# Announcements

## Assignments:

- P2: Optimization
  - Due Thurs 10/5, 10pm
- HW4 (online)
  - Covers LP, IP
  - Due Tues 9/26, 10 pm

## EXAM 1: 9/28!!

# Plan

## Last Time

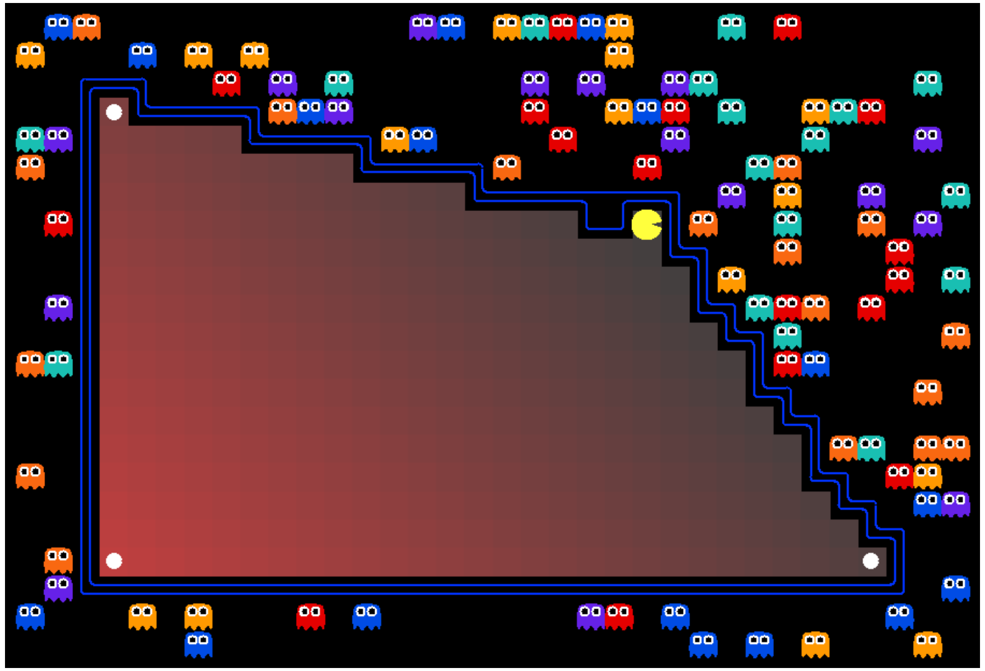
- Linear programming formulation
  - Problem description
  - Graphical representation
  - Optimization representation

## Today

- Solving linear programs
- Higher dimensions than just 2
- Integer programs

# AI: Representation and Problem Solving

## Integer Programming



Instructors: Vincent Conitzer and Aditi Raghunathan

Slide credits: CMU AI with drawings from http://ai.berkeley.edu

# Solving a Linear Program

## Inequality form, with no constraints

$$\min_{\boldsymbol{x}}. \quad \boldsymbol{c}^T \boldsymbol{x}$$

# Solving a Linear Program

## Inequality form, with one constraint

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & a_1 x_1 + a_2 x_2 \leq b \end{aligned}$$

# Poll 1

True or False: A minimizing LP with exactly one constraint, will always have a minimum objective value of $-\infty$.

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & a_1 x_1 + a_2 x_2 \leq b \end{aligned}$$

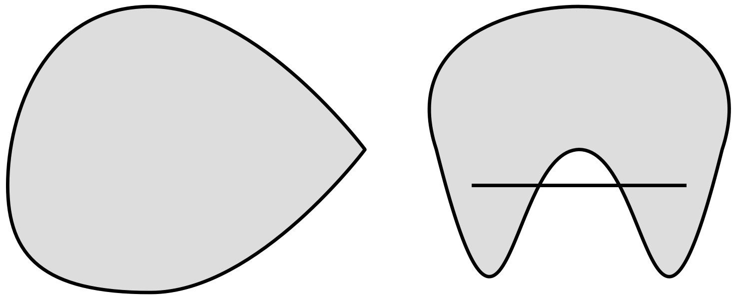
# Question

True or False: A minimizing LP with exactly two constraints, will always have a minimum objective value $> -\infty$.

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 \leq b_2 \end{aligned}$$

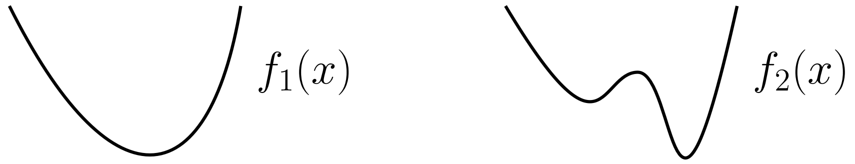
# Convexity

Convex sets are those in which you can draw a line between two points and all the points between them are also in the set



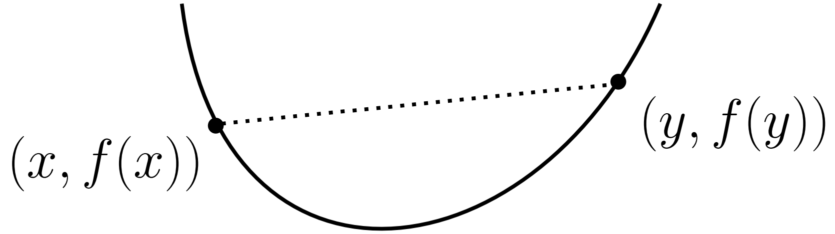
**Convex set** **Nonconvex set**

Convex optimization problems are ones in which the local minimum is also the global minimum




**Convex function** **Nonconvex function**

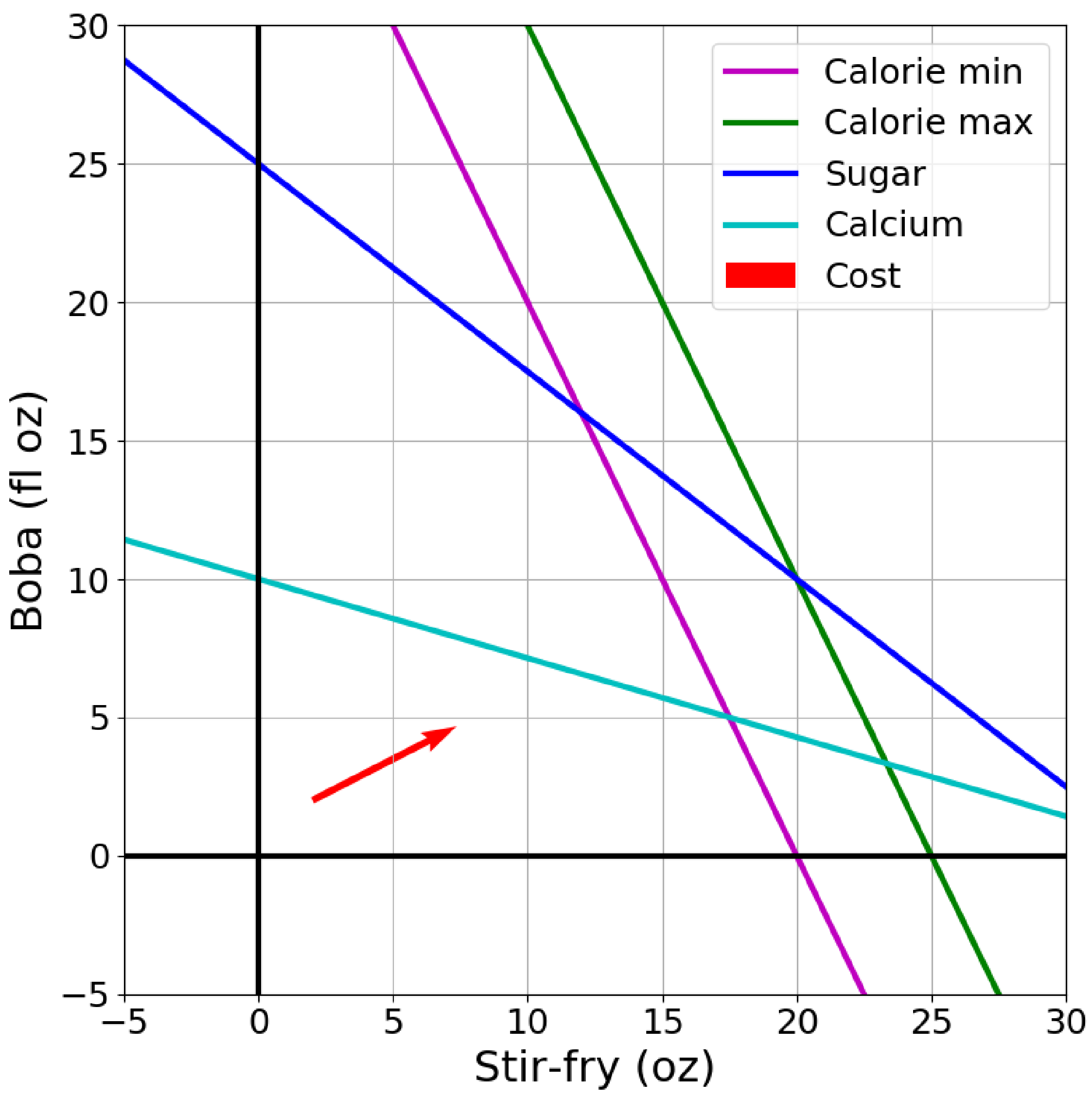
Convex functions have the property that for any point between two points x and y in a convex set:
$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$




Linear functions (like our costs) are convex!

