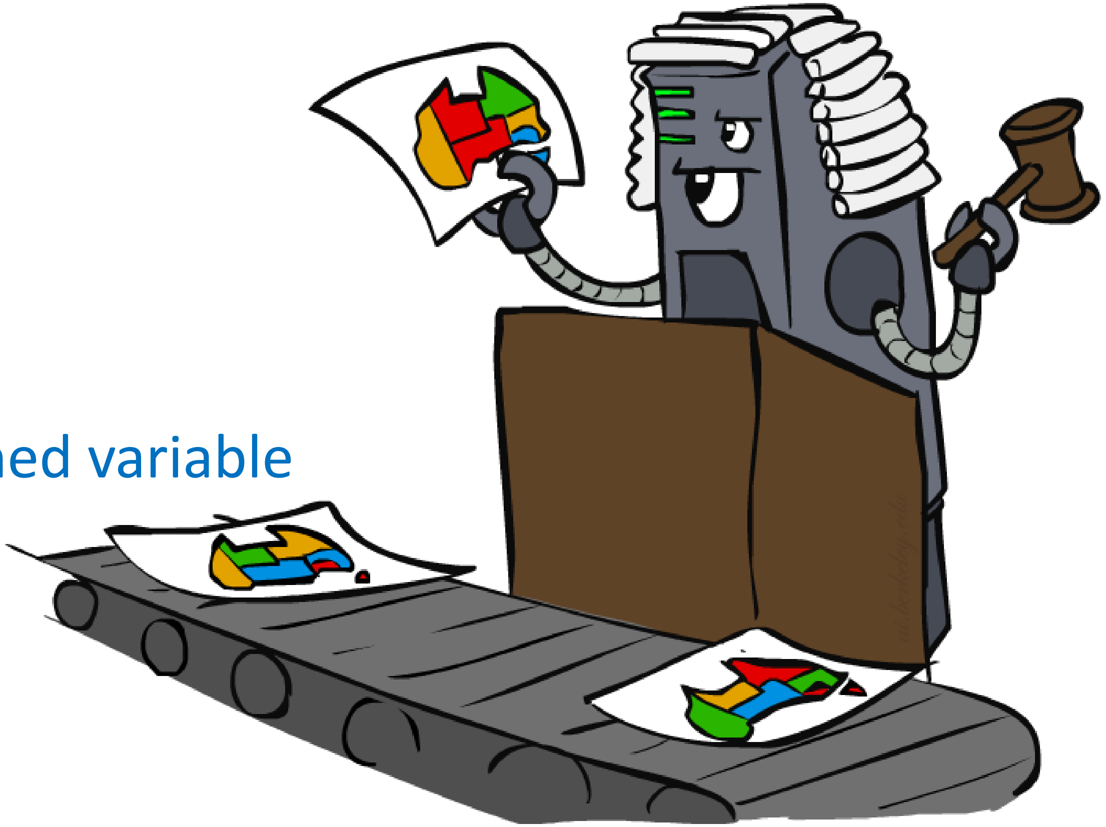
Solving CSPs

Standard Search Formulation

- Standard search formulation of CSPs

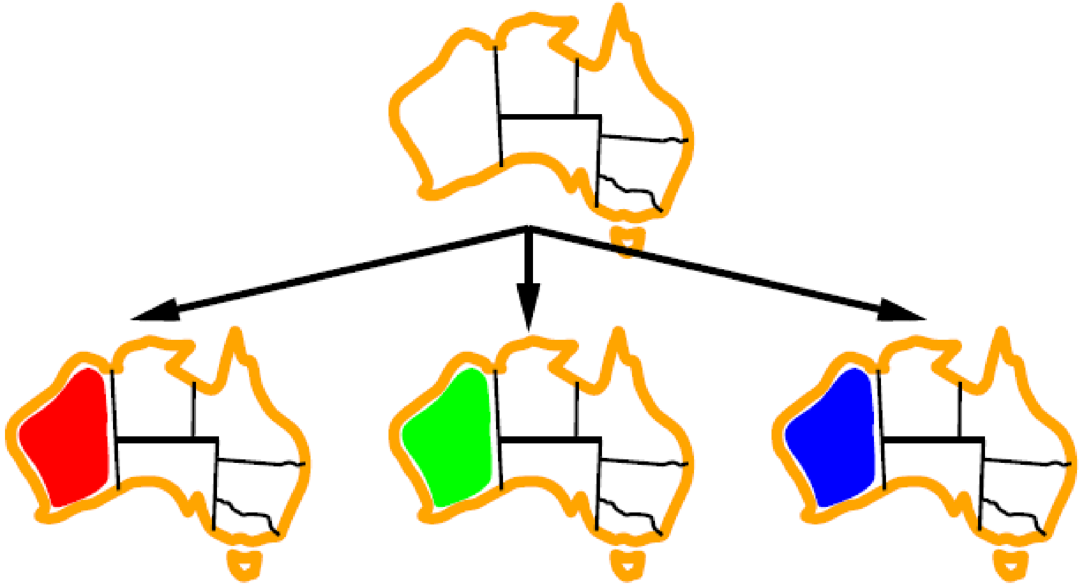
- States defined by the values assigned so far (partial assignments)
  - Initial state: the empty assignment, {}
  - Successor function: assign a value to an unassigned variable →Can be any unassigned variable
  - Goal test: the current assignment is complete and satisfies all constraints

- We'll start with the straightforward, naïve approach, then improve it



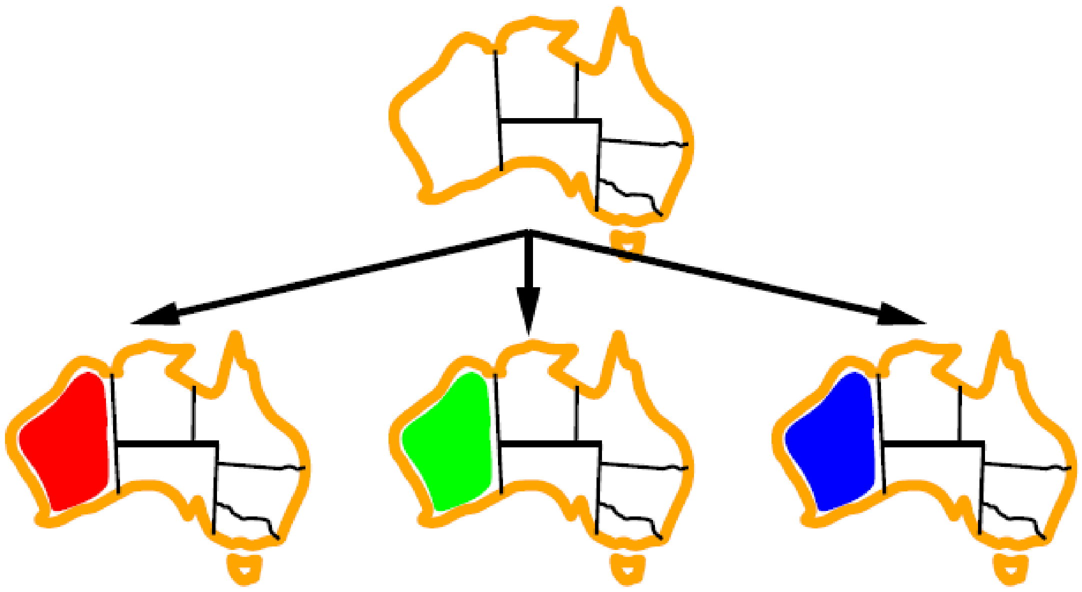
Question: Search for CSPs

Should we use BFS or DFS?

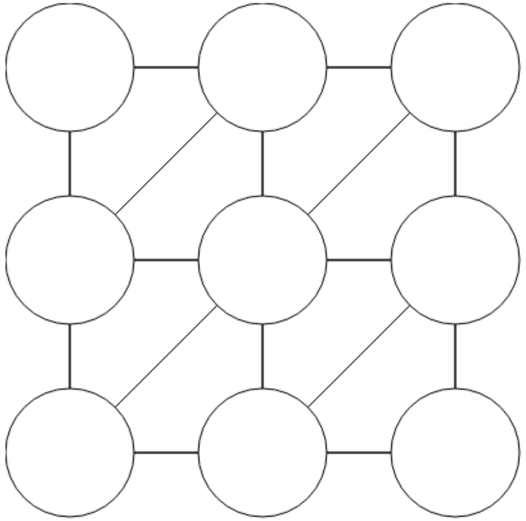


Depth First Search

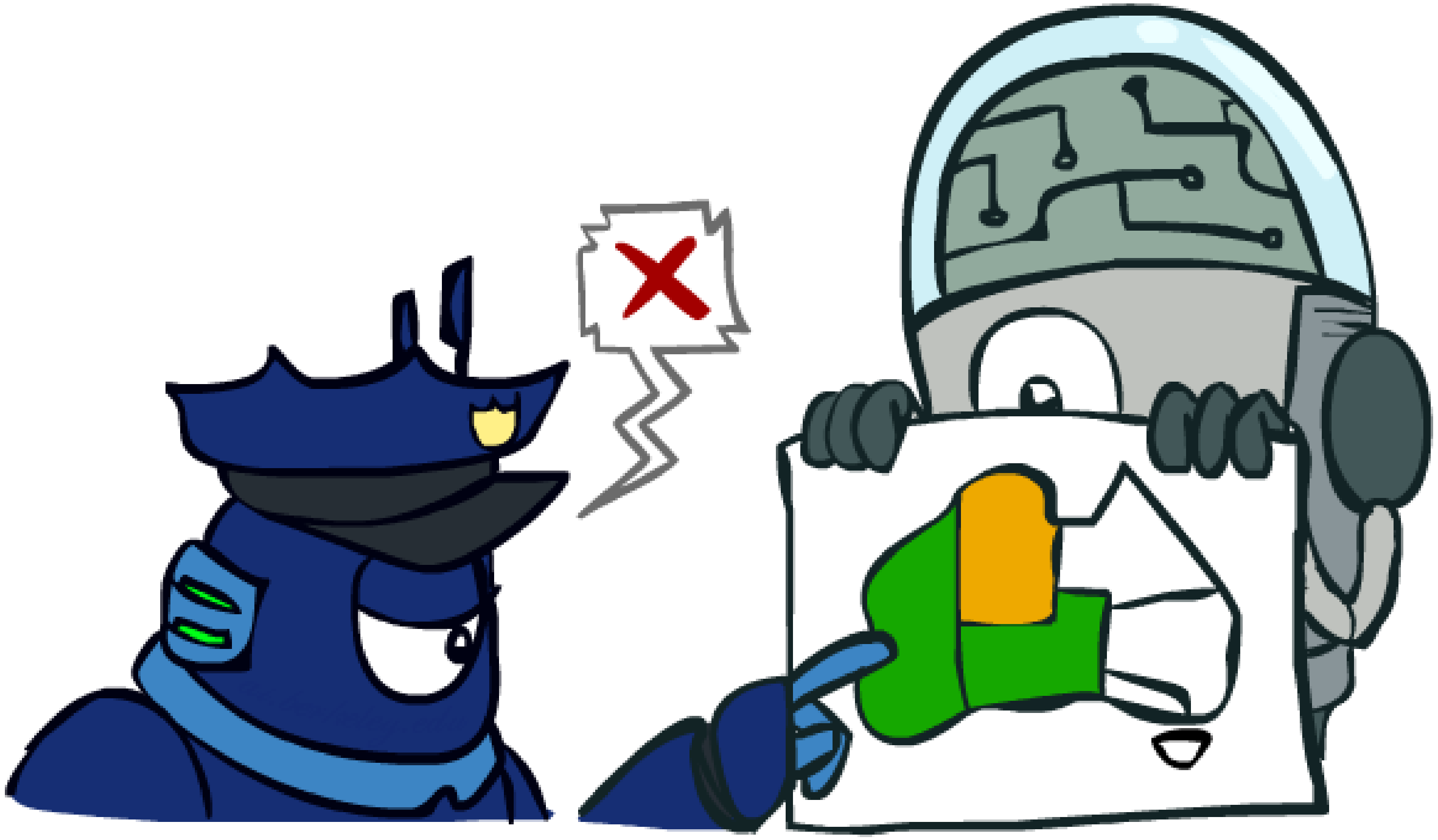
- At each node, assign a value from the domain to the variable
- Check feasibility (constraints) when the assignment is complete



Demo – Naïve Search



Backtracking Search



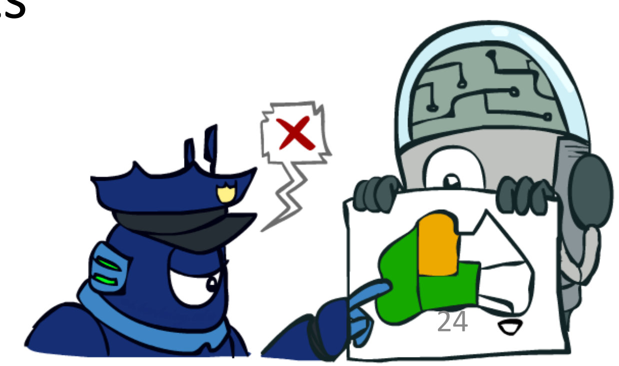
Backtracking Search

- Backtracking search is the basic uninformed algorithm for solving CSPs
- Backtracking search = DFS + two improvements

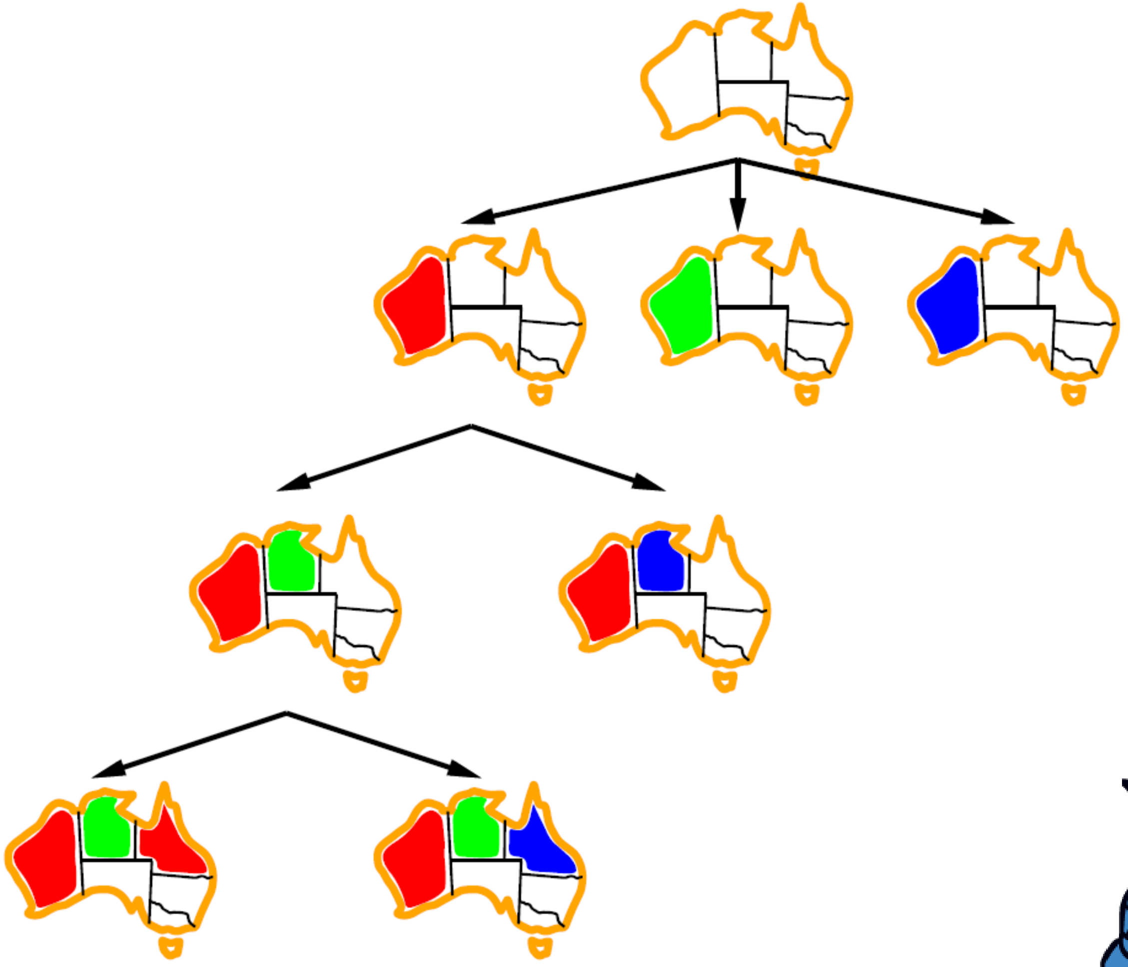
- Idea 1: One variable at a time
  - Variable assignments are commutative
    - [WA = red then NT = green] same as [NT = green then WA = red]
  - Only need to consider assign value to a single variable at each step

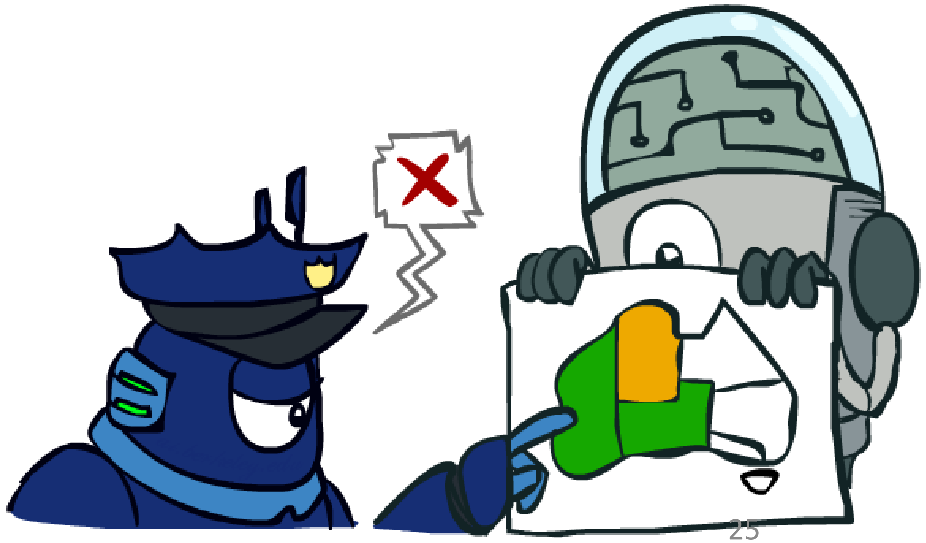
- Idea 2: Check constraints as you go
  - Consider only values which do not conflict previous assignments
  - May need some computation to check the constraints
  - "Incremental goal test"

- Can solve n-queens for n $\approx$ 25



Backtracking Example





Backtracking Search

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
   return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment given CONSTRAINTS[csp] then
             add {var = value} to assignment
             result ← RECURSIVE-BACKTRACKING(assignment, csp)
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No need to check constraints for a complete assignment