# LP Solutions

Solutions are at feasible intersections of constraint boundaries!!



# Solving an LP

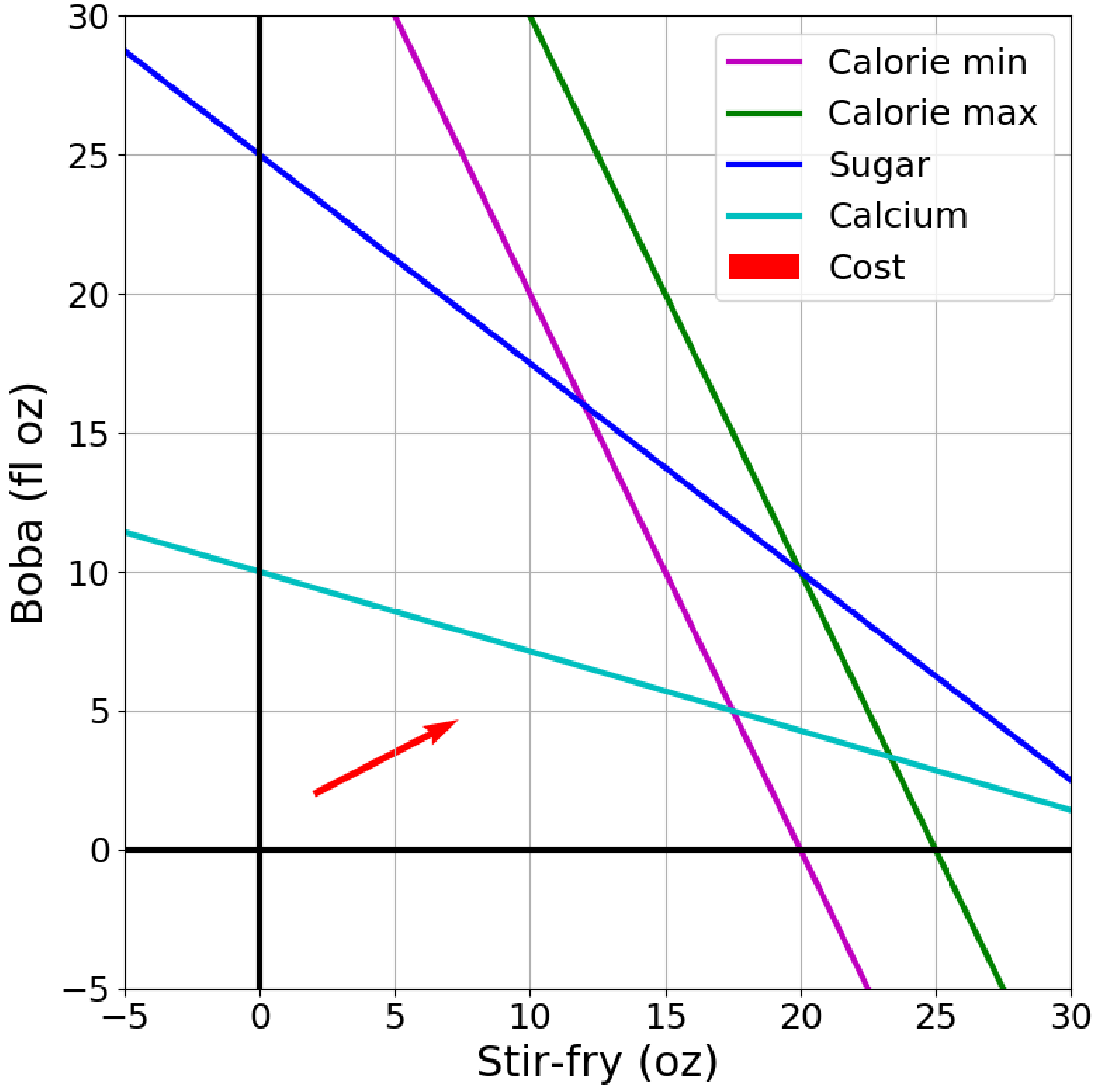
Solutions are at feasible intersections of constraint boundaries!!

Algorithm

- Check objective at all feasible intersections

In more detail:

1. Enumerate all intersections
2. Keep only those that are feasible (satisfy *all* inequalities)
3. Return feasible intersection with the lowest objective value



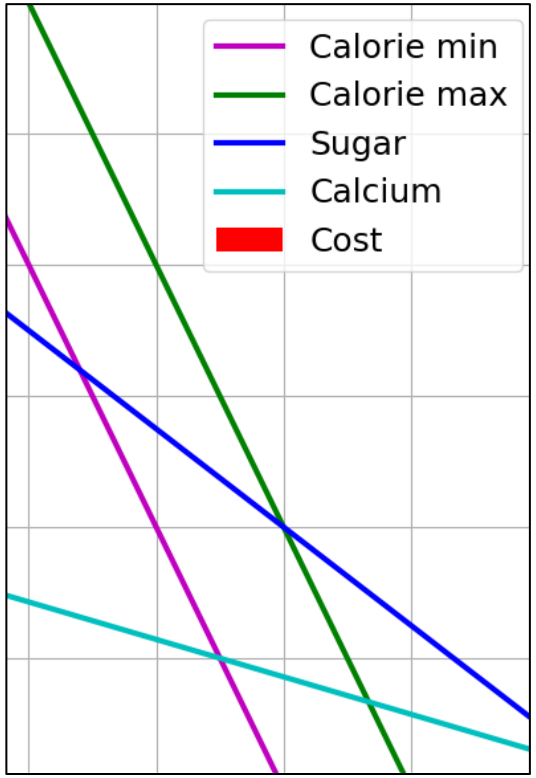
# Solving an LP

But, how do we find the intersection between boundaries?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

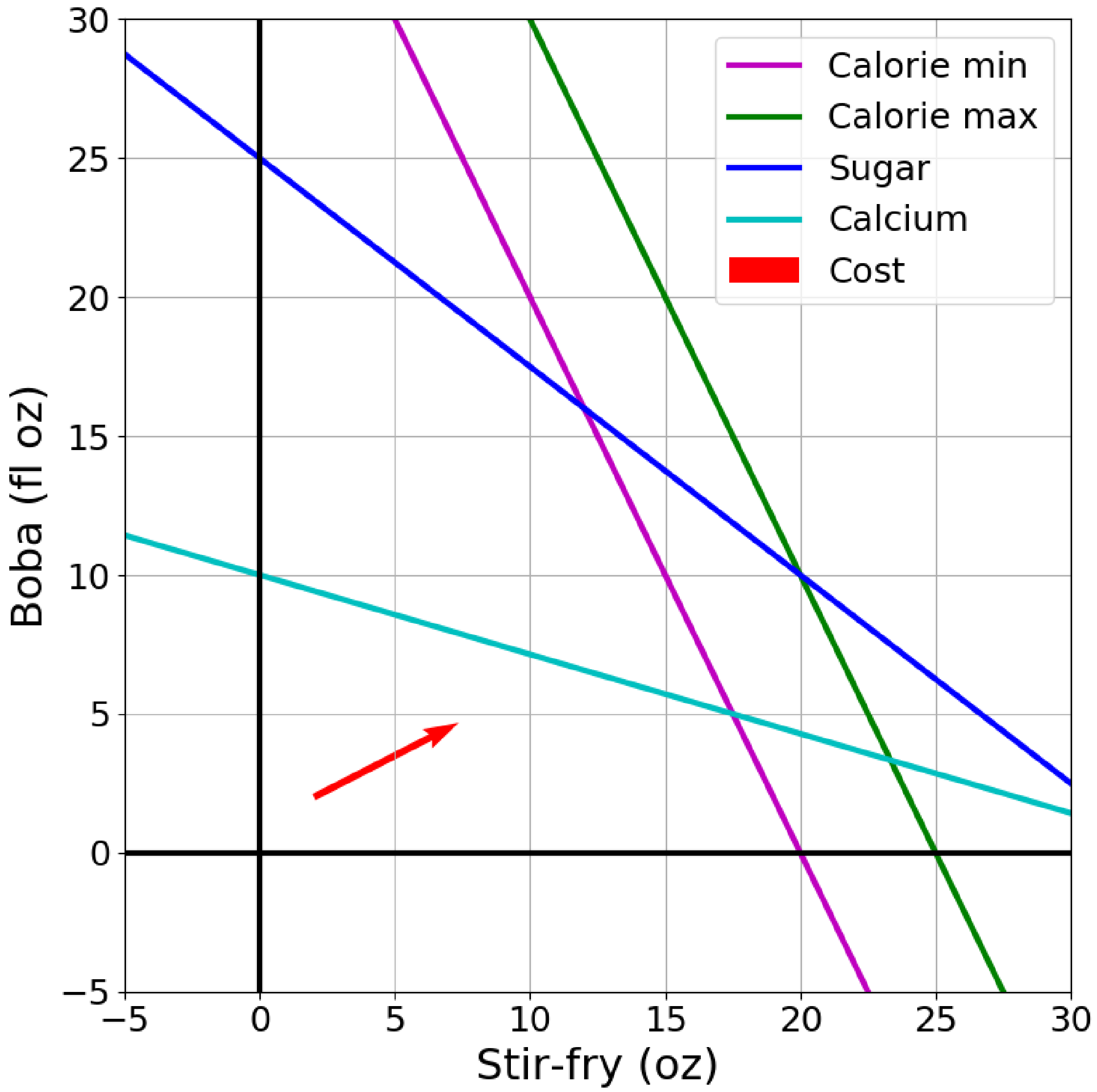
Calorie min
Calorie max
Sugar
Calcium



# Caution

Suppose we **drop** calorie min constraint

What is optimal?