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```

Checks consistency at each assignment

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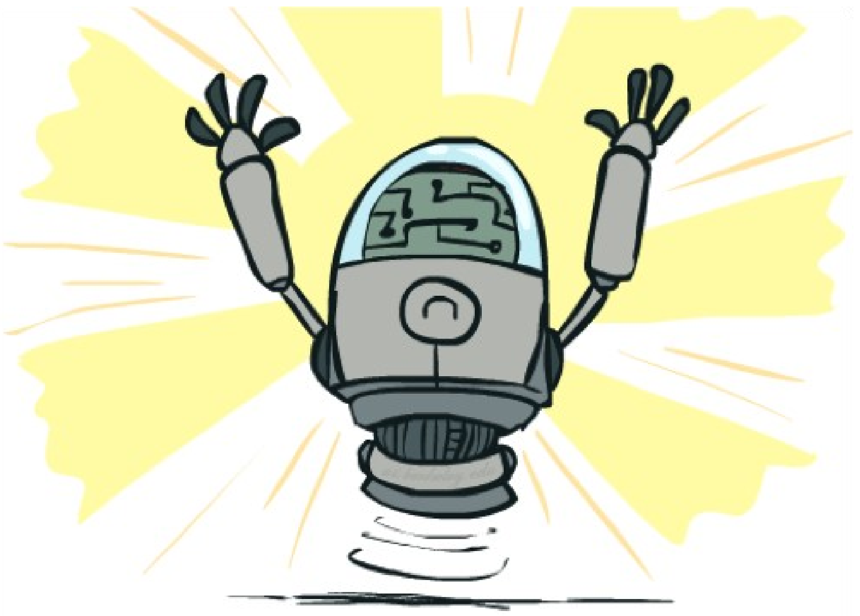
```

- Backtracking = DFS + variable-ordering + fail-on-violation
- What are the decision points?

Improving Backtracking

- General-purpose ideas give huge gains in speed
- Filtering: Can we detect inevitable failure early?
  Today
- Ordering:
  - Which variable should be assigned next?
  - In what order should its values be tried?

  Thursday
- Structure: Can we exploit the problem structure?
  Not going to cover!



Filtering

Filtering: Forward Checking

Filtering: Keep track of domains for unassigned variables and cross off bad options

Forward checking: A simple way for filtering

- After a variable is assigned a value, check related constraints and cross off values of unassigned variables which violate the constraints
- Failure detected if some variables have no values remaining

Filtering: Forward Checking

- Filtering: Keep track of domains for unassigned variables and cross off bad options
- Forward checking: A simple way for filtering
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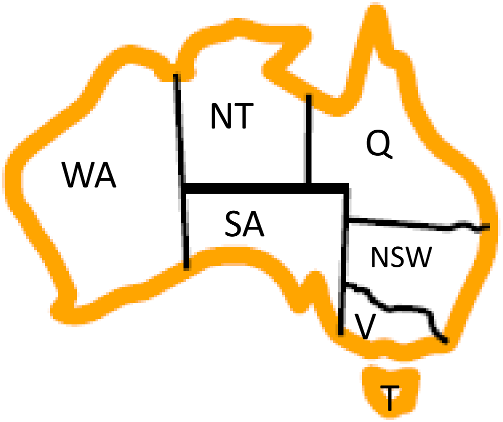

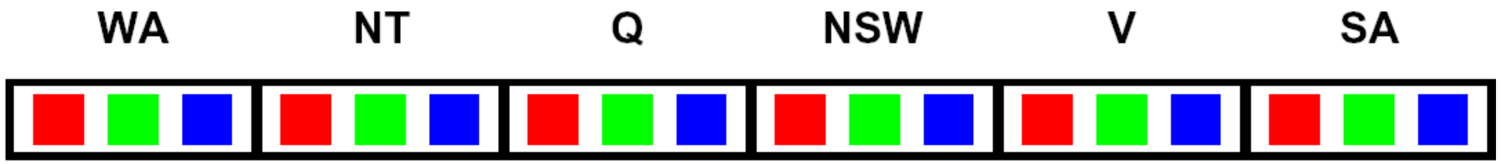

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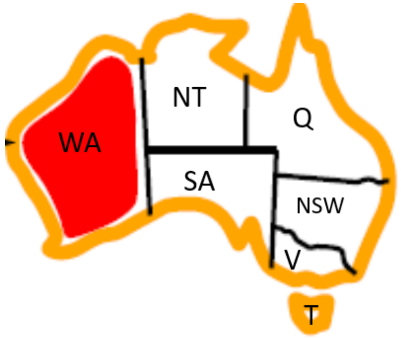

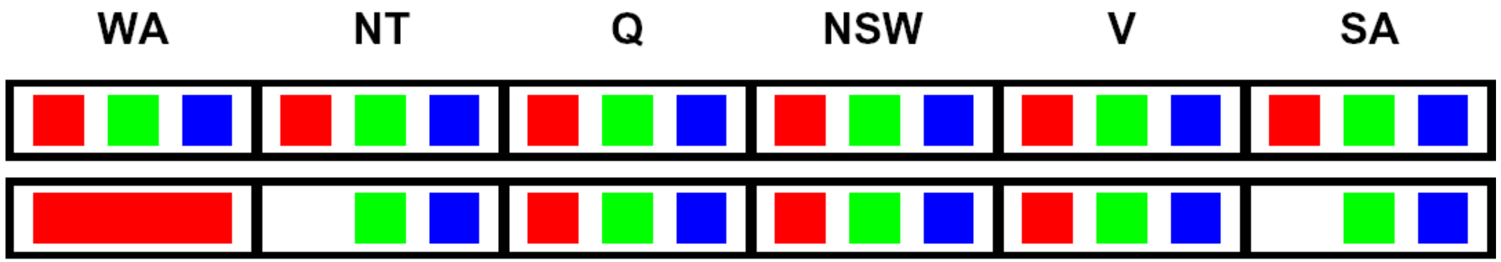

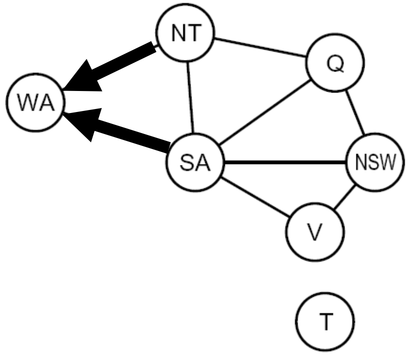

Recall: Binary constraint graph for a binary CSP (i.e., each constraint has most two variables): nodes are variables, edges show constraints

Filtering: Forward Checking

- Filtering: Keep track of domains for unassigned variables and cross off bad options
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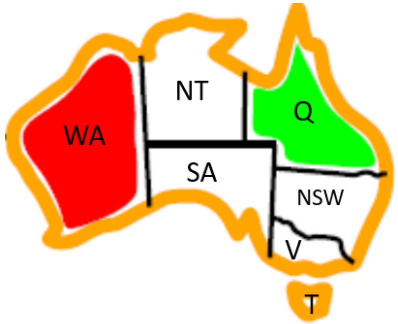

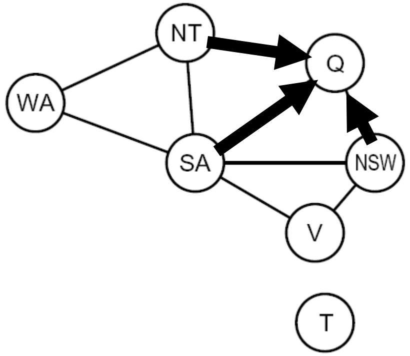

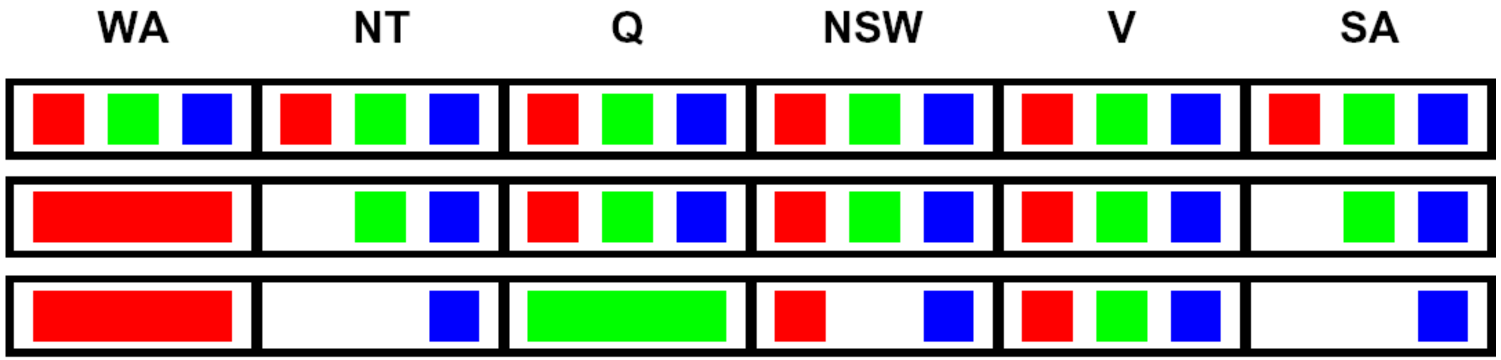

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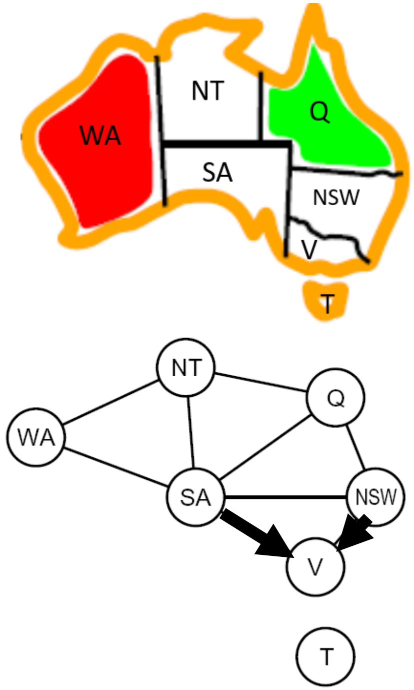
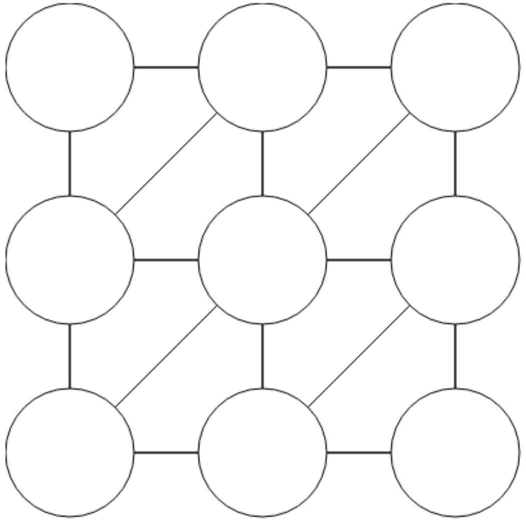


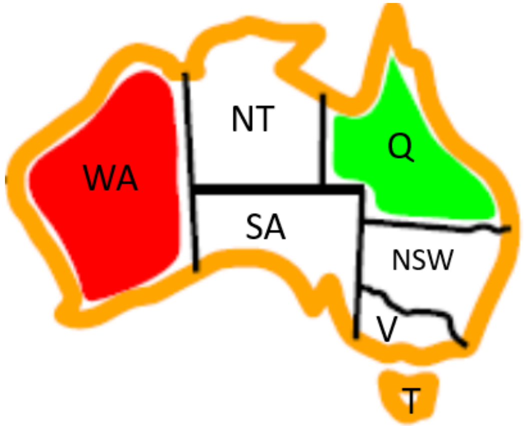
FAIL – variable with no possible values

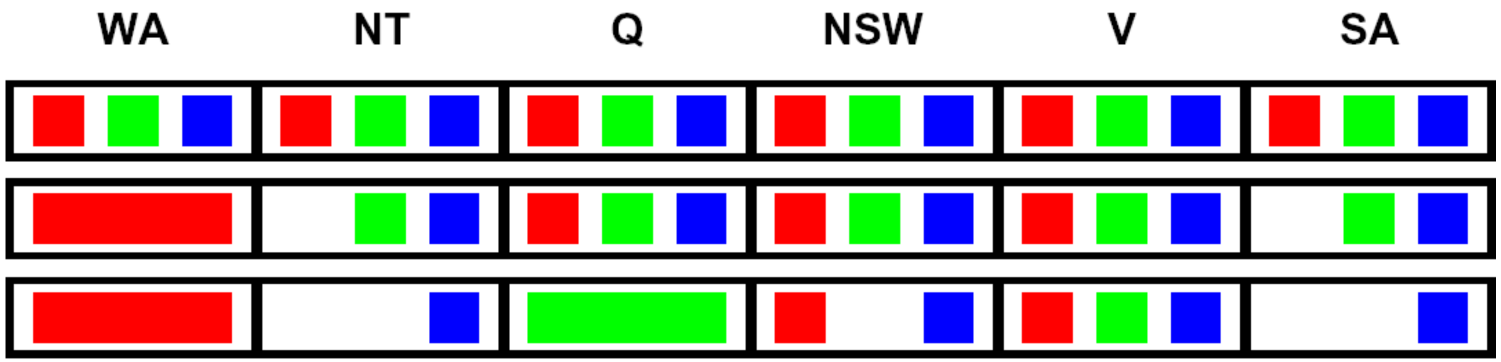
Demo – Backtracking with Forward Checking



Filtering: Constraint Propagation

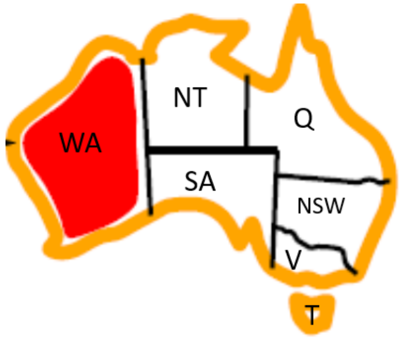
- Limitations of simple forward checking: propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures
  - NT and SA cannot both be blue! Why didn’t we detect this yet?
- *Constraint propagation:* reason from constraint to constraint



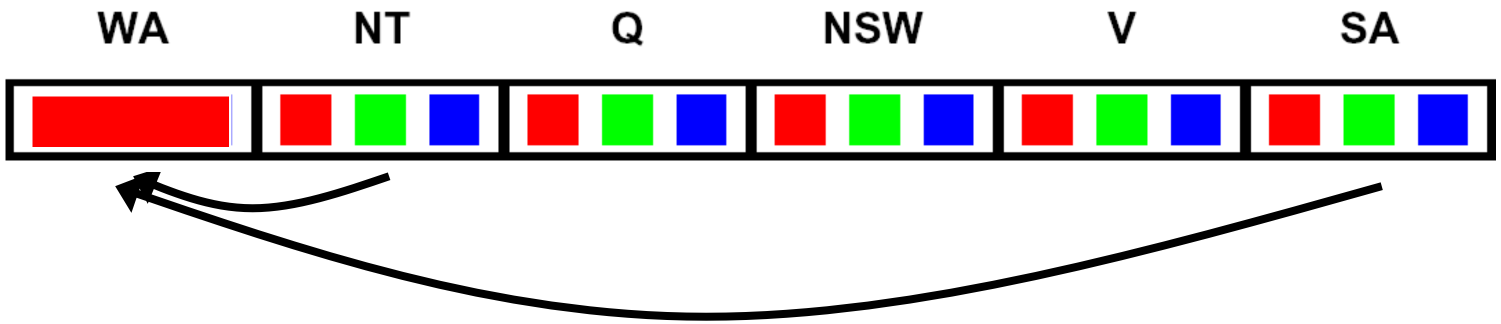


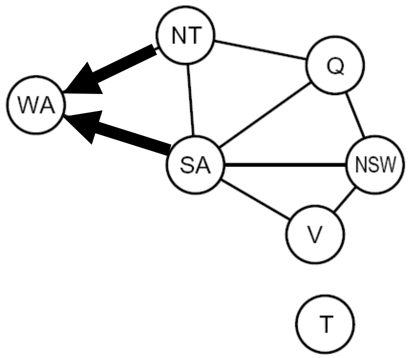
Consistency of A Single Arc

- An arc X → Y is consistent iff for every x in the tail there is some y in the head which could be assigned without violating a constraint
- Enforce arc consistency: Remove values in domain of X if no corresponding legal Y exists
- Forward checking: Only enforce $X \rightarrow Y, \forall (X,Y) \in E$ and Y newly assigned



(Remove values from the tail!)

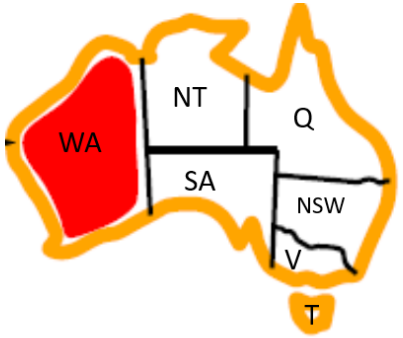


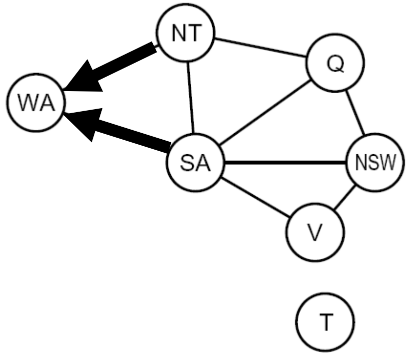


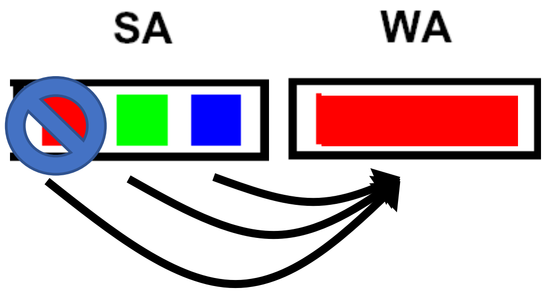
Recall: Binary constraint graph for a binary CSP (i.e., each constraint has most two variables): nodes are variables, edges show constraints

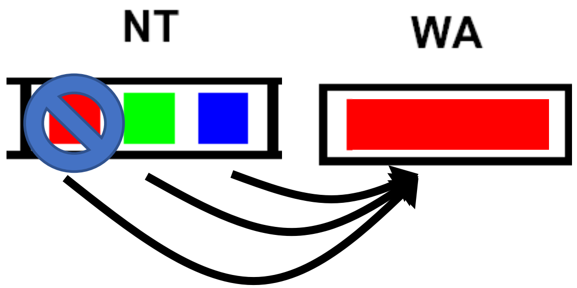
Consistency of A Single Arc

- An arc X → Y is consistent iff for every x in the tail there is some y in the head which could be assigned without violating a constraint
- Enforce arc consistency: Remove values in domain of X if no corresponding legal Y exists
- Forward checking: Only enforce $X \rightarrow Y, \forall (X, Y) \in E$ and Y newly assigned



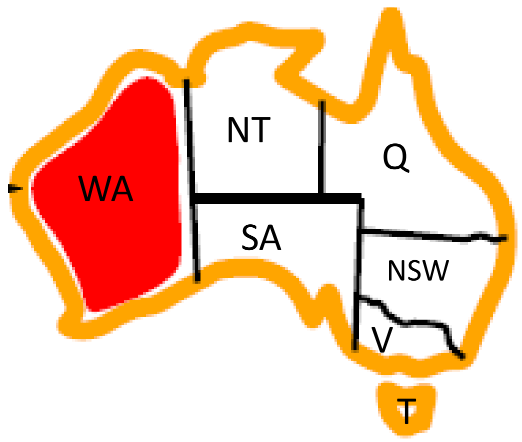


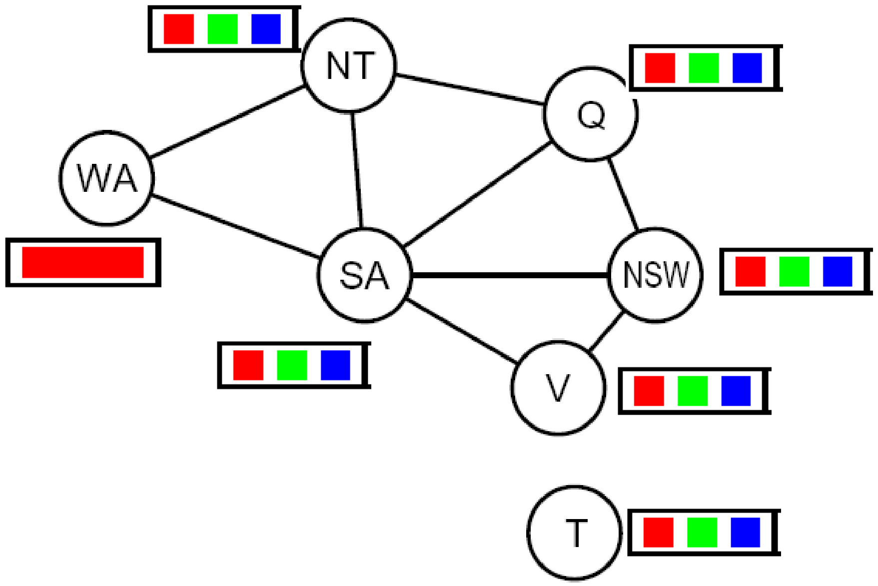




How to Enforce Arc Consistency of Entire CSP

- A simplistic algorithm: Cycle over the pairs of variables, enforcing arc-consistency, repeating the cycle until no domains change for a whole cycle
- AC-3 (short for <u>A</u>rc <u>C</u>onsistency Algorithm #<u>3</u>): A more efficient algorithm ignoring constraints that have not been modified since they were last analyzed





AC-3: Enforce Arc Consistency of Entire CSP

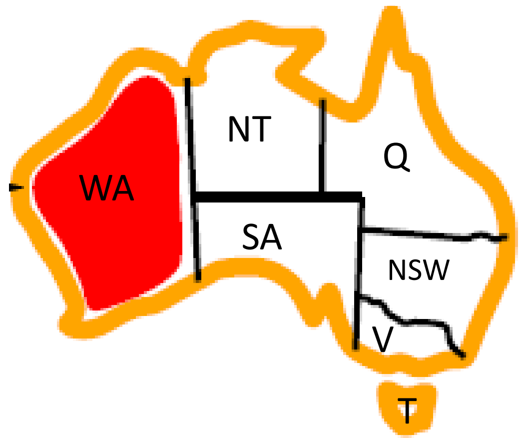
```
function AC-3( csp) returns the CSP, possibly with reduced domains
   inputs: csp, a binary CSP with variables {X1, X2, ..., Xn}
   local variables: queue, a queue of arcs, initially all the arcs in csp

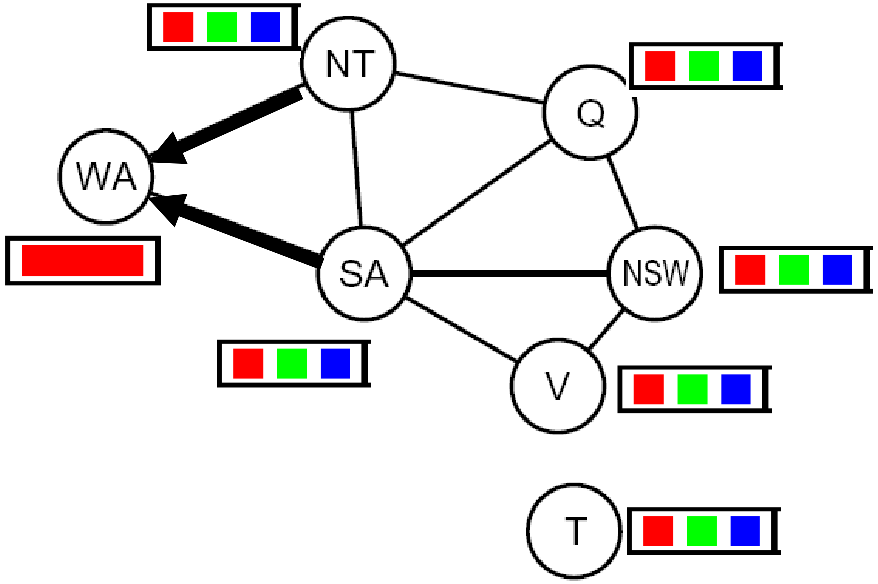
   while queue is not empty do
      (Xi, Xj) ← REMOVE-FIRST(queue)
      if REMOVE-INCONSISTENT-VALUES(Xi, Xj) then
         for each Xk in NEIGHBORS[Xi] do
            add (Xk, Xi) to queue
```