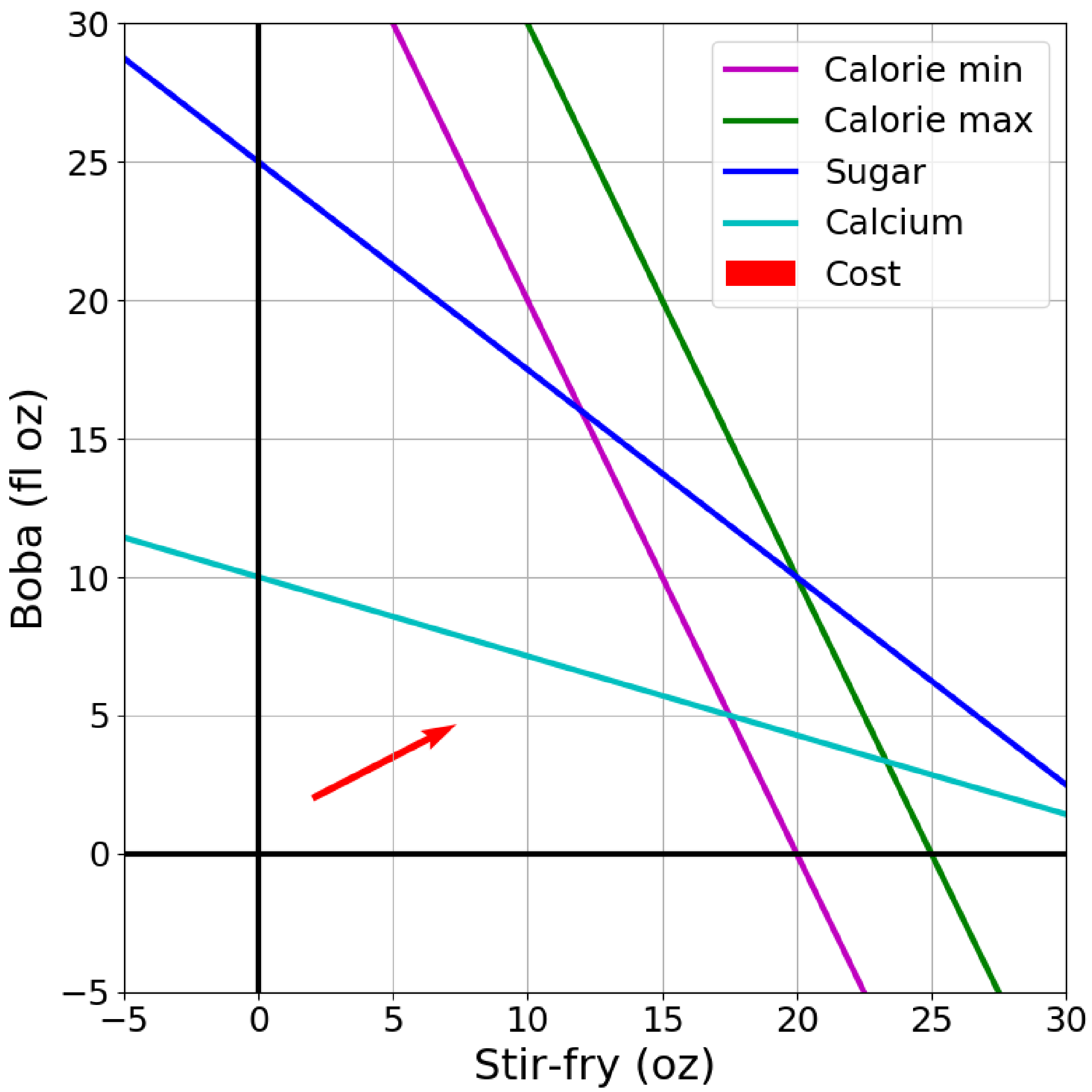


# Solving an LP

Solutions are at feasible intersections of constraint boundaries!!

Algorithms

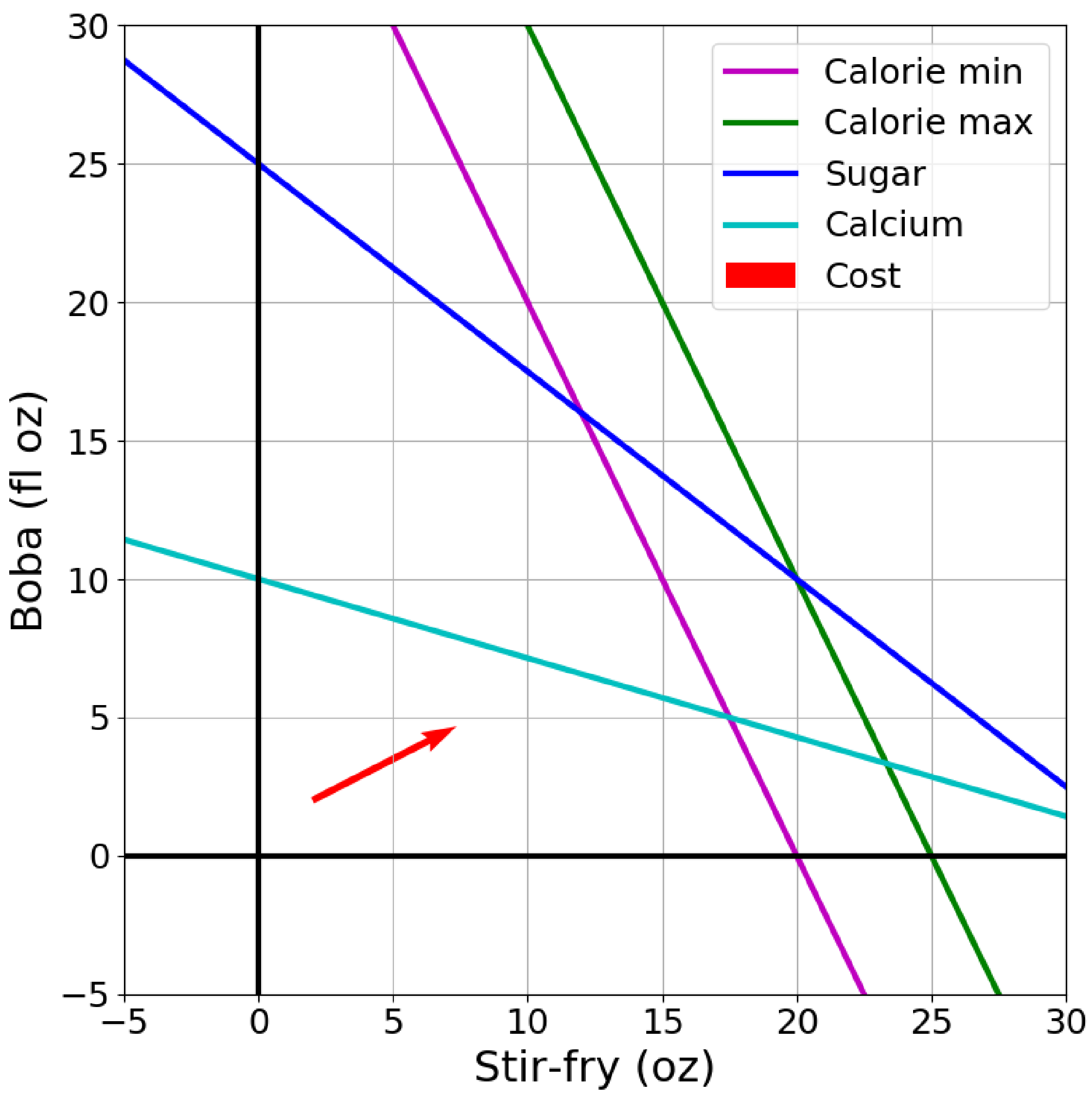
- Check objective at all feasible intersections
  - How many might there be?
- Simplex



# Solving an LP

Simplex algorithm intuition (details missing)

- Start at a feasible intersection (if not trivial, can solve another LP to find one)
- Define successors as “neighbors” of current intersection
  - i.e., remove one row from our square subset of A, and add another row not in the subset; then check feasibility
- Move to any successor with lower objective than current intersection
  - If no such successors, we are done




Greedy local hill-climbing search! … but always finds *optimal* solution (if defined right)

# Solving an LP

Solutions are at feasible intersections of constraint boundaries!!