```
function REMOVE-INCONSISTENT-VALUES( Xi, Xj) returns true iff succeeds
   removed ← false
   for each x in DOMAIN[Xi] do
      if no value y in DOMAIN[Xj] allows (x,y) to satisfy the constraint Xi ↔ Xj
         then delete x from DOMAIN[Xi];  removed ← true
   return removed
```

Constraint Propagation!

AC-3: Enforce Arc Consistency of Entire CSP

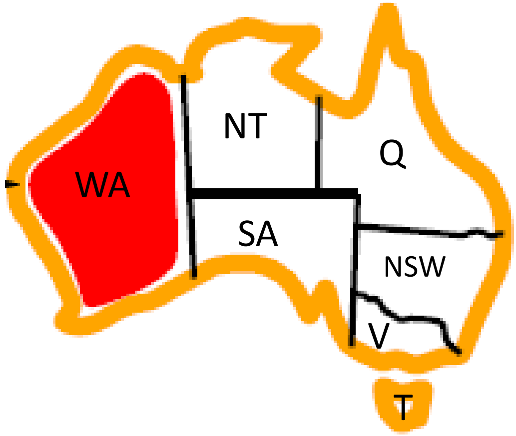


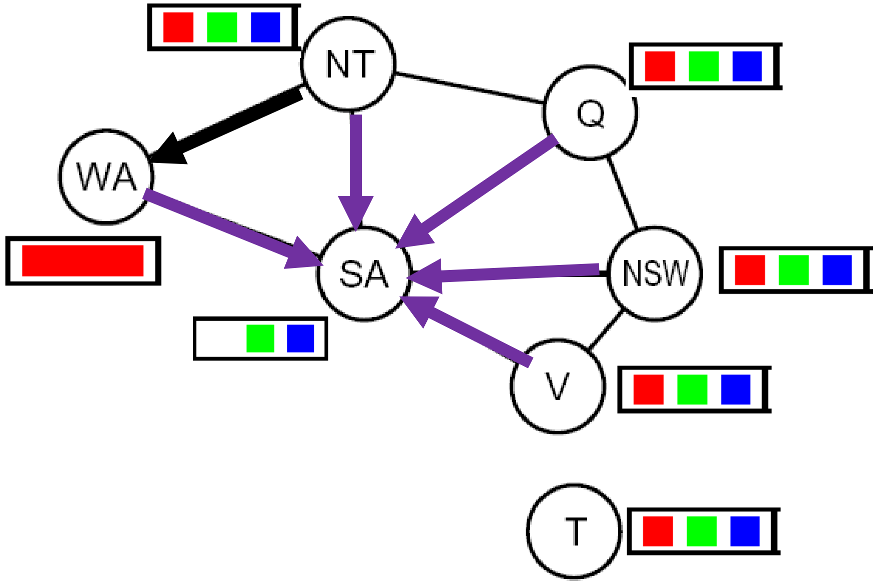


Queue:
SA->WA
NT->WA

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP

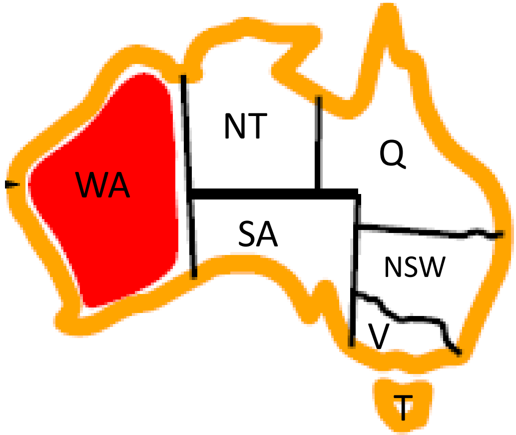


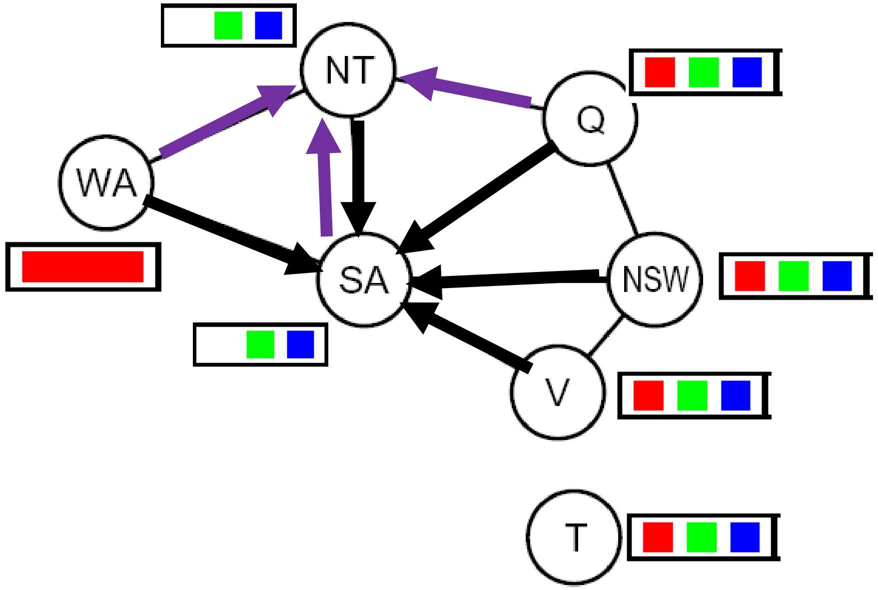


Queue:
NT->WA
WA->SA
NT->SA
Q->SA
NSW->SA
V->SA

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP

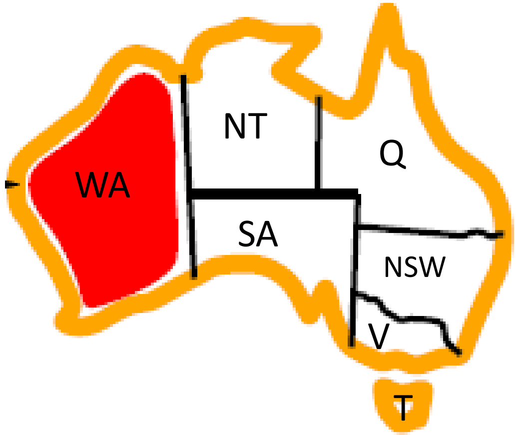


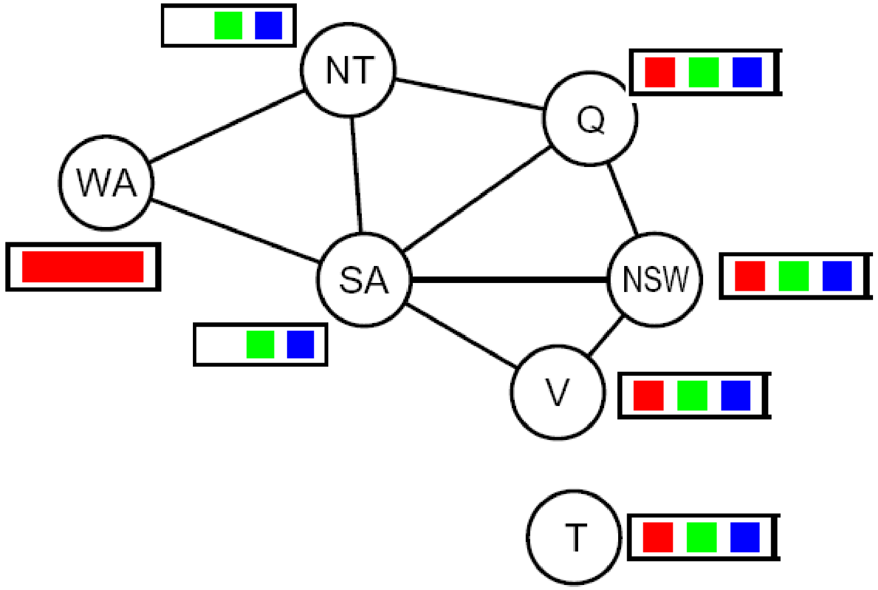


Queue:
WA->SA
NT->SA
Q->SA
NSW->SA
V->SA
WA->NT
SA->NT
Q->NT

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP

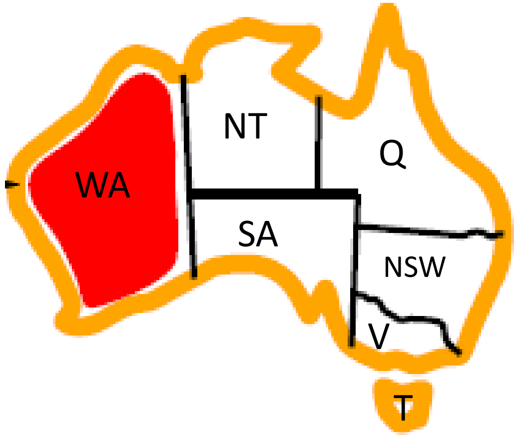


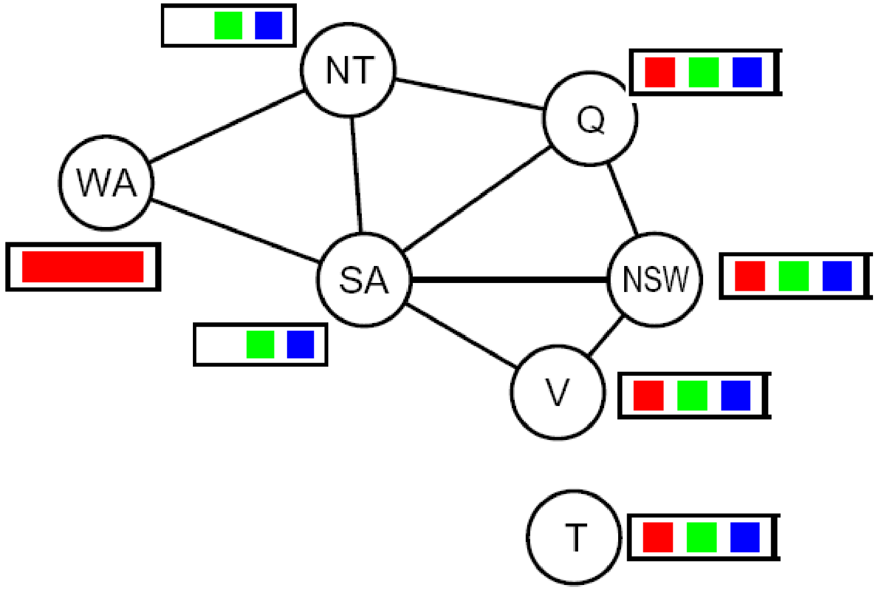


Queue:
WA->SA
NT->SA
Q->SA
NSW->SA
V->SA
WA->NT
SA->NT
Q->NT

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP

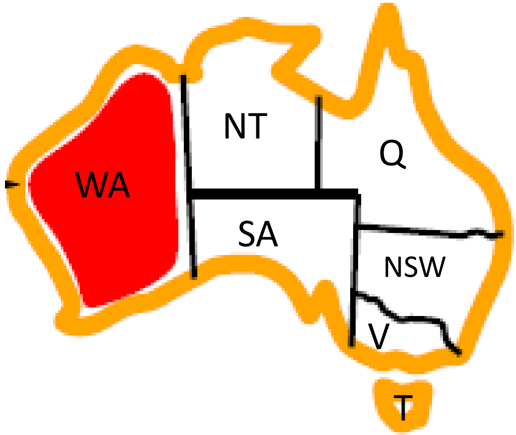


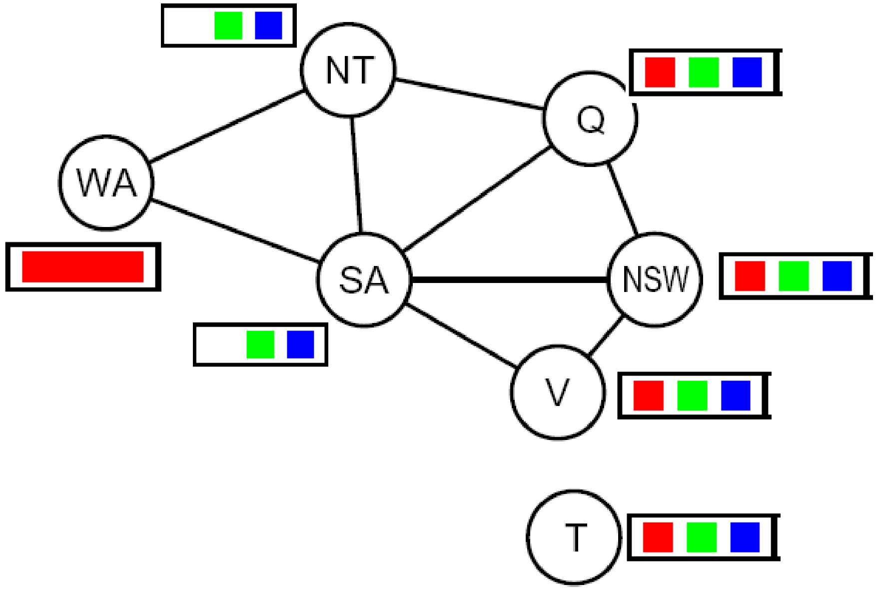


Queue:
NT->SA
Q->SA
NSW->SA
V->SA
WA->NT
SA->NT
Q->NT

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP

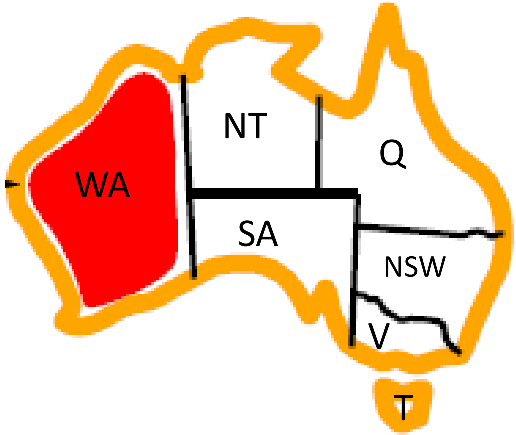


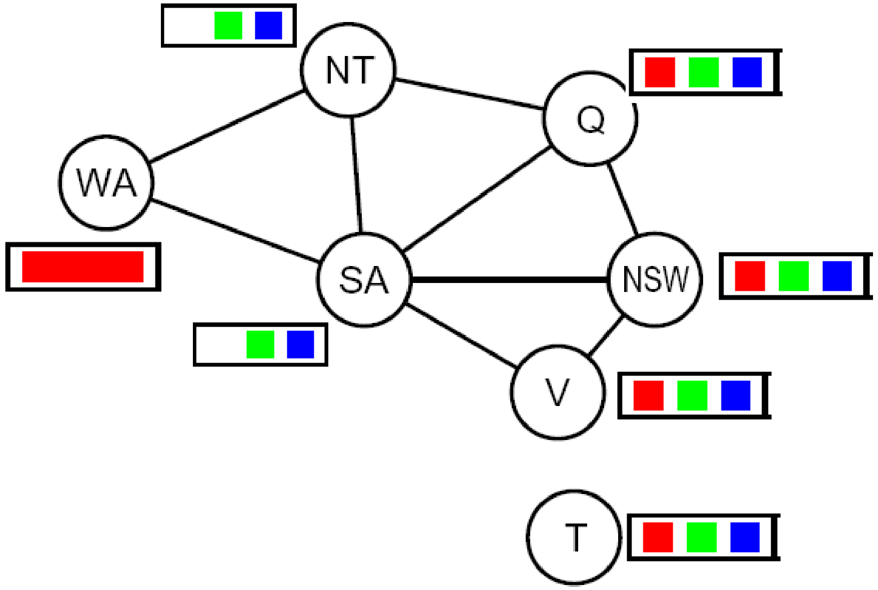


Queue:
Q->SA
NSW->SA
V->SA
WA->NT
SA->NT
Q->NT

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP





Queue:

NSW->SA

V->SA

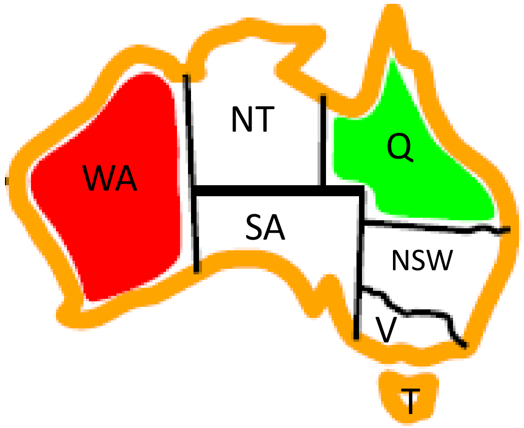
WA->NT

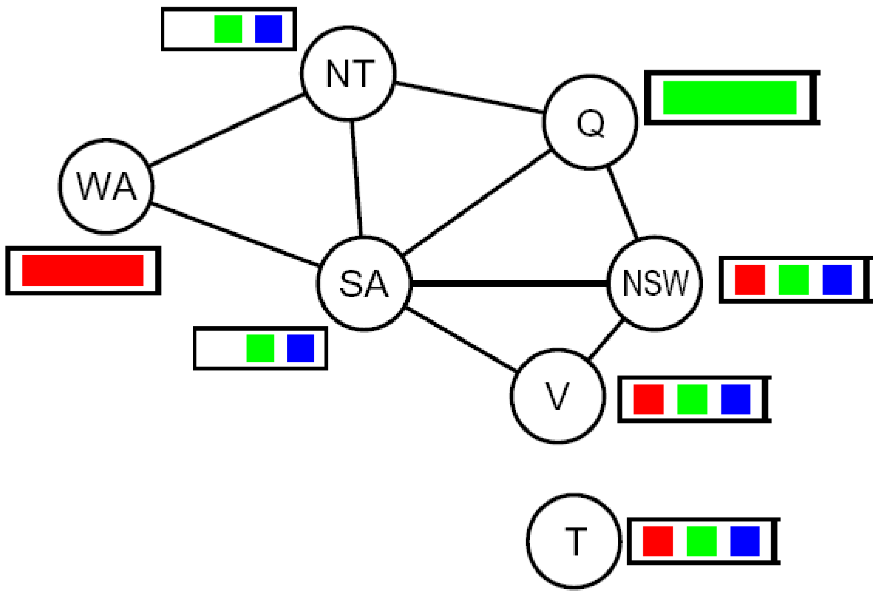
SA->NT

Q->NT

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP

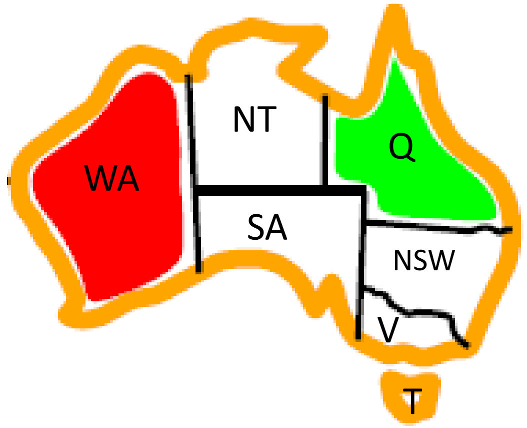


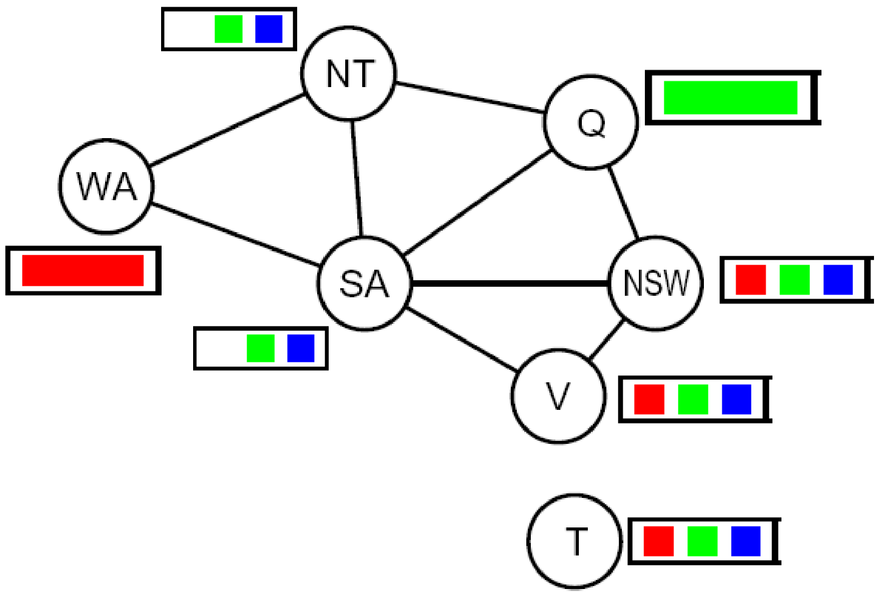


Queue:

Remember: Delete from the tail!

Poll 1: After assigning Q to Green, what gets added to the Queue?



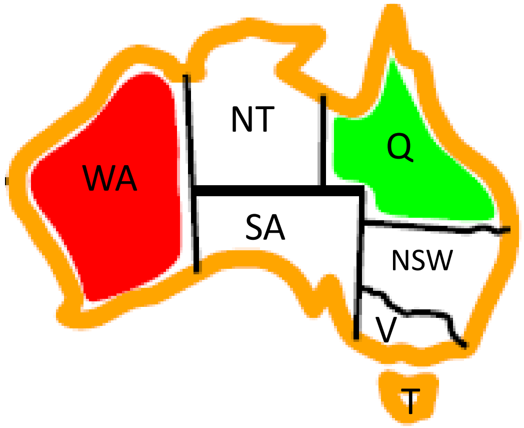


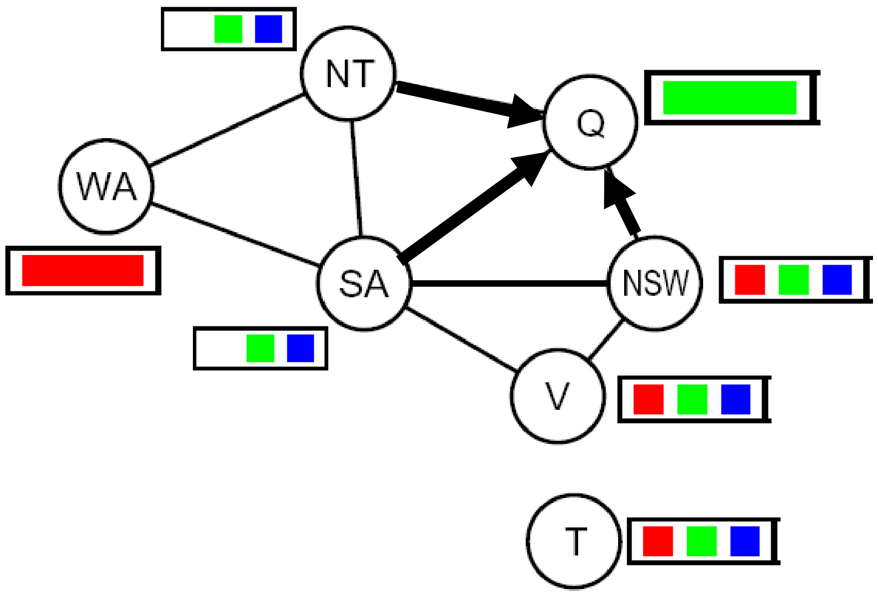
Queue:

A: NSW->Q, SA->Q, NT->Q

B: Q->NSW, Q->SA, Q->NT

AC-3: Enforce Arc Consistency of Entire CSP

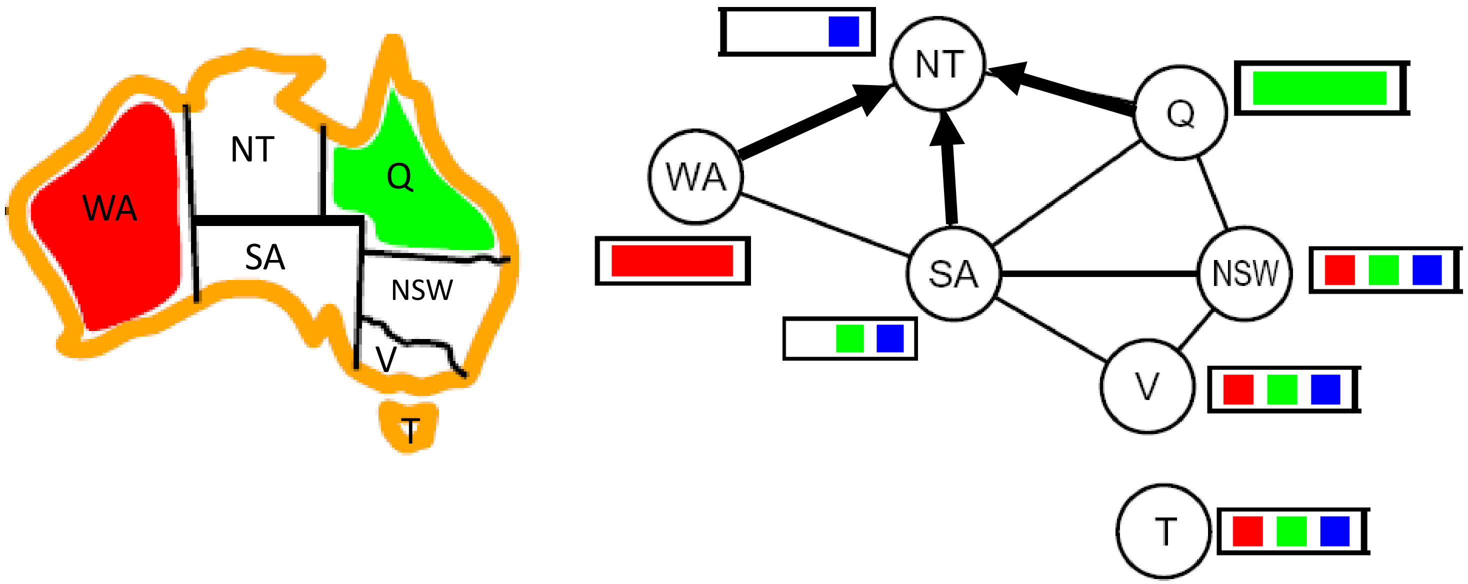




Queue:
NT->Q
SA->Q
NSW->Q

Remember: Delete from the tail!

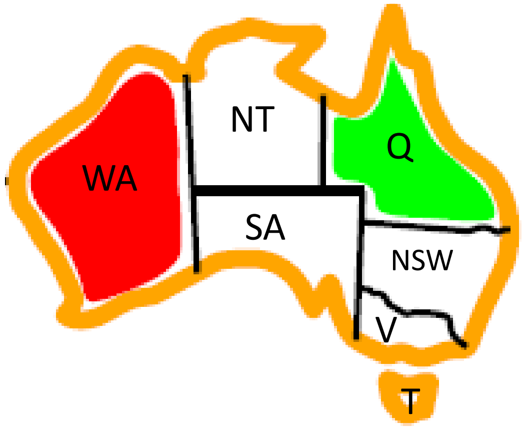
AC-3: Enforce Arc Consistency of Entire CSP

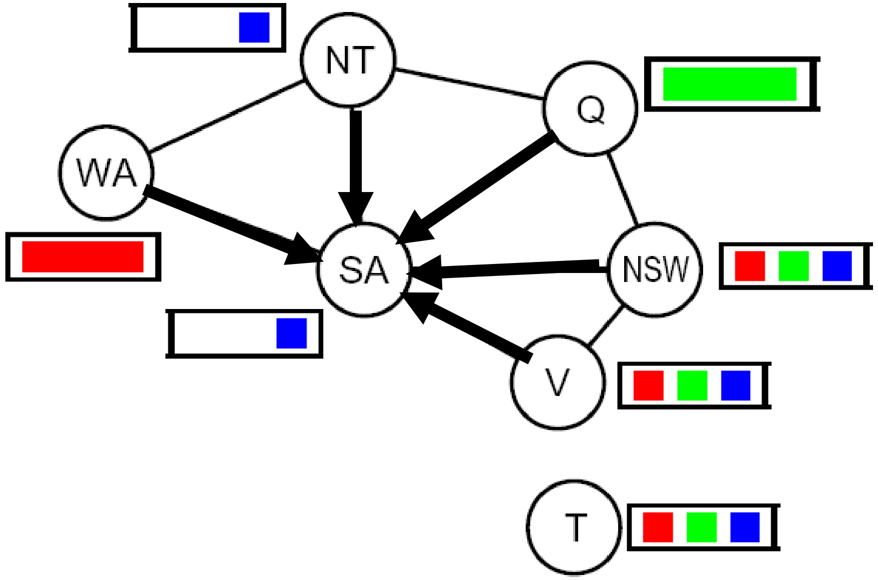


Queue:
SA->Q
NSW->Q
WA->NT
SA->NT
Q->NT

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP

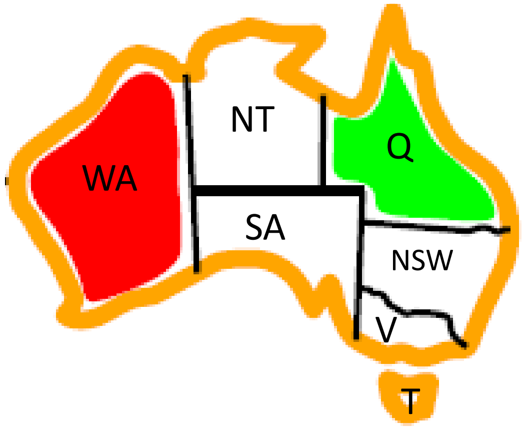


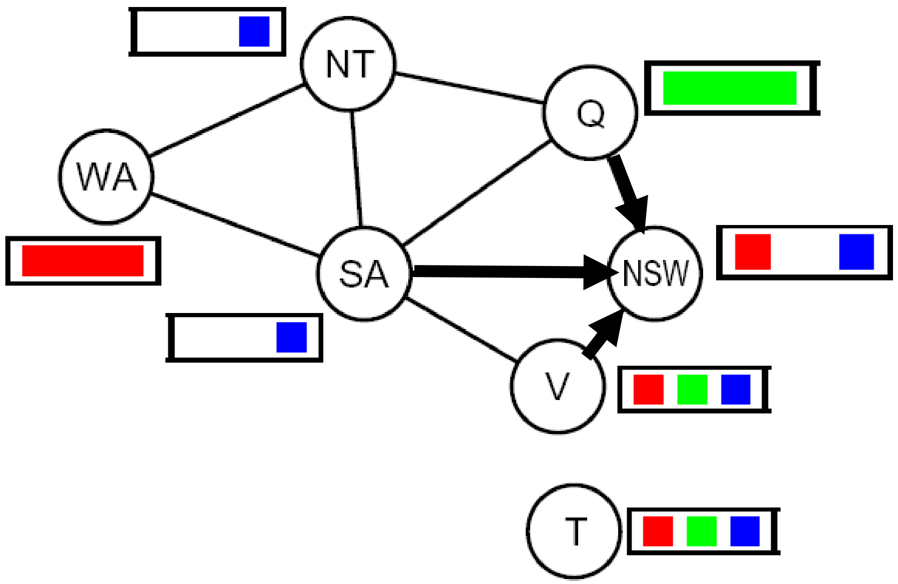


Queue:
NSW->Q
WA->NT
SA->NT
Q->NT
WA->SA
NT->SA
Q->SA
NSW->SA
V->SA

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP

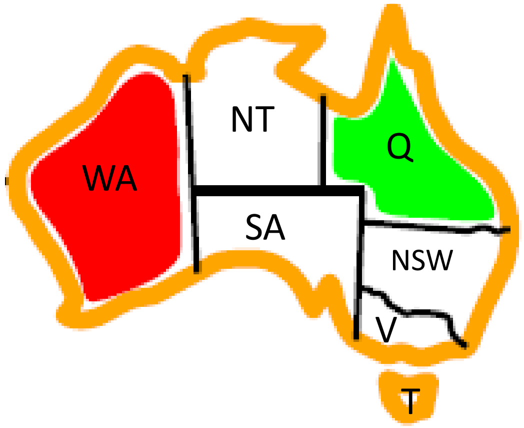


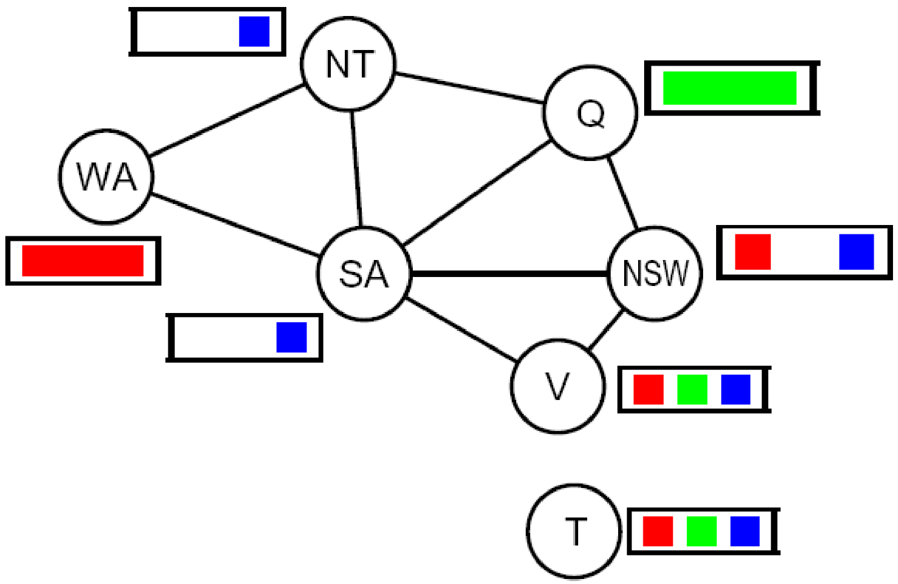


Queue:
WA->NT
SA->NT
Q->NT
WA->SA
NT->SA
Q->SA
NSW->SA
V->SA
V->NSW
Q->NSW
SA->NSW

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP

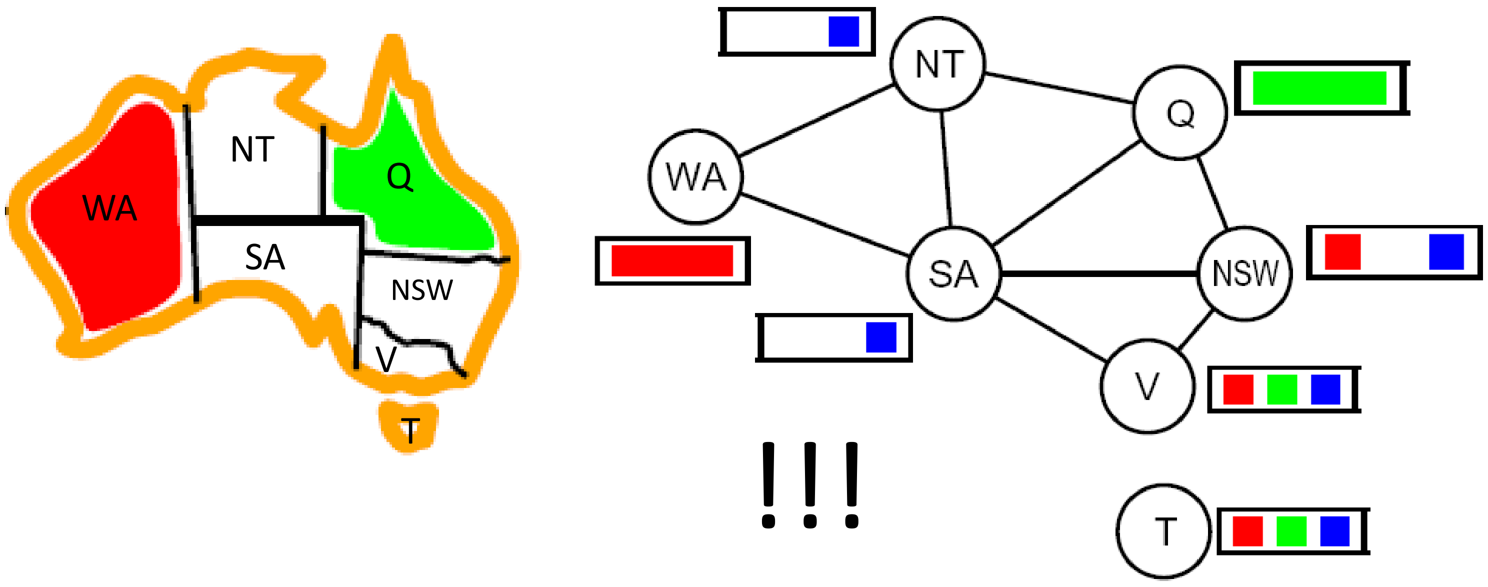




Queue:
WA->NT
SA->NT
Q->NT
WA->SA
NT->SA
Q->SA
NSW->SA
V->SA
V->NSW
Q->NSW
SA->NSW

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP



Queue:

SA->NT

Q->NT

WA->SA

NT->SA

Q->SA

NSW->SA

V->SA

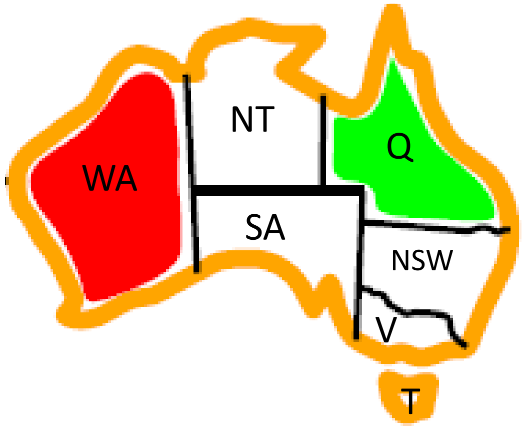
V->NSW

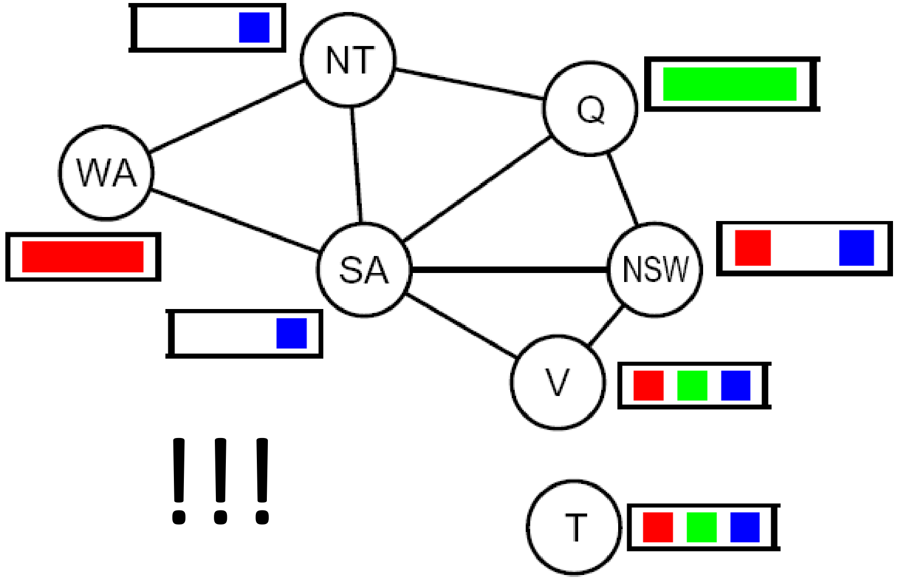
Q->NSW

SA->NSW

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP





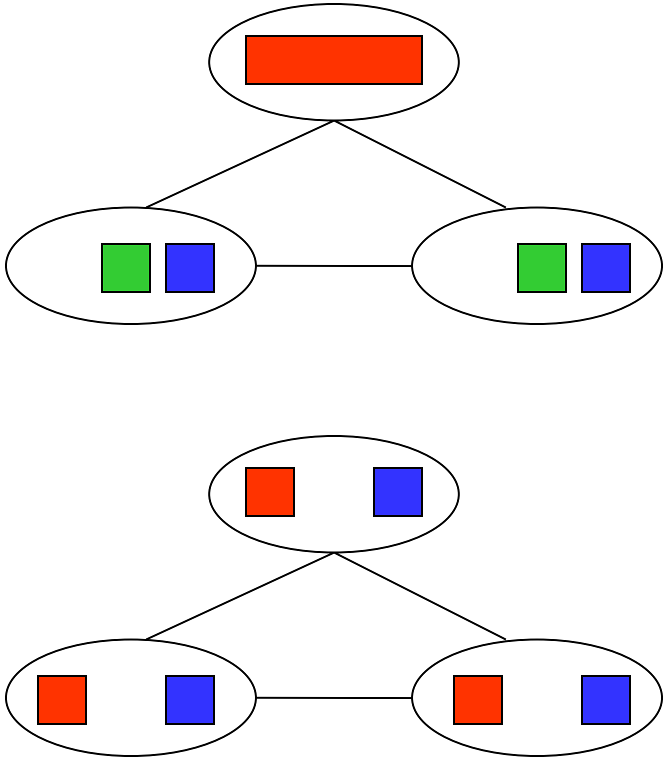
Queue:
SA->NT
Q->NT
WA->SA
NT->SA
Q->SA
NSW->SA
V->SA
V->NSW
Q->NSW
SA->NSW

- Backtrack on the assignment of Q
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment
- What's the downside of enforcing arc consistency?

Remember: Delete from the tail!

Limitations of Arc Consistency

- After enforcing arc consistency:
  - Can have one solution left
  - Can have multiple solutions left
  - Can have no solutions left (and not know it)
- Arc consistency only checks local consistency conditions
- Arc consistency still runs inside a backtracking search!



What went wrong here?

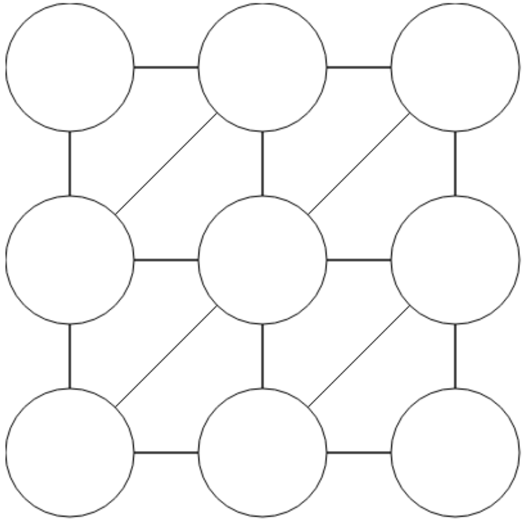
Backtracking Search with AC-3

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
   return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
      if value is consistent with assignment given CONSTRAINTS[csp] then
         add {var = value} to assignment
                                                   AC-3(csp)
         result ← RECURSIVE-BACKTRACKING(assignment, ~~csp~~)
         if result ≠ failure then return result
         remove {var = value} from assignment
   return failure
```