Algorithms

- Check objective at all feasible intersections
- Simplex
- Interior Point

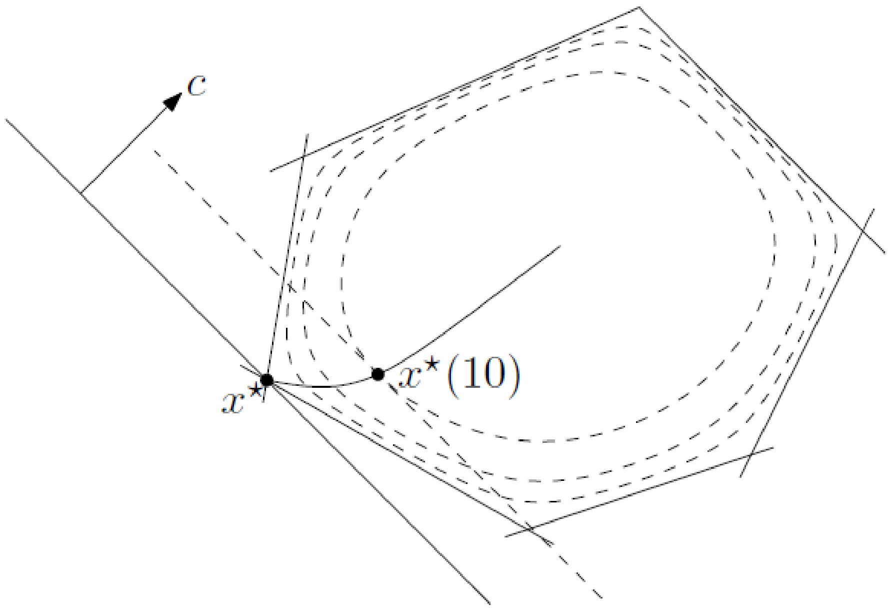
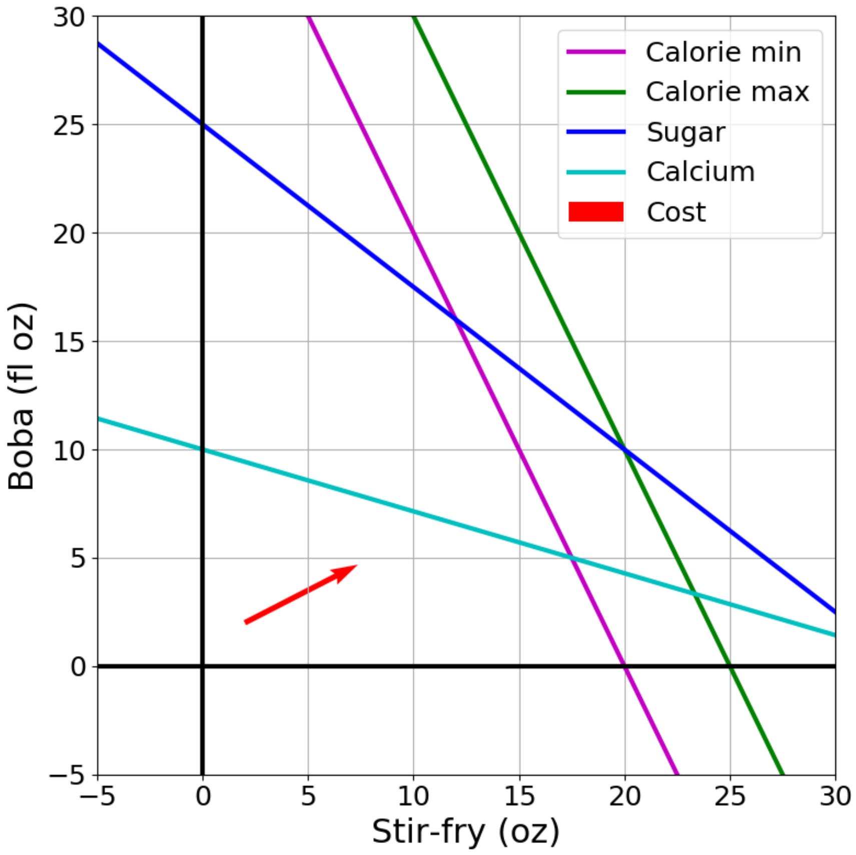



Figure 11.2 from Boyd and Vandenberghe, *Convex Optimization*

# What about higher dimensions?

Problem Description

Graphical Representation




Optimization Representation

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T\boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

# Shapes in higher dimensions

How do these linear shapes extend to 3-D, N-D?

$$a_1\, x_1 + a_2\, x_2 = b_1$$

$$a_1\, x_1 + a_2\, x_2 \le b_1$$

$$\begin{aligned} a_{1,1}\, x_1 + a_{1,2}\, x_2 &\le b_1 \\ a_{2,1}\, x_1 + a_{2,2}\, x_2 &\le b_2 \\ a_{3,1}\, x_1 + a_{3,2}\, x_2 &\le b_3 \\ a_{4,1}\, x_1 + a_{4,2}\, x_2 &\le b_4 \end{aligned}$$

# What are intersections in higher dimensions?

## How do these linear shapes extend to 3-D, N-D?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

$$A = \begin{bmatrix} -100 & -50 & \\ 100 & 50 & \\ 3 & 4 & \\ -20 & -70 & \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Calorie min
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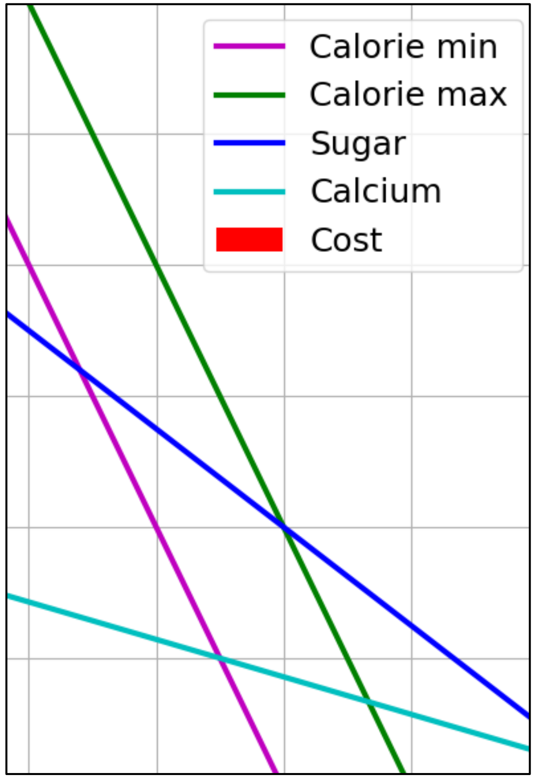
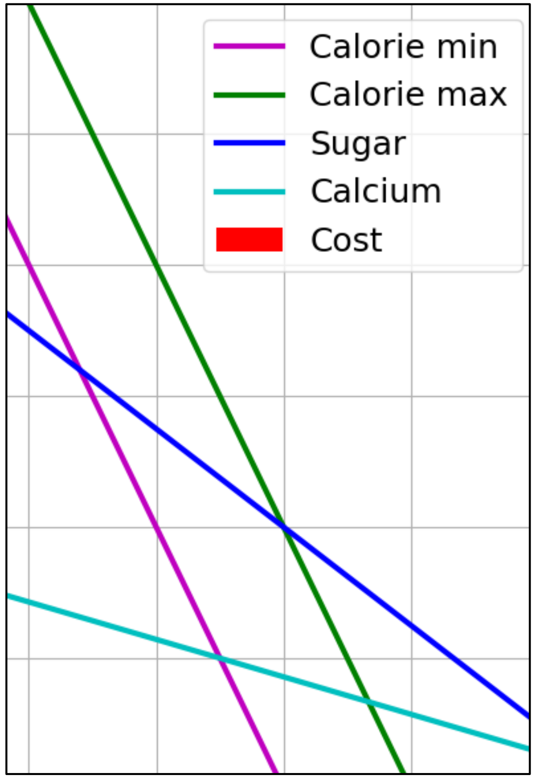


# How do we find intersections in higher dimensions?

## Still looking at subsets of $A$ matrix

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix} \begin{array}{l} \text{Calorie min} \\ \text{Calorie max} \\ \text{Sugar} \\ \text{Calcium} \end{array}$$



# Linear Programming

We are trying to stay healthy by finding the optimal food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

<u>Healthy Squad Goals</u>

- $2000 \leq$ Calories $\leq 2500$
- Sugar $\leq 100$ g
- Calcium $\geq 700$ mg

| Food | Cost | Calories | Sugar | Calcium |
|---|---|---|---|---|
| Stir-fry (per oz) | 1 | 100 | 3 | 20 |
| Boba (per fl oz) | 0.5 | 50 | 4 | 70 |

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

# Linear Programming → Integer Programming

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (bowls) and boba (glasses).