- Where do you run AC-3?

Demo – Backtracking with AC-3



Complexity of a single run of AC-3

function AC-3(*csp*) **returns** the CSP, possibly with reduced domains
  inputs: *csp*, a binary CSP with variables $\{X_1,\ X_2,\ \ldots,\ X_n\}$
  local variables: *queue*, a queue of arcs, initially all the arcs in *csp*

  while *queue* is not empty **do**
    $(X_i,\ X_j) \leftarrow$ REMOVE-FIRST(*queue*)
    if REMOVE-INCONSISTENT-VALUES$(X_i,\ X_j)$ **then**
      for each X_k **in** NEIGHBORS$[X_i]$ **do**
        add $(X_k,\ X_i)$ to *queue*

---

function REMOVE-INCONSISTENT-VALUES($X_i,\ X_j$) **returns** true iff succeeds
  removed $\leftarrow$ *false*
  for each x **in** DOMAIN$[X_i]$ **do**
    if no value y in DOMAIN$[X_j]$ allows (x,y) to satisfy the constraint $X_i \leftrightarrow X_j$
      then delete x from DOMAIN$[X_i]$; *removed* $\leftarrow$ *true*
  return *removed*

Recall that the whole backtracking algorithm with AC-3 will call AC-3 many times

Complexity of a single run of AC-3

```
function AC-3( csp) returns the CSP, possibly with reduced domains
   inputs: csp, a binary CSP with variables {X1, X2, ..., Xn}
   local variables: queue, a queue of arcs, initially all the arcs in csp

   while queue is not empty do
      (Xi, Xj) ← REMOVE-FIRST(queue)
      if REMOVE-INCONSISTENT-VALUES(Xi, Xj) then
         for each Xk in NEIGHBORS[Xi] do
            add (Xk, Xi) to queue
```

```
function REMOVE-INCONSISTENT-VALUES( Xi, Xj) returns true iff succeeds
   removed ← false
   for each x in DOMAIN[Xi] do
      if no value y in DOMAIN[Xj] allows (x,y) to satisfy the constraint Xi ↔ Xj
         then delete x from DOMAIN[Xi];  removed ← true
   return removed
```

- An arc is added after a removal of value at a node
- n node in total, each has $\leq d$ values
- Total times of removal: $O(nd)$

Complexity of a single run of AC-3

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         for each Xk in NEIGHBORS[Xi] do
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```

```
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         then delete x from DOMAIN[Xi];  removed ← true
   return removed
```

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- n node in total, each has $\leq d$ values
- Total times of removal: $O(nd)$
- After a removal, $\leq n$ arcs added
- Total times of adding arcs: $O(n^2 d)$

Complexity of a single run of AC-3

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      if REMOVE-INCONSISTENT-VALUES(Xi, Xj) then
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```

```
function REMOVE-INCONSISTENT-VALUES( Xi, Xj) returns true iff succeeds
   removed ← false
   for each x in DOMAIN[Xi] do
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         then delete x from DOMAIN[Xi]; removed ← true
   return removed
```

- An arc is added after a removal of value at a node
- n node in total, each has $\leq d$ values
- Total times of removal: $O(nd)$
- After a removal, $\leq n$ arcs added
- Total times of adding arcs: $O(n^2 d)$

- Check arc consistency per arc: $O(d^2)$

Complexity of a single run of AC-3 is at most $O(n^2 d^3)$

(Not required) Zhang&Yap (2001) show that its complexity is $O(n^2 d^2)$

Ordering

Backtracking Search

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
   return RECURSIVE-BACKTRACKING({ }, csp)

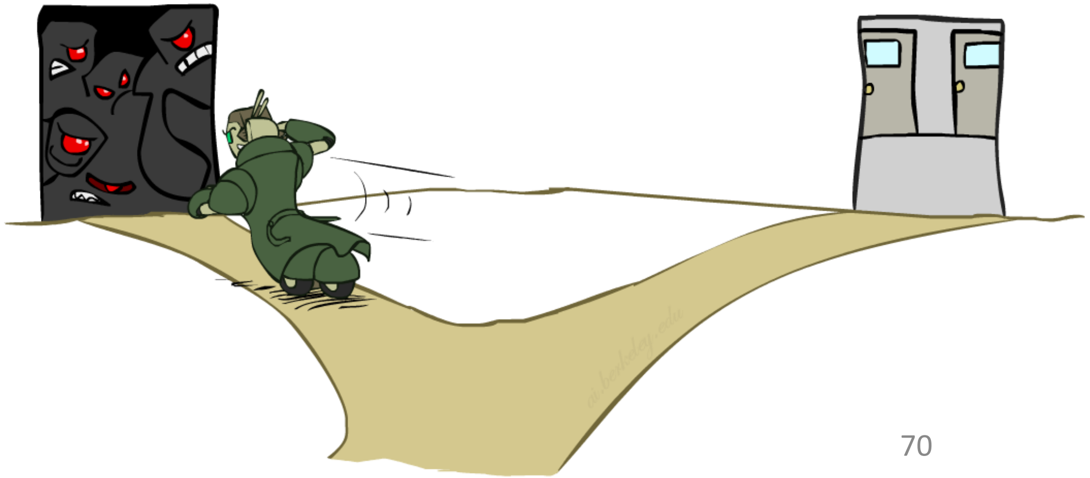
function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
       if value is consistent with assignment given CONSTRAINTS[csp] then
           add {var = value} to assignment
           result ← RECURSIVE-BACKTRACKING(assignment, csp)
           if result ≠ failure then return result
           remove {var = value} from assignment
   return failure
```

- Backtracking = DFS + variable-ordering + fail-on-violation
- What are the decision points?

Ordering: Minimum Remaining Values

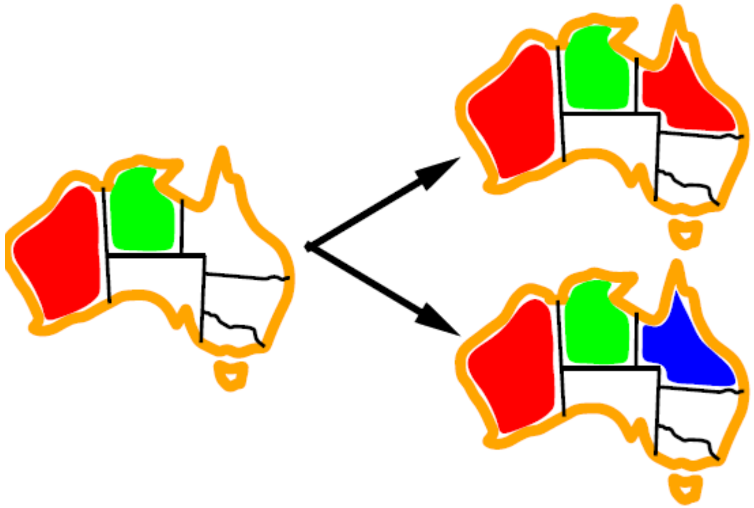
- Variable Ordering: Minimum remaining values (MRV):
  - Choose the variable with the fewest legal left values in its domain

- Why min rather than max?
- Also called "most constrained variable"
- "Fail-fast" ordering



Ordering: Least Constraining Value

- Value Ordering: Least Constraining Value
  - Given a choice of variable, choose the *least constraining value*
  - i.e., the one that rules out the fewest values in the remaining variables
  - Note that it may take some computation to determine this! (E.g., rerunning filtering)

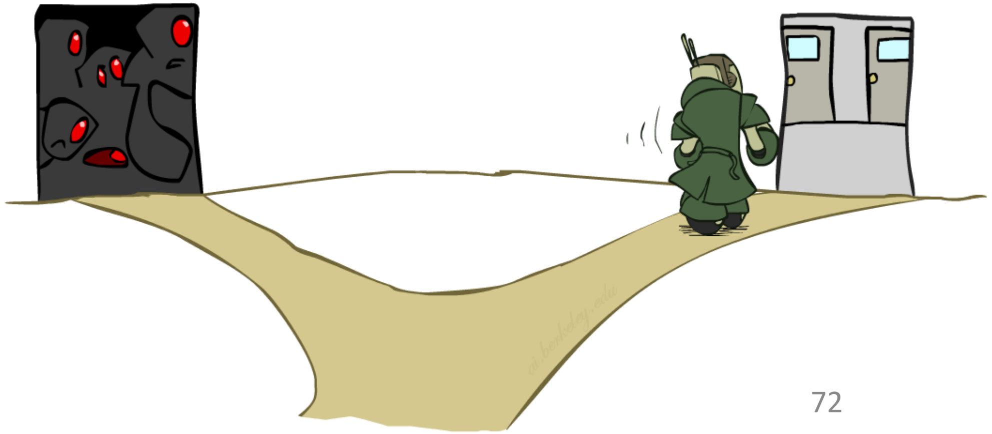


Ordering: Least Constraining Value

- Value Ordering: Least Constraining Value
  - Given a choice of variable, choose the *least constraining value*
  - i.e., the one that rules out the fewest values in the remaining variables
  - Note that it may take some computation to determine this! (E.g., rerunning filtering)

- Why least rather than most?

- Combining these ordering ideas makes 1000 queens feasible



Demo – Coloring with a Complex Graph

Compare

- Backtracking with Forward Checking
- Backtracking with AC-3
- Backtracking + Forward Checking + Minimum Remaining Values (MRV)
- Backtracking + AC-3 + MRV + LCV

How to deal with non-binary CSPs?

- Variables:

  $F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$

- Domains:

  $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

- Constraints:

  $\text{alldiff}(F, T, U, W, R, O)$

  $O + O = R + 10 \cdot X_1$

  $\cdots$

$$
\begin{array}{cccc}
 & \mathrm{T} & \mathrm{W} & \mathrm{O} \\
+ & \mathrm{T} & \mathrm{W} & \mathrm{O} \\
\hline
\mathrm{F} & \mathrm{O} & \mathrm{U} & \mathrm{R}
\end{array}
$$

Constraint graph for non-binary CSPs

- Variable nodes: nodes to represent the variables
- Constraint nodes: auxiliary nodes to represent the constraints
- Edges: connects a constraint node and its corresponding variables

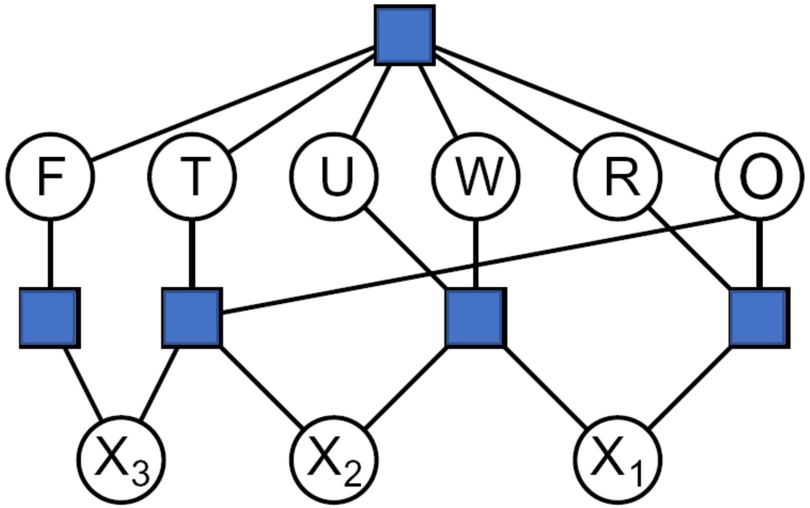
```
    T W O
+   T W O
---------
F   O U R
```

Constraints:

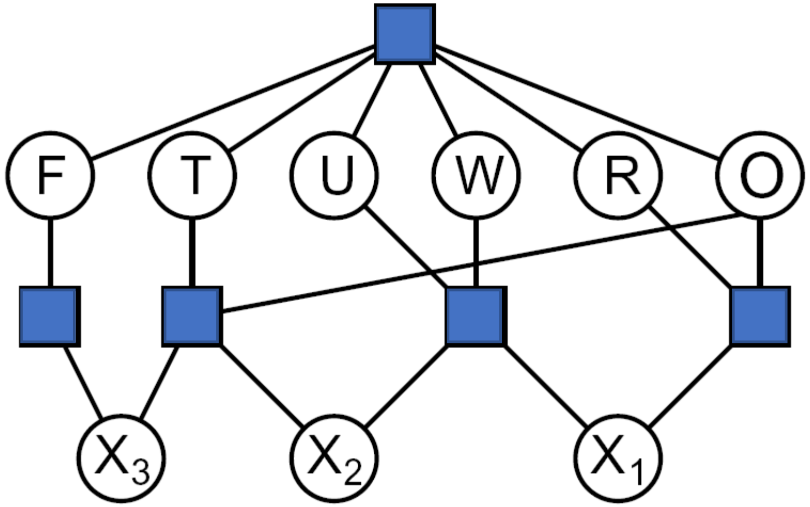
$$\text{alldiff}(F, T, U, W, R, O)$$

$$O + O = R + 10 \cdot X_1$$

$$\cdots$$



Solve non-binary CSPs



- Naïve search?
  - Yes!
- Backtracking?
  - Yes!
- Forward Checking?
  - Need to generalize the original FC operation
  - (nFC0) After a variable is assigned a value, find all constraints with only one unassigned variable and cross off values of that unassigned variable which violate the constraint
  - There exist other ways to do generalized forward checking

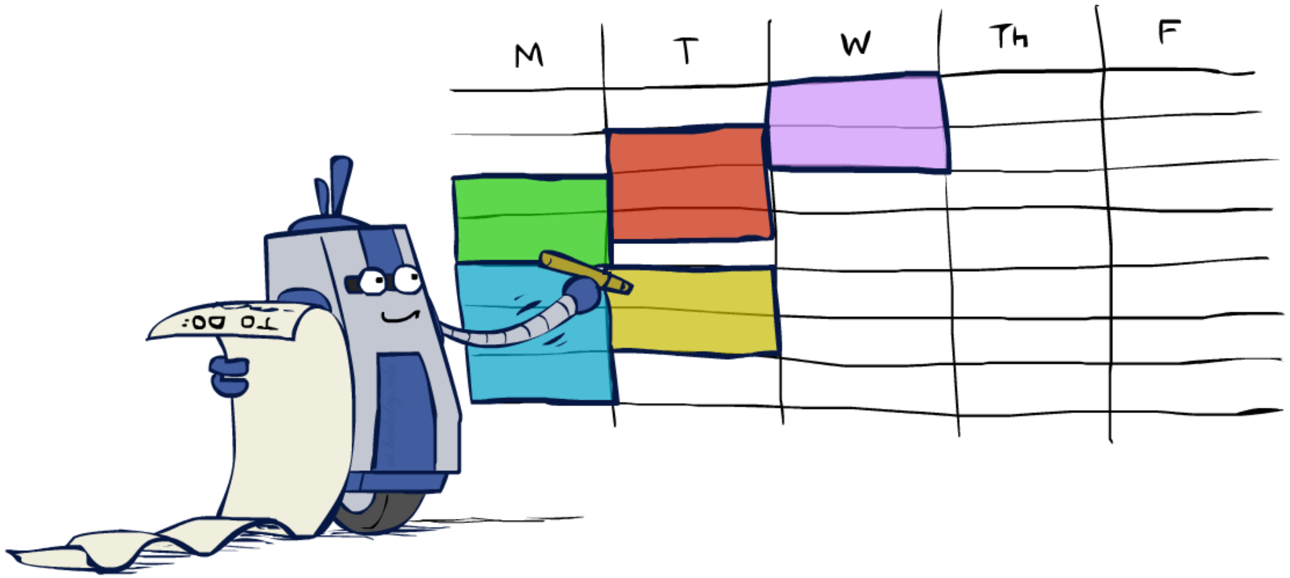
Solve non-binary CSPs

- (Bonus material, not required)
- AC-3? Need to generalize the definition of AC and enforcement of AC
- Generalized arc-consistency (GAC)
  - A non-binary constraint is GAC iff for every value for a variable there exist consistent value combinations for all other variables in the constraint
  - Reduced to AC for binary constraints
- Enforcing GAC
  - Simple schema: enumerate value combination for all other variables
  - O(d^k) on k-ary constraint on variables with domains of size d

- There are other algorithms for non-binary constraint propagation, e.g., (i,j)-consistency [Freuder, JACM 85]

Summary: CSPs

- CSPs are a special kind of search problem:
  - States are partial assignments
  - Goal test defined by constraints
- Basic solution: backtracking search
- Speed-ups:
  - Ordering
  - Filtering
  - Structure



Additional Resources (Not required)

- References

  - Zhang, Yuanlin, and Roland HC Yap. "Making AC-3 an optimal algorithm." In *IJCAI*, vol. 1, pp. 316-321. 2001.
  - Freuder, Eugene C. "A sufficient condition for backtrack-bounded search." *Journal of the ACM (JACM)* 32, no. 4 (1985): 755-761.