Healthy Squad Goals

- $2000 \leq$ Calories $\leq 2500$
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| Food | Cost | Calories | Sugar | Calcium |
|---|---|---|---|---|
| Stir-fry (per bowl) | 1 | 100 | 3 | 20 |
| Boba (per glass) | 0.5 | 50 | 4 | 70 |

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

# Linear Programming vs Integer Programming

Linear objective with linear constraints, but now with additional constraint that all values in $\boldsymbol{x}$ must be integers

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \\ & \boldsymbol{x} \in \mathbb{Z}^N \end{aligned}$$

We could also do:

- Even more constrained: Binary Integer Programming
- A hybrid: Mixed Integer Linear Programming

Notation Alert!

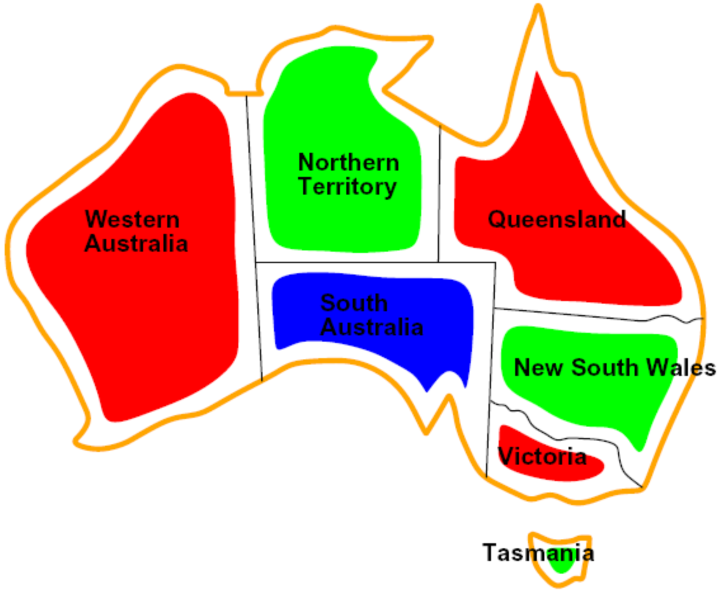
# Integer Programming: Graphical Representation

Just add a grid of integer points onto our LP representation

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \\ & \boldsymbol{x} \in \mathbb{Z}^N \end{aligned}$$

# Integer Programming: Coloring

How would we formulate coloring as an integer program?



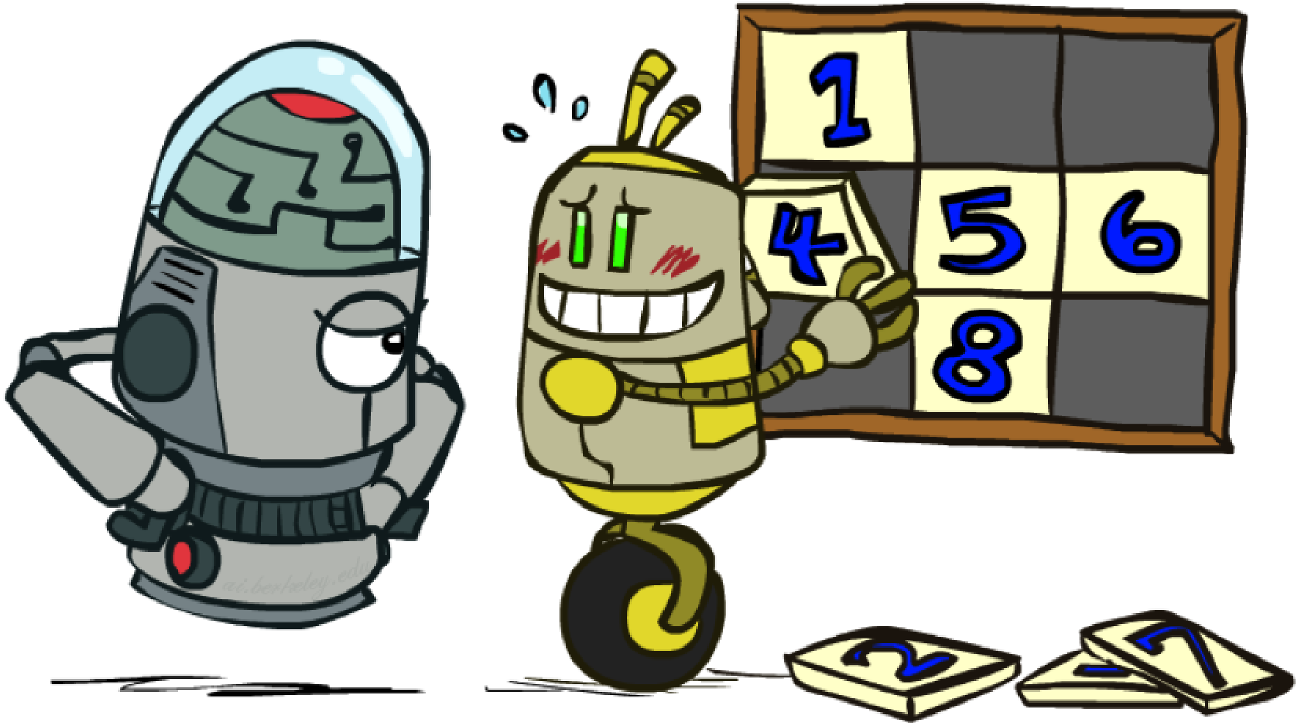
# Convexity and IPs

Integer programs are not convex, but perhaps we can use LP solvers to help find solutions to integer programs?

Relax IP to LP by dropping integer constraints

$$
\begin{aligned}
\min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T \boldsymbol{x} \\
\text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \\
& \cancel{\boldsymbol{x} \in \mathbb{Z}^N}
\end{aligned}
$$

Remember heuristics?



# Poll 2:

True/False: It is sufficient to consider the integer points around the corresponding LP solution?

# Poll 3:

Let $y_{IP}^*$ be the optimal objective of an integer program $P$.

Let $\boldsymbol{x}_{IP}^*$ be an optimal point of an integer program $P$.

Let $y_{LP}^*$ be the optimal objective of the LP-relaxed version of $P$.

Let $\boldsymbol{x}_{LP}^*$ be an optimal point of the LP-relaxed version of $P$.

Assume that $P$ is a minimization problem.

$$\begin{aligned} y_{IP}^* = \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \\ & \boldsymbol{x} \in \mathbb{Z}^N \end{aligned}$$

$$\begin{aligned} y_{LP}^* = \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$

Which of the following are true? Select all that apply.

A) $\boldsymbol{x}_{IP}^* = \boldsymbol{x}_{LP}^*$

B) $y_{IP}^* \le y_{LP}^*$

C) $y_{IP}^* \ge y_{LP}^*$

# Solving an IP

## Branch and Bound algorithm

1. Push current LP (with its solution) into priority queue,

    ordered by objective value of LP solution (heuristic)

2. Repeat:
    - If queue is empty, return that the IP is infeasible
    - Pop candidate solution $\boldsymbol{x}_{LP}^{\star}$ from priority queue
    - If $\boldsymbol{x}_{LP}^{\star}$ is all integer valued, we are done; return solution
    - Otherwise, select a coordinate $x_i$ that is not integer valued, and add two additional LPs (with their solutions) to the priority queue, in each case adding 1 constraint to current LP:

        *Left branch*: Added constraint $x_i \leq floor(x_i)$

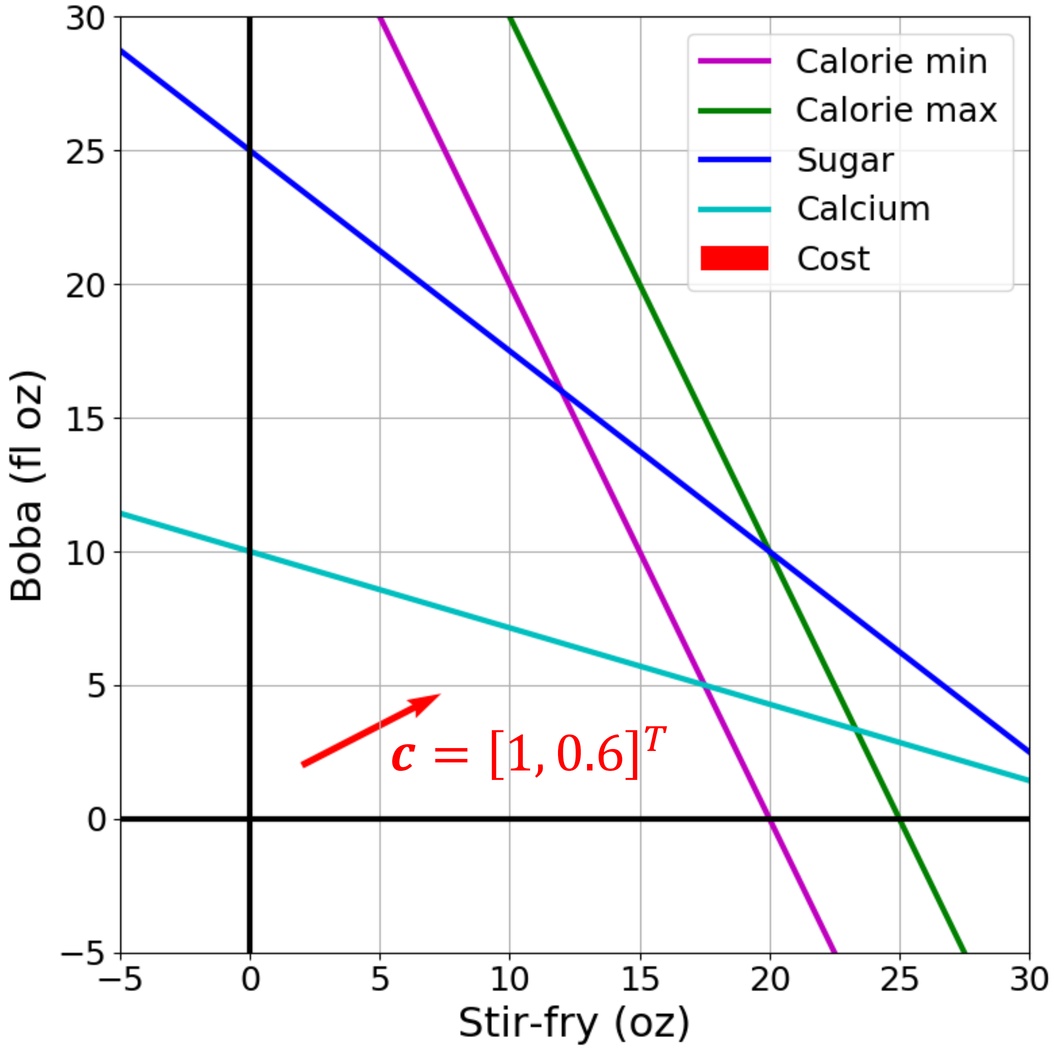
        *Right branch*: Added constraint $x_i \geq ceil(x_i)$

Note: Only add LPs to the queue if they are feasible

# Branch and Bound Example

$\boldsymbol{x}^{\star} = (17.5, 5)$
$y^{\star} = 20.5$

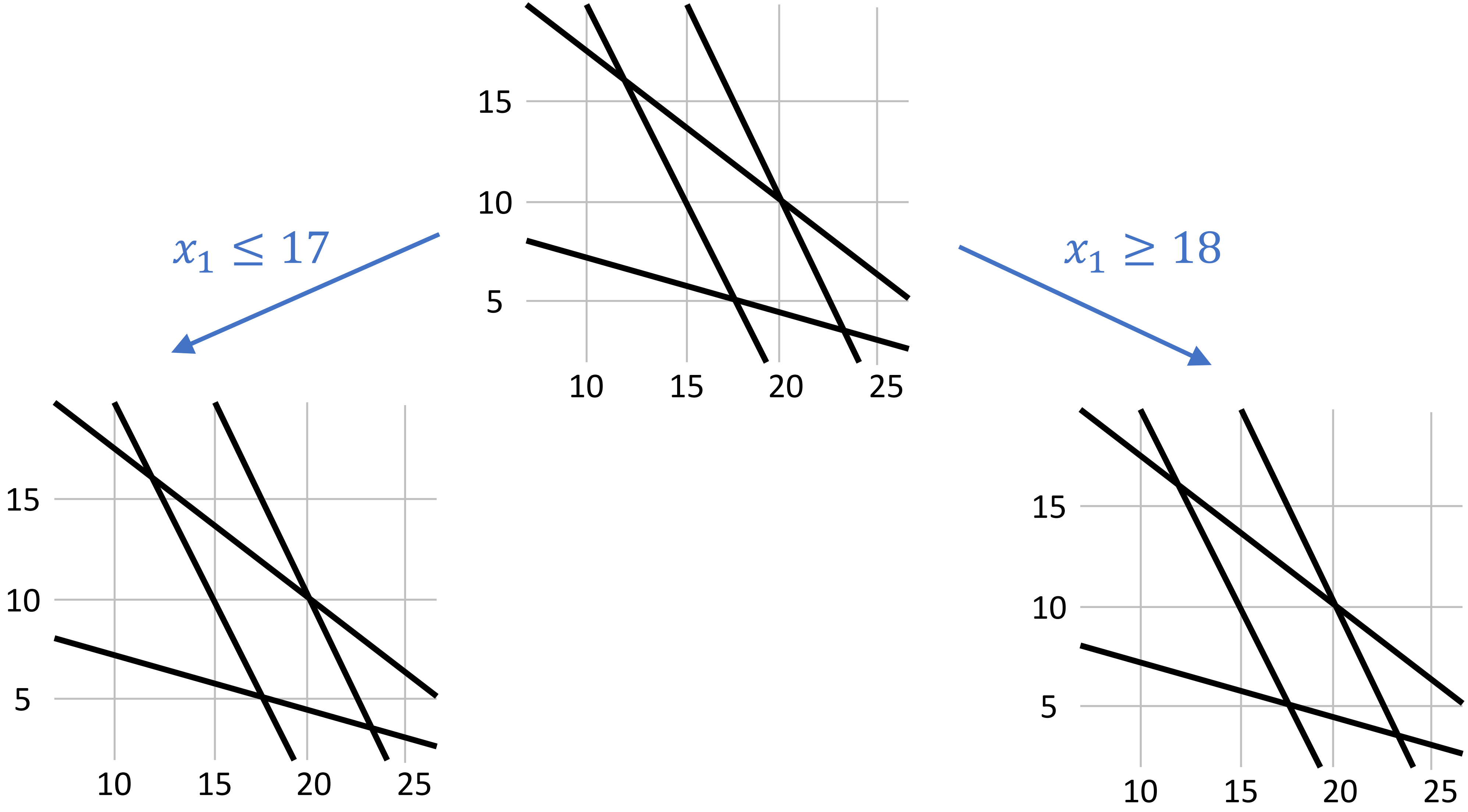
$x_1 \leq 17$ $\quad$ $x_1 \geq 18$




Priority Queue:

1. $\boldsymbol{x}^{\star} = (17.5, 5),\ y^{\star} = 20.5$

# Branch and Bound Example



# Branch and Bound Example

$$\boldsymbol{x}^{\star} = (17.5,5)$$
$$y^{\star} = 20.5$$

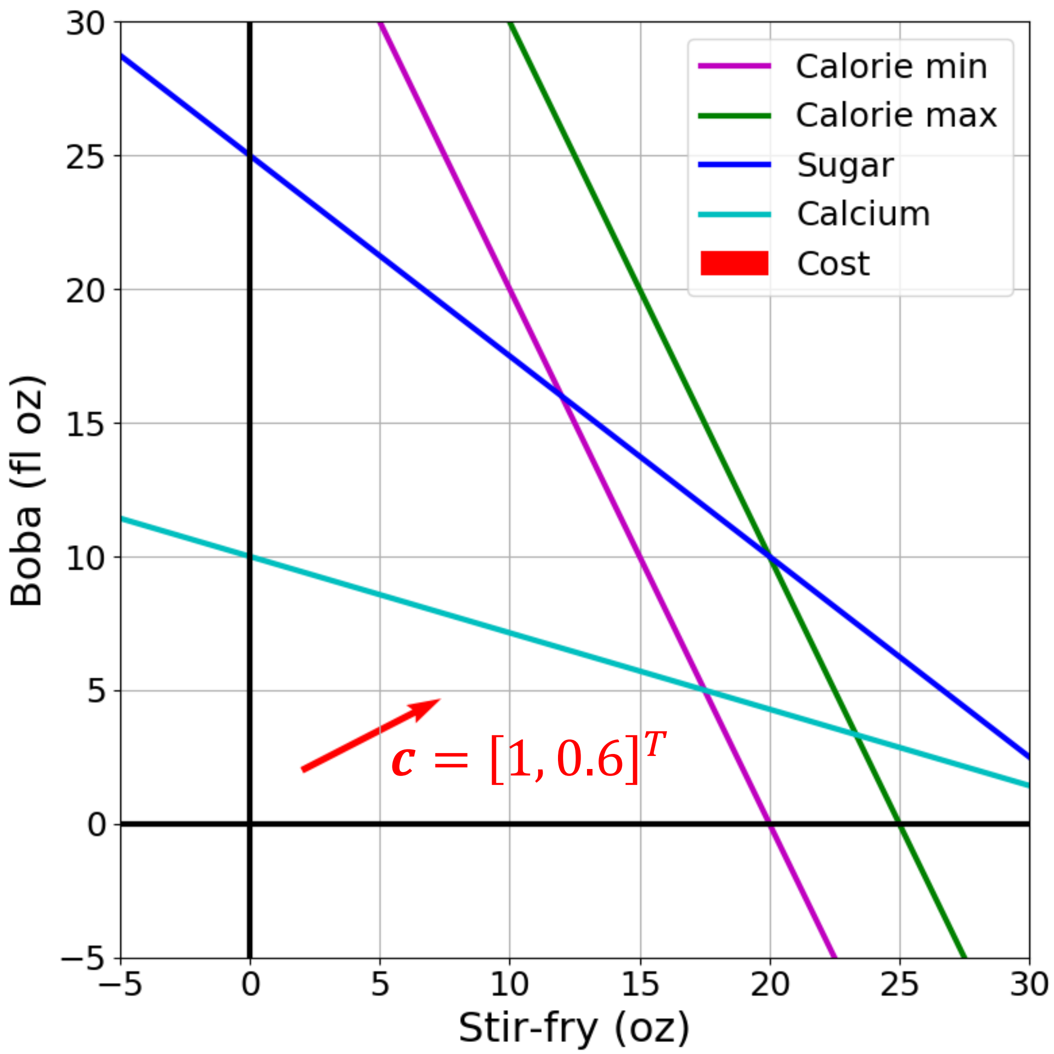
$x_1 \leq 17$ | $x_1 \geq 18$

$$\boldsymbol{x}^{\star} = (17,6)$$
$$y^{\star} = 20.6$$

$$\boldsymbol{x}^{\star} = (18,4.85)$$
$$y^{\star} = 20.91$$

Priority Queue:

1. $\boldsymbol{x}^{\star} = (17,6)$, $y^{\star} = 20.6$
2. $\boldsymbol{x}^{\star} = (18,4.85)$, $y^{\star} = 20.91$

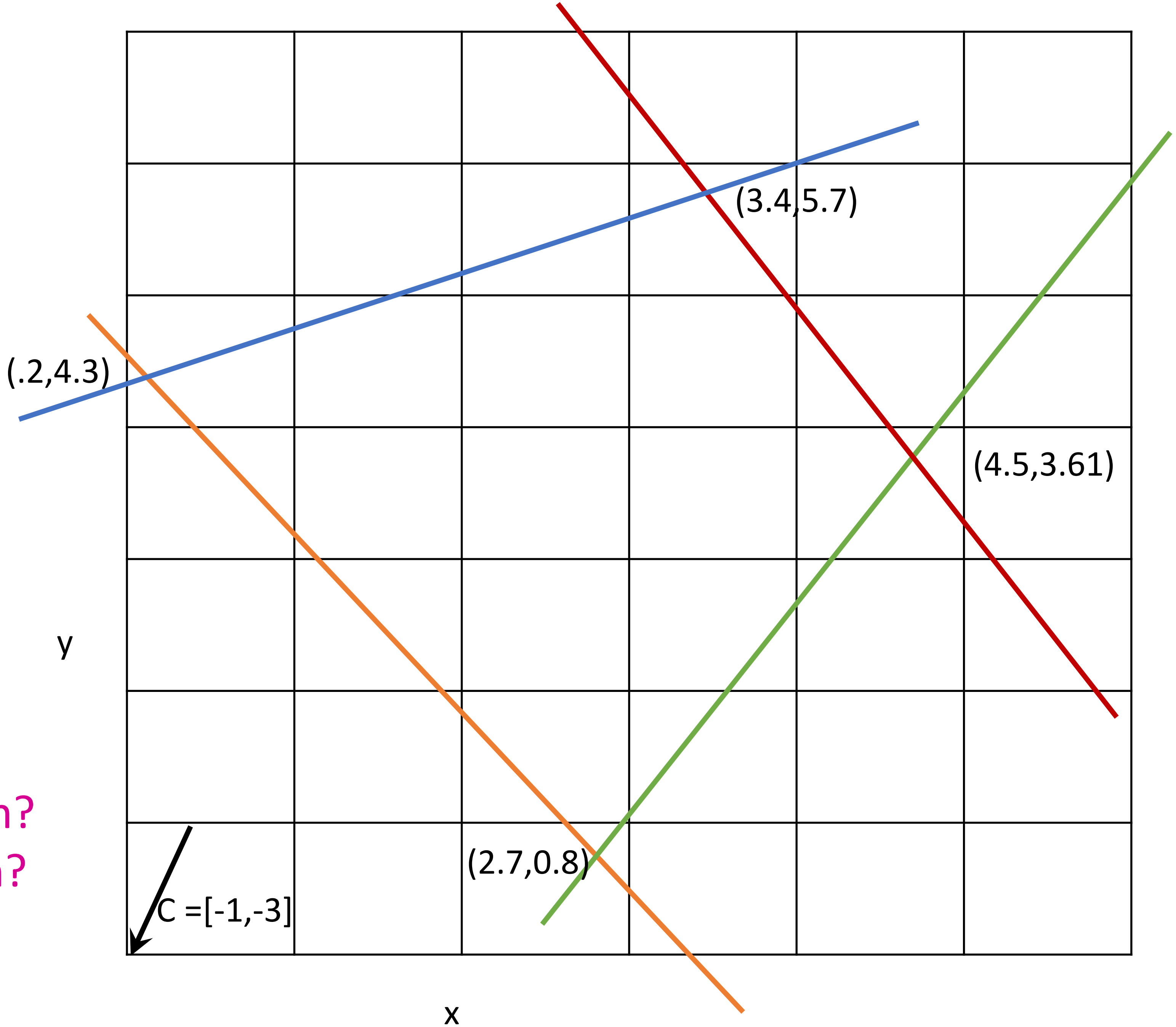


# Activity + Poll

Constraints :

$$y = -1.4x + 4.58$$
$$y = 1.56x + 3.41$$
$$y = -1.9x + 12.16$$
$$y = .44x + 4.21$$

Priority Queue:




Poll 4: What is the LP solution?

Poll 5: What is the IP solution?

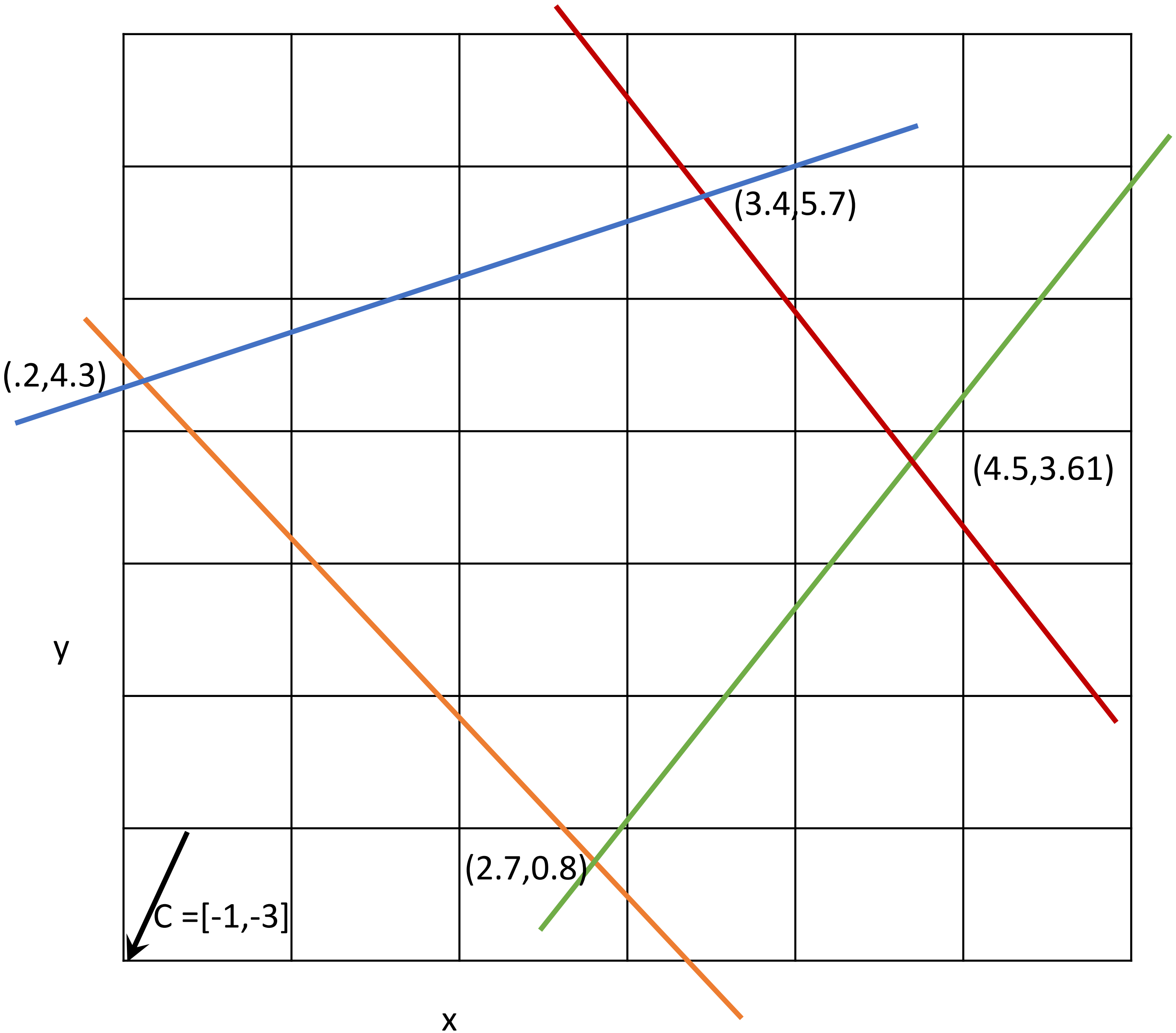
# Activity

**Constraints :**

$$y = -1.4x + 4.58$$
$$y = 1.56x + 3.41$$
$$y = -1.9x + 12.16$$
$$y = .44x + 4.21$$

<u>Priority Queue</u>:

-20.5: (3.4,5.7)



# Activity

Constraints :

$$y = -1.4x + 4.58$$
$$y = 1.56x + 3.41$$
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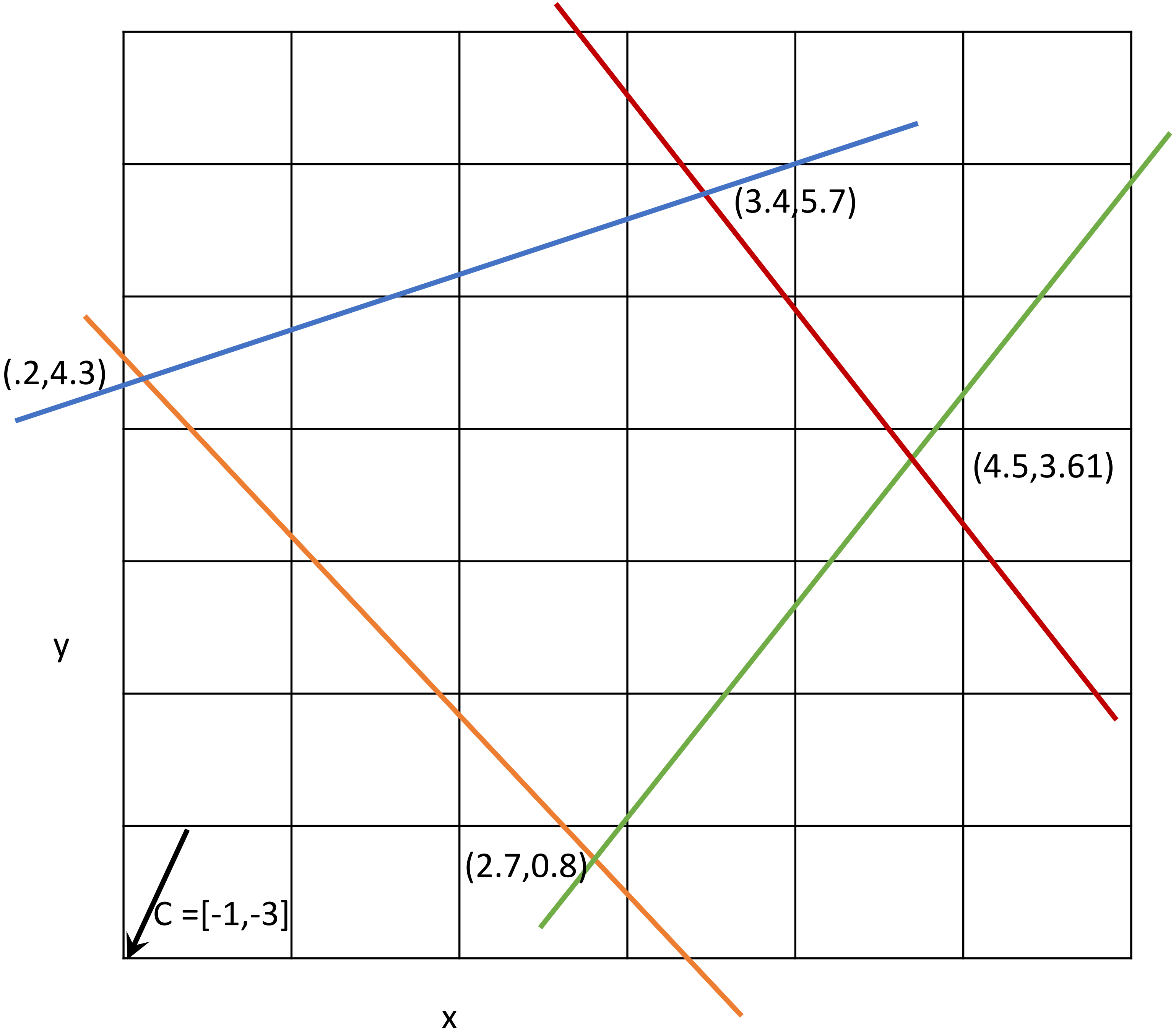



Priority Queue:

~~-20.5: (3.4,5.7)~~

-19.6: (3,5.53) (x <= 3)

-17.7: (4,4.56) (x >=4)

# Activity

Constraints :

$$y = -1.4x + 4.58$$
$$y = 1.56x + 3.41$$
$$y = -1.9x + 12.16$$
$$y = .44x + 4.21$$

<u>Priority Queue</u>:

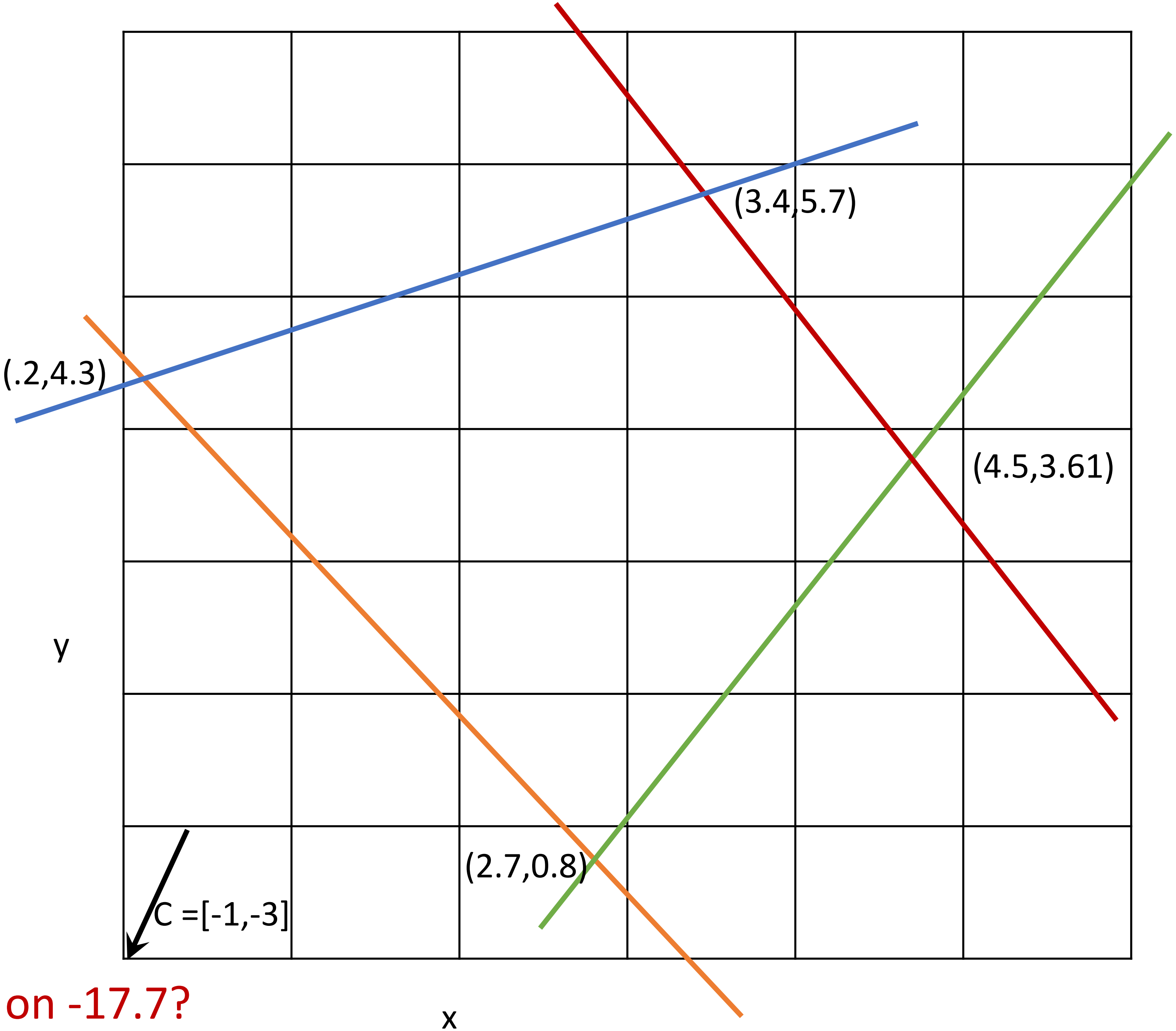
~~-20.5: (3.4,5.7)~~

~~-19.6: (3,5.53) (x <= 3)~~

-17.7: (4,4.56) (x >=4)

-18.0: (3,5) (x<=3,y<=5)

Inf: (x<=3,y>=6)




Why do we not need to recurse on -17.7?